SYLLABUS OF PLANE GEOMETRY

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SYLLABUS OF PLANE GEOMETRY

(Corresponding to Euclid, Book I - VI)

PREPARED AS AN INTRODUCTION TO ABSOLUTE GEOMETRY BY
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PREFACE

THE logical order in geometry has held the attention of mathematicians for more than two thousand years, but no essential progress was made towards its solution, after the time of Euclid, until the nineteenth century witnessed the creation and development of the non-Euclidean geometry. A radically new departure was taken when Lobatschewzky and Bolyai demonstrated that a consistent geometry could be constructed under an assumption which contradicts the parallel axiom. This took place before the year 1830, yet no serious attempt was made to reclassify the subject matter of geometry until the close of the century. The present essay, prepared as a thesis for the master's degree in the year 1898, attempts a classification based upon the three hypotheses which may be made with reference to the existence of parallels. It makes no use of the other independent postulates of geometry enunciated for the first time by Hilbert in the year 1899. A more minute classification than the one here presented is therefore possible, but what order is best adapted to the systematic unfolding of geometry for the beginner has not yet been determined. Meanwhile we have at hand in this essay a working plan which may be used to good effect, before the study of geometry is abandoned by the pupil, in exhibiting its logical structure. A course of geometry without some such syllabus is like an arch without a keystone; it is certain to fall into fragments.

IRVING STRINGHAM.

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Syllabus of Plane Geometry

DEFINITIONS

Def. 1. Any physical object takes up room. The room occupied by an object, considered apart from the object, is called a solid.

Def. 2. That which separates a solid from any part of the room surrounding it, but itself no solid, is called a surface.

Def. 3. That which separates one part of a surface from an adjacent part is called a line.

Def. 4. That which separates one part of a line from an adjacent part is called a point.

A point is not divisible; it has position only.

The intersection of two lines is a point. The intersection of a surface and a line is a point. The intersection of two surfaces is a line.

Def. 5. Points, lines, surfaces, or solids, or any combination of them, are called figures.

Def. 6. Any assumption concerning the relations of figures to one another is called a postulate.*

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* See Helmholtz: The Origin and Significance of Geometrical Axioms, Popular Scientific Lectures.

Def. 7. When a series of figures existing in accordance with certain given postulates form only other figures belonging to the same series, they are said to constitute a geometry.

A geometry can yield no relation between its figures that is not already contained in its postulates, but the statement of certain relations is essential to the complete determination of its figures. Such statements, capable of being established from previous assumptions, are called theorems. These previous assumptions may, themselves, be theorems or postulates.

It is evident that only those theorems are fundamental to a geometry which are required to establish its integrity. The other theorems, practically unlimited in number, express relations between figures in general.

Def. 8. The figure which contains the figures, as they exist in accordance with the postulates of a geometry, is called the space, or space form of that geometry.

The effect of introducing into a geometry a new postulate is to define more completely the character of its space-form.

Any postulate may be introduced into a

geometry which does not destroy the integrity of the geometry.

Def. 9. The space-form of a geometry is completely determined when no postulate can be added to the geometry which is not an equivalent of a postulate already included in it.

Def. 10. When the space-form of a geometry is completely determined, the geometry may be said to be logically complete.

Def. 11. The geometries which are logically complete for the same series of figures together constitute what is called the absolute geometry of those figures.

PLANE GEOMETRY

Book I.

Post. 1.* Any figure may be moved from one position in space to another, without change of size or shape.

Figures may be compared with respect to extent, or size, and shape. Figures which may be considered as extending in either of two opposite directions may, also, be compared in respect to sense. The sense of the figure is called positive when the figure is thought to extend in one direction and negative in the opposite direction. It is optional which of the two opposite directions is to be regarded as positive, and which negative. But, the selection once made, that convention must be observed throughout the entire case.

The sense of figures is usually disregarded except where its introduction is necessary for a complete generalization of the theorem.

Def. 12. If two figures are so related that points in the one take the place of lines in the

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* For a full discussion of this postulate, see Clifford's lectures on the "Philosophy of the Pure Sciences," Lectures and Essays (p. 222.)

other and lines in the one, of points in the other, the two figures are called reciprocal figures and corresponding theorems concerning the two figures, reciprocal theorems.*

SECTION I.

The Straight Line.

Def. 13. If a line is uniquely determined by any two points within it, which have no special relation to one another, it is called a straight line.

Def. 14. Any portion of a straight line cut off by two fixed points is called a segment.

In certain spaces there may be points so related that any number of straight lines can be drawn between two of them, as in the case of the great circles joining the poles of a sphere. But these are exceptional cases occurring only when the points bear a special relation to one another. In general, one straight line, and only one, can be drawn through two points.

Def. 15. If a surface is uniquely determined by any three points within it, which have no

* Olaus Henrici: Elementary Geometry, Congruent Figures.

special relation to one another, it is called a plane surface, or plane.*

Def. 16. A figure formed by straight lines is called a rectilineal figure. A rectilineal figure of one straight line is called an unilateral; of two straight lines, a bilateral; of three straight lines, a trilateral; of four straight lines, a quadrilateral; etc.

Def. 17. The figure which has for its reciprocal, a segment is called an angle.

A segment is determined by two points, its extremities lying in a straight line; hence an angle must be determined by two lines meeting in a point. The lines are called the sides of the angle and their common point, the vertex.

As a segment may be formed by the movement of a point from one position in a straight line to another, an angle may be formed by the movement of a straight line about a point in it from one position to another. The size of the segment depends upon the amount of movement of the point, likewise the size of the angle depends upon the amount of turning of the line and is

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* In the theorems and propositions which follow the words, "in the same plane," are to be understood when any other possibility might arise.

entirely independent of the length of the line or the extent of surface passed over.

Def. 18. To fix that sense of an angle which shall be regarded as positive, the rotation in a direction opposite to that of the hands of a watch is called positive.

No cognizance is taken of angles formed by the turning of a line through more than one complete rotation.

Def. 19. Two angles are said to be equal if the bilaterals forming them can be brought into coincidence in such a way that the rotating lines pass over the same surface.

Def. 20. The bisector of an angle is the straight line that divides it into two equal angles.

Def. 21. An angle is said to be a straight angle or a perigon, according as its arms lie in the same straight line on opposite sides or on the same side of the vertex.

Def. 22. Angles having the same vertex and a common arm are called adjacent angles.*

Def. 23. The angle formed by a line rotating from one exterior arm to the other and passing through the common arm is called the sum of the two adjacent angles

Def. 24. When two straight lines intersect so as

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* Halsted: Elementary Synthetic Geometry.

to make two of the adjacent angles equal, each of the angles formed is called a right angle.

Def. 25. A perpendicular to a straight line is a straight line that makes a right angle with it.

Def. 26. An angle less than a right angle is said to be acute; an angle greater than a right angle but less than a straight angle is said to be obtuse; an angle greater than a straight angle but less than a perigon is said to be reflex.

Def. 27. Two angles are said to be complements of each other, or complemental, if their sum is a right angle.

Def. 28. Two angles are said to be supplements of each other, or supplemental, if their sum is a straight angle.

Def. 29. Two angles are said to be conjugates of each other if their sum is a perigon.

Def. 30. The opposite angles of a bilateral are called vertical angles.

Theor. A.* All straight angles are equal.

(By post. 1 and the def. of a straight line.)

Cor. 1. All right angles are equal.

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* The theorems denoted by letters are considered to be fundamental. See def. 7.)

Cor. 2. If a straight line stands upon another straight line, it makes the adjacent angles together equal to two right angles.

Cor. 3. All the angles made by any number of straight lines drawn from a point, each with the next following in order, are together equal to four right angles.

Cor. 4. The complements of equal angles are equal.

Cor. 5. The supplements of equal angles are equal.

Cor. 6. The vertical angles of a bilateral are equal.

Prop. 1.* At a given point in a given straight line not more than one perpendicular can be drawn to that line. (Indirect method.)

Prop. 2. If the adjacent angles made by one straight line with two others, all three meeting at a point, are together equal to two right angles, these two straight lines are in one straight line.

Def. 31. Two figures are said to be congruent if

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* In addition to the theorems selected as fundamental, the proofs of many others are to be found in the usual text-books on Euclid. Those here given under the head of propositions complete the list as given in the Syllabus prepared by the Association for the Improvement of Geometrical Teaching. They are added for the purpose of showing how the subject matter of geometry distributes itself under the plan here presented.

they can be made by superposition to coincide with one another.

Def. 32. If in a rectilinear figure, points of intersection, to the number of its lines, are so selected that two, and only two, points fall on each line, the figure formed by the segments thus cut off on the lines is called a polygon.

The points of intersection which determine the polygon are called its vertices, and the segments joining them, its sides.

The surfaces enclosed, or the sum of the surfaces enclosed, is called the area of the polygon. (Negative areas are to be considered.)

If the sides of a polygon are produced in succession, either way, the angles determined by the sides produced and the following sides are called the exterior angles of the polygon; the supplements of the exterior angles are called the interior angles, or simply angles, of the polygon.

A straight line, other than a side, joining any two vertices of a polygon is called a diagonal.

The sum of the sides is called the perimeter of a polygon.

Def. 33. A polygon, no side of which cuts another when produced, is called a convex polygon.

Def. 34. A polygon of three angles is a trigon, or triangle; of four angles, a tetragon; of five angles, a pentagon; of six, a hexagon; etc.

Theor. B. If two triangles have two sides and the included angle of the one equal respectively to two sides and the included angle of the other, the triangles are congruent.

(If the sides of one triangle are arranged in reverse order to the corresponding sides of a second triangle, one of the triangles must be turned over before it can be brought into coincidence with the other. Now, under certain conditions it may be necessary for a proper understanding of the problem to draw the figures on a surface other than a plane. For example, if the question of the possible intersection of two lines in a second point should arise, a more suggestive figure can be obtained by using the surface of a sphere. In the turning of such a figure the surface must undergo flexure. This is not true, however, of the plane surface in which the figure, thus represented, actually lies.)

Theor. C. If two triangles have two angles and the included side of the one equal respectively to two angles and the included side of the other, the triangles are congruent.

Def. 35. If a triangle has two equal sides, it is called an isosceles triangle.

Def. 36. The line drawn from any vertex of a triangle to the middle point of the opposite side is called the median to that side.

Prop. 3. In an isosceles triangle, the angles opposite the equal sides are equal.

Prop. 4. An equilateral triangle is also equiangular.

Prop. 5. If two angles of a triangle are equal, the triangle is isosceles.

Prop. 6. An equiangular triangle is also equilateral.

Prop. 7. If two triangles have the three sides of the one respectively equal to the three sides of the other, the triangles are congruent.

Prop. 8. If two triangles have two sides of the one equal respectively to two sides of the other and the angles opposite one pair of equal sides equal, then the angles opposite the other pair of equal sides are either equal or supplemental, and, if equal, the triangles are congruent.

Def. 37. A line cutting two or more lines is called a transversal of those lines.

The angles formed by the transversal with either line on the side opposite to the other are called exterior angles.

The angles formed by the transversal with either line on the same side as the other are called interior angles.

The interior angles, or the exterior angles, on opposite sides of the transversal are called alternate angles.

An exterior angle and the non-adjacent interior angle on the same side of the transversal are called corresponding angles.

Prop. 9. If a transversal of two lines makes a pair of alternate angles equal, then any angle is equal to its alternate angle, any angle is equal to its corresponding angle, and any two interior or any two exterior angles on the same side of the transversal are supplemental.

Prop. 10. The same conclusions follow, if two corresponding angles are equal, or if two interior or two exterior angles, on the same side of the transversal are supplemental.

Section 2.

The Circle.

Def. 38. The circle is a portion of a surface bounded by one line called the circumference and is such that all points on that line are equidistant from a point within the figure called the center of the circle.

Def. 39. A straight line drawn from the center to the circumference is called a radius.

Def. 40. A straight line drawn through the center and terminated both ways by the circumference is called a diameter.

Prop. 11. Circles having equal radii are congruent. (Superposition.)

Prop. 12. Any diameter of a circle divides it into two congruent parts called semicircles.

Prop. 13. Any two diameters at right angles to one another divide the circle into four congruent parts called quadrants.

Def. 41. The figure formed by all points satisfying a given condition is called the locus of all points satisfying that condition.

Prop. 14. The circumference of a circle is the locus of all points at the distance of the radius from the center.

Def. 42. An arc is a part of a circumference.

Two arcs which together form a circumference are said to be conjugate. The greater of the two arcs is called the major arc, and the smaller, the minor arc.

Def. 43. An angle at the center is said to stand upon the arc which lies within the angle and is cut off by the sides of the angle.

Def. 44. A portion of a circle cut off by an arc and two radii drawn to its extremities is called a sector, and the angle at the center which stands upon that arc is called the angle of the sector.

Theor. D. In the same circle, or in equal circles, if two angles at the center are equal, the arcs on which they stand are also equal, and of two unequal angles at the center, the greater stands on the greater arc.

Prop. 15. In the same circle, or in equal circles, sectors which have equal angles are equal, and of two such sectors which have unequal angles, the greater is that which has the greater angle.

Prop. 16. In the same circle, or in equal circles, if two arcs are equal, the angles which they subtend at the center are also equal, and of two unequal arcs, the greater subtends the greater angle at the center.

Prop. 17. In the same circle, or in equal circles, equal sectors have equal angles, and of two unequal sectors, the greater has the greater angle.

Def. 45. The straight line joining any two points on a circumference is called a chord.

Prop. 18. In the same circle, or in equal circles, equal arcs are subtended by equal chords.

Prop. 19. In the same circle, or in equal circles, equal chords subtend equal arcs.

Prop. 20. The straight line drawn from the center to the middle point of a chord is perpendicular to the chord.

Prop. 21. The straight line drawn perpendicular to a chord through its middle point passes through the center of the circle.

Def. 46. If all the vertices of a polygon lie on a circumference, the polygon is said to be inscribed in the circle, and the circle is said to be circumscribed about the polygon.

Def. 47. A polygon is said to be regular when it is both equilateral and equiangular.

Prop. 22. If the whole circumference of a circle is divided into any number of equal arcs, the inscribed polygon formed by the chords of these arcs is regular. (Superposition.)

Section 3.

*Proportion.**

Def. 48. A figure is called a magnitude when its size alone is considered.

Def. 49. Two figures are said to be magnitudes of the same kind when they can be compared with one another.

Def. 50. A greater magnitude is said to be a

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* For a geometrical treatment of the subject of Proportion see Hall and Stevens' Text Book of Euclid's Elements, Book V.

multiple of a less, when the greater contains the less an exact number of times.

The following property of multiples is assumed as axiomatic:

m A > = or < m B according as A > = or < B, where *A* and *B* represent two magnitudes and *m* is any integer.

The converse proposition necessarily follows:

A > = or < B according as m A > = or < m B.

Def. 51. The ratio of one magnitude to another of the same kind is the relation which the first bears to the second.

The ratio of magnitude *A* to magnitude *B* is denoted thus, A: B, and *A* is called the antecedent, *B*, the consequent.

If *A* is equal to *B*, the ratio *A*:*B* is called a ratio of equality. If *A* is greater than *B*, the ratio *A*:*B* is called a ratio of greater inequality. If *A* is less than *B*, the ratio *A*:*B* is called a ratio of less inequality.

The ratio of *A*:*B* may be estimated by examining how the multiples of *A* are distributed among the multiples of *B* in their relative scale.

Def. 52. The ratio of two magnitudes is said to be equal to a second ratio of two other magnitudes (whether of the same or of a different kind from the former), when the multiples of the antecendent of the first ratio are distributed among those of its consequent in the same order as the multiples of the antecedent of the second ratio among those of its consequent.

Def. 53. When the ratio of A to B is equal to that of P to Q the four magnitudes, A, B, P, Q, are said to be proportionals, or to form a proportion. The proportion is denoted thus:

$$A : B :: P : Q,$$

which is read A is to B as P is to Q. A and Q are called the extremes; B and P, the means.

Def. 54. The ratio of one magnitude to another is greater that that of a third magnitude to a fourth, when it is possible to find equimultiples of the antecedents and equimultiples of the consequents, such that while the multiple of the antecedent of the first ratio is greater than, or equal to, that of its consequent, the multiple of the antecedent of the second ratio is not greater or is less, than that of its consequent.

Def. 55. Two ratios are said to be reciprocal when the antecedent and the consequent of one are the consequent and the antecedent of the other, respectively.

Theor. E. Ratios which are equal to the same ratio are equal to one another.

Prop. 23. If two ratios are equal, their reciprocal ratios are equal.

Prop. 24. Equal magnitudes have the same ratio to the same magnitude; and the same magnitude has the same ratio to equal magnitudes.

Prop. 25. Of two unequal magnitudes, the greater has a greater ratio to a third magnitude than the less has; and the same magnitude has a greater ratio to the less of two magnitudes than it has to the greater.

Prop. 26. Magnitudes which have the same ratio to the same magnitude are equal to one another; and those to which the same magnitude has the same ratio are equal to one another.

Prop. 27. That magnitude which has a greater ratio than another has to the same magnitude is the greater of the two, and that magnitude to which the same magnitude has a greater ratio than it has to another magnitude is the less of the two.

Prop. 28. Magnitudes have the same ratio one to another which their equimultiples have.

Prop. 29. If four magnitudes of the same kind are proportionals, the first is greater than, equal to, or less than, the third according as the second is greater than, equal to, or less than, the fourth.

Prop. 30. If four magnitudes of the same kind are proportionals, the first will have to the third the same ratio as the second to the fourth. (Alternando).

Theor. F. In the same circle, or in equal circles, angles at the center and sectors are proportional to the arcs on which they stand.

Book II.

Post 2. Through a given point there is at least one straight line which does not cut a given straight line however far produced.*

Section 1.

The Straight Line.

Theor. G. If any side of a triangle is produced, the exterior angle is greater than either of the interior angles not adjacent to it.

Prop. 31. Any two angles of a triangle are together less than two right angles.

Prop. 32. If a triangle has one right angle, or one obtuse angle, its remaining angles are acute.

Prop. 33. From a point outside a given line not more than one perpendicular can be drawn to that line.

Def. 56. A triangle, one of whose angles is a right angle, is called a right-angled triangle.

Def. 57. A triangle, one of whose angles is an obtuse angle, is called an obtuse-angled triangle.

Def. 58. A triangle, all of whose angles are acute, is called an acute-angled triangle.

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* The assumption here that space is infinite precludes the possibility of two straight lines intersecting more than once, also the return unto itself of an unlimited straight line.

Def. 59. The side opposite the right angle of a right-angled triangle is called the hypotemuse.

Prop. 34. If two sides of a triangle are unequal, the greater side has the greater angle opposite to it.

Prop. 35. If two angles of a triangle are unequal, the greater angle has the greater side opposite to it.

Prop. 36. Of all the straight lines that can be drawn to a given straight line from a given point outside it, the perpendicular is the shortest; of others, those making equal angles with the perpendicular are equal; and of two others, that which makes the greater angle with the perpendicular is the greater.

Prop. 37. Not more than two equal oblique lines can be drawn from a given point to a given straight line.

Prop. 38. The sum of any two sides of a triangle is greater than the third side.

Prop. 39. The difference of any two sides of a triangle is less than the third side.

Prop. 40. If from a point within a triangle, two lines are drawn to the extremities of a side, their sum is less than the sum of the other two sides of the triangle, but they contain the greater angle.

Prop. 41. If two triangles have two sides of the one equal respectively to two sides of the other, but the included angles unequal, then the third sides are unequal, the greater side being opposite the greater angle.

Prop. 42. If two triangles have two sides of the one equal respectively to two sides of the other, but the third sides unequal, then the included angles are unequal, the greater angle being opposite the greater third side.

Prop. 43. If two triangles have two angles of the one equal respectively to two angles of the other, and the sides opposite one pair of equal angles equal, the triangles are congruent.

Theor. H. If a transversal of two straight lines make a pair of alternate angles equal, the two straight lines will not meet.

Prop. 44. If a transversal of two straight lines makes a pair of corresponding angles equal, or a pair of interior angles on the same side supplemental, the straight lines will not meet.

Section 2.

The Circle.

Prop. 45. A diameter perpendicular to a chord bisects the chord and its subtended arc.

Prop. 46. All points in a chord lie within a circle, and all points in the chord produced lie without the circle.

Prop. 47. A straight line cannot meet the circumference of a circle in more than two points.

Def. 60. A straight line cutting the circumference of a circle in two points is called a secant.

Prop. 48. Two circles whose circumferences have three points in common coincide wholly.

Prop. 49. The circumferences of two circles which do not coincide cannot meet in more than two points.

Prop. 50. If from a point within a circle, more than two straight lines drawn to the circumference are equal, that point is the center.

Prop. 51. In the same circle, or in equal circles, the greater of two unequal minor arcs is subtended by the greater chord.

Prop. 52. In the same circle, or in equal circles, if two chords are unequal, the greater subtends the greater minor and the less major arc.

Prop. 53. In the same circle, or in equal circles, equal chords are equidistant from the center; and of two unequal chords, the greater is nearer the center.

Prop. 54. In the same circle, or in equal circles, chords that are equidistant from the center are equal; and of two chords unequally distant, the one nearer the center is the greater.

Prop. 55. The diameter is the greatest chord in a circle.

Prop. 56. Of all straight lines passing through a point on the circumference of a circle, the only one that does not meet the circumference again is perpendicular to the radius at that point.

Def. 61. The straight line which meets the circumference of a circle in but one point is said to touch, or be tangent to, the circle at that point. The point is called the point of contact or the point of tangency.

Prop. 57. The center of a circle lies in the perpendicular to any tangent at the point of contact.

Prop. 58. The radius perpendicular to a tangent meets it at the point of contact.

Prop. 59. A straight line cuts, touches, or does not meet a circumference, according as its distance from the center is less than, equal to, or greater than the radius.

Def. 62. The perpendicular to a given straight line from a given point, external to it, is called the distance of that point from the straight line.

Prop. 60. The distance of a straight line from the center of a circle is less than, equal to, or greater than, the radius according as the straight line cuts, touches, or does not meet the circumference.

Def. 63. Two circles are said to touch or be tangent when their circumferences have one, and only one, point in common.

Prop. 61. If two circumferences meet in a point which is not on the line joining their centers, they meet in one other point; their common chord is bisected at right angles by the line joining their centers; and the distance between their centers is greater than the difference and less than the sum of the radii.

Prop. 62. If two circles meet in one point only, that point lies in the line joining their centers.

Prop. 63. If two circles meet in a point which is on the line joining their centers, they are tangent.

Prop. 64. If two circles intersect, neither point of intersection is on the line joining their centers.

Prop. 65. If two circles touch one another, they have a common tangent line at the point of contact.

Def. 64. If all the lines of a rectilineal figure are tangent to a circle, the figure is said to be circumscribed about the circle and the circle

is called an inscribed or escribed circle of any polygon formed by the figure, according as it lies within or without that polygon.

Prop. 66. **If the whole circumference of a circle is divided into any number of equal arcs, the circumscribed polygon formed by the tangents drawn at all the points of division is regular.**

Book III.

Def. 65. Two straight lines which do not meet, however far produced, are said to be parallel.*

Post. 3. Through a given point there can not be more than one straight line parallel to a given straight line.

Section 1.

The Straight Line.

Theor. I. The alternate angles made by a transversal with two parallels are equal.

Cor. The corresponding angles made by a transversal with two parallels are equal.

Prop. 67. A transversal perpendicular to one of two parallels is perpendicular to the other also.

Prop. 68. The interior angles on the same side of a transversal of two parallels are supplemental.

Prop. 69. If the interior angles on the same side of a transversal of two straight lines are not supplemental, the straight lines are not parallel, but meet on that side of the trans-

* The term "parallel" thus defined includes the non-intersectors of hyperbolic space.

versal on which the sum of the interior angles is less than a straight angle.

Theor. J. Straight lines parallel to the same straight line are parallel to each other.*

Prop. 70. Any exterior angle of a triangle is equal to the sum of the two opposite interior angles, and the sum of the three interior angles is equal to two right angles.

Prop. 71. The acute angles of a right-angled triangle are supplemental.

Prop. 72. The sum of the interior angles of a polygon is equal to two right angles, taken as many times less two as the figure has sides.

Prop. 73. The sum of the exterior angles of a polygon is four right angles.

Def. 66. A quadrilateral whose opposite sides are parallel is called a parallelogram.

Def. 67. A quadrilateral that has one pair of opposite sides parallel is called a trapezoid.

Prop. 74. In a parallelogram, any two opposite angles are equal, and any two consecutive angles are supplemental.

Prop. 75. If one angle of a parallelogram is a right angle, all of its angles are right angles.

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* This theorem is true for hyperbolic space if a distinction is made between parallels (proper) and non-intersectors, but it did not seem advisable to make such a distinction in an Elementary Geometry.

Def. 68. A parallelogram, one of whose angles is a right angle, is called a rectangle.

Theor. K. In any parallelogram, either diagonal divides it into two congruent triangles, also the opposite sides are equal.

Prop. 76. If two adjacent sides of a parallelogram are equal, all of its sides are equal.

Def. 69. A rhombus is a parallelogram that has two adjacent sides equal.

Def. 70. A square is a rectangle that has two adjacent sides equal.

Prop. 77. If two parallelograms have two adjacent sides of the one equal respectively to two adjacent sides of the other, and an angle of the one equal to an angle of the other, the parallelograms are congruent.

Prop. 78. Two rectangles are congruent if two adjacent sides of the one are equal respectively to any two adjacent sides of the other.

Prop. 79. Two squares are congruent if a side of the one is equal to a side of the other.

Prop. 80. If a quadrilateral has two opposite sides equal and parallel, it is a parallelogram.

Prop. 81. If there are two pairs of straight lines all of which are parallel, and if the intercepts made by each pair on any transversal are equal, then the intercepts on any other transversal are also equal.

Prop. 82. If a system of parallels cuts off equal segments on any transversal, it cuts off equal segments on every transversal.

Prop. 83. The straight line drawn through the middle point of one side of a triangle parallel to another side bisects the third side.

Prop. 84. The straight line joining the middle points of two sides of a triangle is parallel to the third side.

SECTION 2.

Equality of Polygons.

Def. 71. The altitude of a parallelogram with respect to a given side as base is the perpendicular distance between the base and the opposite side.

Def. 72. The altitude of a triangle with respect to a given side as base is the perpendicular distance from the opposite vertex to the line of that base.

Theor. L. Parallelograms on the same base, or on equal bases, and between the same parallels are equal.

Cor. A parallelogram is equal to a rectangle of the same base and altitude.

Prop. 85. Parallelograms having equal bases and altitudes are equal; and of two parallelograms having equal altitudes, that is the greater which has the greater base; and also of two parallelograms having equal bases, that is the greater which has the greater altitude.

Theor. M. Triangles on the same base, or on equal bases, and between the same parallels, are equal.

Cor. A triangle is equal to half of a rectangle of the same base and altitude.

Prop. 86. Triangles having equal bases and equal altitudes are equal.

Prop. 87. Equal triangles on the same base, or on equal bases, have equal altitudes.

Theor. N. Equal triangles on the same base, or on equal bases in the same straight line, and on the same side of it, are between the same parallels.

Prop. 88. A trapezoid is equal to a rectangle whose base is half the sum of the two parallel sides, and whose altitude is the perpendicular distance between them.

Prop. 89. The straight lines, drawn through any point in a diagonal of a parallelogram parallel to the sides, form equal parallelograms on opposite sides of the diagonal.

Def. 73. All rectangles being congruent which have two adjacent sides equal to two given straight lines, any such rectangle is spoken of as the rectangle of those lines.

Def. 74. In like manner, any square whose side is equal to a given straight line is spoken of as the square of that line.

Def. 75. A point in a segment of a straight line is said to divide it internally; a point in a segment produced is said to divide it externally.

Prop. 90. The square of the sum of two lines is equal to the sum of the squares of those lines and twice their rectangle.

Prop. 91. The square of the difference of two lines is equal to the sum of the squares of those lines less twice their rectangle.

Prop. 92. The difference of the squares of two lines is equal to the rectangle of the sum and difference of those lines.

Def. 76. The projection of a point on a straight line is the foot of the perpendicular let fall from the point to the straight line.

Def. 77. The projection of a finite line on a straight line is the segment cut off on the latter by the projection of the extremities of the former.

Prop. 93. The square on any side of a triangle is equal to the sum of the squares on the other two sides together with twice the rectangle formed by one of those sides with the projection of the other upon it. (By the use of zero and negative magnitudes.)

Prop. 94. The sum of the squares on any two sides of a triangle is equal to twice the square on half of the third side, increased by twice the square on the median to that side.

Prop. 95. The sum of the squares on the sides of a quadrilateral is equal to the sum of the squares on the diagonals together with four times the square of the lines joining the middle points of the diagonals.

Section 3.

The Circle.

Def. 78. A segment of a circle is either of the two figures into which a circle is divided by a chord.

Def. 79. The segments are called major or minor segments according as their arcs are major or minor arcs.

Def. 80. An angle is said to be inscribed in a circle when its vertex lies on the circumference and its sides form chords of the circle. An inscribed angle is said to stand upon the arc between its arms. An inscribed angle is said to be an angle in that segment whose arc is the conjugate of the arc upon which it stands.

Prop. 96. An inscribed angle equals half the angle at the center standing on the same arc.

Prop. 97. Angles in the same segment are equal to one another.

Prop. 98. An angle in a segment is greater than, equal to, or less than a right angle, according as the segment is less than, equal to, or greater than a semicircle.

Prop. 99. The opposite angles of a quadrilateral inscribed in a circle are supplementary.

Prop. 100. Two tangents, and only two, can be drawn to a circle from the same external point.

Prop. 101. Tangents to a circle from the same external point are equal.

Prop. 102. An angle formed by a tangent and a chord is equal to half the angle at the center standing on the intercepted arc.

Prop. 103. An angle formed by two secants intersecting within or without a circle is equal to half the sum of the angles at the center subtended by the intercepted arcs. (By using negative quantities.)

Prop. 104. The arcs intercepted on a circumference by two parallel lines are equal.

Prop. 105. A circle can be circumscribed about, or inscribed in, a regular polygon.

Prop. 106. If a chord of a circle is divided into two segments, either internally or externally, the rectangle contained by these segments is equal to the difference of the squares on the radius and on the line joining the given point with the center of the circle.

SECTION 4.

Proportion.

Theor. O. Rectangles having equal altitudes are to one another as their bases.

Cor. Parallelograms or triangles of the same altitude are to one another as their bases.

Prop. 107. A line parallel to one side of a triangle divides the other two sides proportionally. The two triangles thus formed have their corresponding sides proportional.

Prop. 108. A line can be divided, internally or externally, into segments having a given ratio, except that if it is divided externally the ratio cannot be one of equality; and if the line is given in both length and sense, the point of division is unique.

Prop. 109. The line which divides two sides of a triangle proportionally is parallel to the third side.

Theor. P. If on two straight lines, cut by two parallel straight lines, equimultiples of the intercepts respectively are taken; then the line joining the points of section is parallel to either of the two parallels.

Theor. Q. If two straight lines are cut by three parallel straight lines, the intercepts on the one are proportional to the corresponding intercepts on the other.

Def. 81. Two figures are said to be placed in perspective when the straight lines joining corresponding points of the figures pass through a common point, called the center of perspective.

Def. 82. Two figures are said to be similar when they can be so placed that the center of perspective divides each line joining corresponding points into segments having a constant ratio.

Def. 83. The center of perspective is then called the center of similitude and the ratio into which it divides the lines, the ratio of similitude.

Prop. 110. Mutually equiangular triangles are similar.

Prop. 111. If two triangles have an angle of the one equal to an angle of the other, and the including sides proportional, they are similar.

Prop. 112. If two triangles have their corresponding sides proportional, they are similar.

Prop. 113. Similar triangles have their corresponding angles equal and their corresponding sides proportional.

Prop. 114. Similar polygons have their corresponding angles equal and their corresponding sides proportional.

Prop. 115. Similar polygons may be divided into the same number of similar triangles.

Prop. 116. Polygons similar to the same polygon are similar to each other.

Prop. 117. If in a right-angled triangle, a perpendicular is drawn from the vertex of the right angle to the hypotemuse, it divides the triangle into two right triangles which are similar to the whole triangle, and also to each other.

Prop. 118. Each side of a right-angled triangle is a mean proportional between the hypotemuse and its adjacent segment.

Prop. 119. The perpendicular from the vertex of the right angle to the hypotemuse of a right-angled triangle is a mean proportional between the segments of the hypotemuse.

Prop. 120. Similar triangles are to one another as the squares of their corresponding sides.

Prop. 121. Similar polygons are to one another as the squares of their corresponding sides.

Prop. 122. Any polygon described on the hypotemuse of a right-angled triangle is equal to the sum of the two similar and similarly described polygons on the sides.