Finkel's Mathematical Solution Book

A MATHEMATICAL

SOLUTION BOOK.

CONTAINING

SYSTEMATIC SOLUTIONS OF MANY OF THE MOST DIFFICULT PROBLEMS

Taken from the Leading Authors on Arithmetic and Algebra, Many Problems and Solutions from Geometry, Trigonometry, and Calculus, Many Problems and Solutions from the Leading Mathematical Journals of the United States, and Many Original Problems and Solutions,

WITH

NOTES AND EXPLANATIONS.

BY

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FOURTH EDITION—REVISED AND ENLARGED.

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DEDICATED

TO MY FRIEND,

H. S. LEHR, A. M., PH. D.

PRESIDENT OF THE

OHIO NORMAL UNIVERSITY.

PREFACE TO THE FIRST EDITION.

This work is the outgrowth of a number of years' experience in teaching in the Public Schools, during which time I have observed that a work presenting a systematic treatment of solutions of problems would be serviceable to both teachers and pupils.

It is not intended to serve as a key to any work on mathematics; but the object of its appearance is to present, for use in the schoolroom, such an accurate and logical method of solving problems as will best awaken the latent energies of pupils, and teach them to be original investigators in the various branches of science.

It will not be denied by any intelligent educator that the so-called "Short Cuts" and "Lightning Methods" are positively injurious to beginners in mathematics. All the "whys" are cut out by these methods and the student robbed of the very object for which he is studying mathematics; *viz.*, the development of the reasoning faculty and the power to express his thoughts in a forcible and logical manner. By pursuing these methods, mathematics is made a mere memory drill and when the memory fails, all is lost; whereas, it should be presented in such a way as to develop the memory, the imagination, and the reasoning faculty. By following out the method pursued in this book, the mind will be strengthened in these three powers, besides a taste for neatness and a love of the beautiful will be cultivated.

Any one who can write out systematic solutions of problems can resort to "Short Cuts" at pleasure; but, on the other hand, let a student who has done all his work in mathematics by formulæ, "Short Cuts," and "Lightning Methods" attempt to write out a systematic solution — one in which the work explains itself — and he will soon convince one of his inability to express his thoughts in a logical manner. These so-called "Short Cuts" should not be used at all, in the schoolroom. After pupils and students have been drilled on the systematic method of solving problems, they will be able to solve more problems by short methods than they could by having been instructed in all the "Short Cuts" and "Lightning Methods" extant.

It can not be denied that more time is given to, and more time wasted in the study of arithmetic in the public schools than in any other branch of study; and yet, as a rule, no better results are obtained in this branch than in any other. The reason of this, to my mind, is apparent. Pupils are allowed to combine the numbers in such a way as "to get the answer" and that is all that is required. They are not required to tell why they do this, or why they do that, but, "did you get the answer?" is the question. The art of "ciphering" is thus developed at the expense of the reasoning faculty.

The method of solving problems pursued in this book is often called the "Step Method." But we might, with equal propriety, call any orderly manner of doing any thing, the "Step Method." There are only two methods of solving problems—a right method and a wrong method. That is the right method which takes up, in logical order, link by link, the chain of reasoning and arrives at the correct result. Any other method is wrong and hurtful when pursued by those who are beginners in mathematics.

One solution, thoroughly analyzed and criticised by a class, is worth more than a dozen solutions the difficulties of which are seen through a cloud of obscurities.

This book can be used to a great advantage in the classroom—the problems at the end of each chapter affording ample exercise for supplementary work.

Many of the Formulæ in Mensuration have been obtained by the aid of the Calculus, the operation alone being indicated. This feature of the work will not detract from its merits for those persons who do not understand the Calculus; for those who do understand the Calculus it will afford an excellent drill to work out all the steps taken in obtaining the formulæ. Many of the formulæ can be obtained by elementary geometry and algebra. But the Calculus has been used for the sake of presenting the beauty and accuracy of that powerful instrument of mathematics.

In cases in which the formulæ lead to series, as in the case of the circumference of the ellipse, the rule is given for a near approximation.

It has been the aim to give a solution of every problem presenting anything peculiar, and of those which go the rounds of the country. Any which have been omitted will receive space in future editions of this work. The limits of this book have compelled me to omit much curious and valuable matter in Higher Mathematics.

I have taken some problems and solutions from the *School Visitor*, published by John S. Royer; the *Mathematical Magazine*, and the *Mathematical Visitor*, published by Artemas Martin, A. M., Ph. D., LL. D.; and the *Mathematical Messenger*, published by G. H. Harvill, by the kind permission of these distinguished gentlemen.

It remains to acknowledge my indebtedness to Prof. William Hoover, A. M., Ph. D., of the Department of Mathematics and Astronomy in the Ohio University at Athens, for critically reading the manuscript of the part treating on Mensuration, and to William G. Williams, LL. D., Wright—Professor of the Greek Language and Literature in the Ohio Wesleyan University, for his aid in extending the names of polygons one pages **236–237**.

Hoping that the work will, in a measure, meet the object for which it is written, I respectfully submit it to the use of my fellow teachers and co-laborers in the field of mathematics.

Any correction or suggestion will be thankfully received by communicating the same to

THE AUTHOR.

PREFACE TO SECOND EDITION.

In bringing out a second edition of this work, I am greatly indebted to Dr. G. B. M. Zerr for critically reading the work with a view to eliminating all errors.

THE AUTHOR.

Drury College, Feb. 19, 1897.

PREFACE TO THIRD EDITION.

The hearty reception accorded this book, as is attested by the fact that two editions of 1,200 copies each have already been sold, encouraged me to bring out this third edition.

In doing so, I have availed myself of the opportunity of making some important corrections, and such changes and improvements as experience and the suggestions of teachers using the book have dictated. The very favorable comments on the work by some of the most eminent mathematicians in this country confirm the opinion that the book is a safe one to put into the hands of teachers and students.

While mathematics is the exact science, yet not every book that is written upon it treats of it as though it were such. Indeed, until quite recently, there were very few books on Arithmetic, Algebra, Geometry, or Calculus that were not mere copies of the works written a century ago, and in this way the method, the spirit, the errors and the solecisms of the past two hundred years were preserved and handed down to the present generation. At the present time the writers on these subjects are breaking away from the beaten paths of tradition, and the result, though not wholly apparent, is a healthier and more vigorous mathematical philosophy. Within the last twenty-five years there has set in, in America, a reaction against the spirit and the method of previous generations, so that C. A. Laisant, in his *La Mathématique; Philosophie.—Enseignement,* Paris, 1898, says, "No country has made greater progress in mathematics during the past twenty-five years than the United States." Most of the text-books on Arithmetic, Algebra, Geometry, and the Calculus, written within the last five years, are evidence of this progress.

The reaction spoken of was brought about, to some extent, by the introduction into our higher institutions of learning of courses of study in mathematics bearing on the wonderful researches of Abel, Cauchy, Galois, Riemann, Weierstrass, and others. This reaction, it may be said, started as early as 1832, the time when Benjamin Peirce, the first American worthy to be ranked with Legendre, Wallis, Abel, and the Bernouillis, became professor of mathematics and natural philosophy at Harvard University. Since that time the mathematical courses in our leading Universities have been enlarged and strengthened, until now the opportunity for research work in mathematics as offered, for example, at the Universities of Chicago, Harvard, Yale, Cornell, Johns Hopkins, Princeton, Columbia and others, is as good as is to be found anywhere in the world. For example, the following are the subjects offered at Harvard for the Academic year 1899-1900: Logarithms, Plane and Spherical Trigonometry; Plane Analytical Geometry; Plane and Solid Analytical Geometry; Algebra; Theory of Equations.—Invariants; Differential and Integral Calculus; Modern Methods in Geometry.—Determinants; Elements of Mechanics; Quaternions with application to Geometry and Mechanics; Theory of Curves and Surfaces; Dynamics of a Rigid Body; Trigonometric Series.—Introduction to Spherical Harmonics.—Potential Function; Hydrostatics.—Hydrokinematics.—Force Functions and Velocity-Potential Functions and their uses.—Hydrokinetics; Infinite Series and Products; The Theory of Functions; Algebra.—Galois's Theory of Equations; Lie's Theory as applied to Differential Equations; Riemann's Theory of Functions; The Calculus of Variations; Functions Defined by Linear Differential Equations; The Theory of Numbers; The Theory of Planetary Motions; Theory of Surfaces; Linear Associative Algebra; the Algebra of Logic; the Plasticity of the Earth; Elasticity; and the Elliptic and the Abelian Transcendants.

While great activity and real progress in mathematics is going on in our higher institutions of learning, a like degree of activity is not yet being manifested in many of our colleges and academies and the Public Schools in general. It is not desirable that the quantity of mathematics studied in our Public Schools be increased, but it is desirable that the quality of the teaching should be greatly improved. To bring about this result is the aim of this book.

It does not follow, as is too often supposed, that any one familiar with the multiplication table, and able, perhaps, to solve a few problems, is quite competent to teach Arithmetic, or "Mathematics," as arithmetic is popularly called. The very first principles of the subject are of the utmost importance, and unless the correct and refined notions of these principles are presented at the first, quite as much time is lost by the student in unlearning and freeing himself from erroneous conceptions as was required in acquiring them. Moreover, no advance in those higher modern developments in Mathematics is possible by any one having false notions of its first principles.

As a branch for mental discipline, mathematics, when properly taught, has no superior. Other subjects there are that are equally beneficial, but none superior. The idea entertained by many teachers,—generally those who have prepared themselves to teach other subjects, but teach mathematics until an opportunity to teach in their special line presents itself to them,—that mathematics has only commercial value and only so much of it should be studied as is needed by the student in his business in after life, is pedagogically and psychologically wrong. Mathematics has not only commercial value, but educational and ethical value as well, and that to a degree not excelled by any other science. No other science offers such rich opportunity for original investigation and discovery. So far from being a perfected and complete body of doctrine "handed down from heaven" and incapable of growth, as many suppose, it is a subject which is being developed at such a marvelous rate that it is impossible for any but the best to keep in sight of its ever-increasing and receding boundary. Because, therefore, of the great importance of mathematics as an agent in disciplining and developing the mind, in advancing the material comforts of man by its application in every department of art and invention, in improving ethical ideas, and in cultivating a love for the good, the beautiful, and the true, the teachers of mathematics should have the best training possible. If this book contributes to the end, that a more comprehensive view be taken of mathematics, better services rendered in presenting its first principles, and greater interest taken in its study, I shall be amply rewarded for my labor in its preparation.

In this edition I have added a chapter on Longitude and Time, the biographies of a few more mathematicians, several hundred more problems for solution, an introduction to the study of Geometry, and an introduction to the study of Algebra.

The list of biographies could have been extended indefinitely, but the student who becomes interested in the lives of a class of men who have contributed much to the advancement of civilization, will find a short sketch of the mathematicians from the earliest times down to the present day in Cajori's *History of Mathematics* or Ball's *A Short History of Mathematics.*

The biographies which have been added were taken from the *American Mathematical Monthly.* I have received much aid in my remarks on Geometry from *Study and Difficulties of Mathematics,* by Augustus De Morgan.

It yet remains for me to express my thanks to my colleague and friend, Prof. F. A. Hall, of the Department of Greek, for making corrections in the Greek terms used in this edition.

THE AUTHOR.

Drury College, July, 1899.

PREFACE TO THE FOURTH EDITION.

In bringing out this edition, I have been guided somewhat by the suggestions of many teachers who have used the book, and, in view of these suggestions, I have inserted Tables of Compound Numbers; Factoring; Arithmetical and Geometrical Progressions; and Specific Gravity,— giving each of these subjects the fullest possible treatment compatible with the limits of the book. It did not seem wise to include the Extraction of Square and Cube Roots as these are fully discussed in most arithmetics.

I have taken this opportunity to insert the Metric System of Weights and Measures and have called special attention to the importance of this simple, beautiful, and convenient system. Fractions have been more fully treated and the problems with their solutions in Analysis have been classified under familiar titles, viz., Age Problems, Fish Problems, Fox and Hound Problems, Time Problems, etc. Also the problems at the end of Mensuration have been placed in their appropriate places in the body of that subject. Another Examination Test with answers and biographies of Newton and Gauss have been inserted. Many other minor changes and additions have been made.

I have endeavored to eliminate all errors, and in this I have received much help from W. J. Greenstreet, editor of the *Mathematical Gazette*, Stroud, Eng., to whom I hereby acknowledge my indebtedness.

THE AUTHOR.

Drury College, July 17, 1902.

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CHAPTER I.

DEFINITIONS.

1. ***Mathematics*** (μαθηματική, science) is that science which treats of quantity.

- MATHEMATICS.
  - I. Pure.
    - (1.) Arithmetic.
    - (2.) Algebra...
      - 1. Calculus..........
        - a. Differential.
        - b. Integral.
        - c. Calculus of Variations.
      - 2. Quaternions.
    - (3.) Geometry..
      - 1. Platonic Geometry..
        - a. Pure Geometry.
        - b. Conic Sections.
        - c. Trigonometry..
          - 1. Plane Trigon'y.
          - 2. Analytical Trig.
          - 3. Spherical "
      - 2. Analytical Geometry.
      - 3. Descriptive Geometry.
  - II. Applied.
    - (1.) Mensuration.
    - (2.) Surveying.
    - (3.) Navigation.
    - (4.) Mechanics.
    - (5.) Astronomy.
    - (6.) Optics.
    - (7.) Gunnery.
    - (8.) &c., &c.

2. ***Pure Mathematics*** treats of magnitude or quantity without relation to matter.

3. ***Applied Mathematics*** treats of magnitude as subsisting in material bodies.

4. ***Arithmetic*** (ἀριθμητική, from ἀριθμός, a number) is the science of numbers and the art of computing by them.

5. ***Algebra*** (*Ar. al*, the, and *geber*, philosopher) is that method of mathematical computation in which letters and other symbols are employed.

6. ***Geometry*** (γεωμετρία, from γεωμετρεῖν to measure land, from γέα, γῆ, the earth, and μετρεῖν, to measure) is the science of position and extension.

7. ***Calculus*** (*Calculus*, a pebble) is that branch of mathematics which commands, by one general method, the most difficult problems of geometry and physics.

8. *Differential Calculus* is that branch of Calculus which investigates mathematical questions by measuring the relation of certain infinitely small quantities called *differentials*.

9. *Integral Calculus* is that branch of Calculus which determines the functions from which a given differential has been derived.

10. *Calculus of Variations* is that branch of calculus in which the laws of dependence which bind the variable quantities together are themselves subject to change.

11. *Quaternions* (*quaternis*, from *quaterni* four each, from *quattuor*, four) is that branch of algebra which treats of the relations of magnitude and position of lines or bodies in space by means of the quotient of two direct lines in space, considered as depending on a system of four geometrical elements, and as expressed by an algebraic symbol of quadrinominal form.

12. *Platonic Geometry* is that branch of geometry in which the argument is carried forward by a direct inspection of the figures themselves, delineated before the eye, or held in the imagination.

13. *Pure Geometry* is that branch of Platonic geometry in which the argument may be practically tested by the aid of the compass and the square only.

14. *Conic Sections* is that branch of Platonic geometry which treats of the curved lines formed by the intersection of a cone and a plane.

15. *Trigonometry* (*τρίγωνον*, triangle, *μέτρον*, measure) is that branch of Platonic geometry which treats of the relations of the angles and sides of triangles.

16. *Plane Trigonometry* is that branch of trigonometry which treats of the relations of the angles and sides of plane triangles.

17. *Analytical Trigonometry* is that branch of trigonometry which treats of the general properties and relations of trigonometrical functions.

18. *Spherical Trigonometry* is that branch of trigonometry which treats of the solution of spherical triangles.

19. *Analytical Geometry* is that branch of geometry in which the properties and relations of lines and surfaces are investigated by the aid of algebraic analysis.

20. *Descriptive Geometry* is that branch of geometry which seeks the graphic solution of geometrical problems by means of projections upon auxiliary planes.

21. ***Mensuration*** is that branch of applied mathematics which treats of the measurment of geometrical magnitudes.

22. ***Surveying*** is that branch of applied mathematics which treats of the art of determining and representing distances, areas, and the relative position of points upon the earth's surface.

23. ***Navigation*** is that branch of applied mathematics which treats of the art of conducting ships from one place to another.

24. ***Mechanics*** is that branch of applied mathematics which treats of the laws of equilibrium and motion.

25. ***Astronomy*** (ἀστρονομία, from ἄστρον, star and νόμος law) is that branch of applied mathematics in which mechanical principles are used to explain astronomical facts.

26. ***Optics*** (ὀπτική, from ὄψις sight,) is that branch of applied mathematics which treats of the laws of light.

27. ***Gunnery*** is that branch of applied mathematics which treats of the theory of projectiles.

28. ***A Proposition*** is a statement of something proposed to be done.

29. Prop't'n.
- 1. Demonstrable.
  - *a.* Theorem.
    - 1. Lemma.
    - 2. Corollary.
  - *b.* Problem.
- 2. Indemonstrable.
  - *a.* Axiom.
  - *b.* Postulate.

30. ***A Demonstrable Proposition*** is one that can be proved by the aid of reason.

31. ***A Theorem*** is a truth requiring a proof.

32. ***A Lemma*** is a theorem demonstrated for the purpose of using it in the demonstration of another theorem.

33. ***A Corollary*** is a subordinate theorem, the truth of which is made evident in the course of the demonstration of a more general theorem.

34. ***A Problem*** is a question proposed for solution.

35. ***An Indemonstrable Proposition*** can not be proved by any manner of reasoning.

36. ***An Axiom*** is a self-evident truth.

37. ***A Postulate*** is a proposition which states that something can be done, and which is so evidently true as to require no process of reasoning to show that it is possible to be done.

38. ***A Demonstration*** is the process of reasoning, proving the truth of a proposition.

39. ***A Solution*** of a problem is an expressed statement showing clearly how the result is obtained.

40. ***An Operation*** is a process of finding, from given quantities, others that are known, by simply illustrating the solution.

41. ***A Rule*** is a general direction for solving all problems of a particular kind.

42. ***A Formula*** is the expression of a general rule or principle in algebraic language.

43. ***A Scholium*** is a remark made at the close of a discussion, and designed to call attention to some particular feature or features of it.

CHAPTER II.

NUMERATION AND NOTATION.

1. ***Numeration*** is the art of reading numbers.

2. There are two methods of numeration; the *French* and the *English.*

3. The *French* method is that in general use. In this method, we begin at the right hand and divide the number into periods of three figures each, and give a distinct name to each period.

4. The *English* method is that formerly used in Great Britain. In this method, we divide the number (if it consists of more than six figures) into periods of six figures each, and give a distinct name to each period. The following number illustrates the two methods; the upper division showing how the number is read by the English method, and the lower division showing how it is read by the French method.

	4th period, Trillions.	3d period, Billions.		2d period, Millions.		1st period. Units.	
English.	845	678	904	325	147	434	913
French.	7th period, Quintillions.	6th period, Quadrillions.	5th period, Trillions.	4th period, Billions.	3d period, Millions.	2d period, Thousands.	1st period, Units.

5. The number expressed in words by the English method, reads thus:

Eight hundred forty-five trillion, six hundred seventy-eight thousand nine hundred four billion, three hundred twenty-five thousand one hundred forty-seven million, four hundred thirty-four thousand nine hundred thirteen.

Remark.—Use the conjunction *and*, only in passing over the decimal point. It is incorrect to read 456,734 four hundred and fifty-six thousand, seven hundred and thirty-four. Omit the *and's* and the number will be correctly expressed in words.

6. The following are the names of the Periods, according to the common, or French method:

First	Period,	Units.	Sixth	Period,	Quadrillions.
Second	"	Thousands.	Seventh	"	Quintillions.
Third	"	Millions.	Eighth	"	Sextillions.
Fourth	"	Billions.	Ninth	"	Septillions.
Fifth	"	Trillions,	Tenth	"	Octillion.

Other periods in order are, Nonillions, Decillions, Undecillions, Duodecillions, Tredecilions, Quatuordecillions Quindecillions, Sexdecillions, Septendecillions, Octodecillions, Novendecillions, Vigintillions, Primo-Vigintillions, Secundo-vigintillions, Tertio-vigintillions, Quarto-vigintillions, Quinto-vigintillions, Sexto-vigintillions, Septo-vigintillions, Octo-vigintillions, Nono-vigintillions, Trigillions; Primo-Trigillions, Secundo-Trigillions, and so on to Quadragillions; Primo-quadragillions, Secundo-quadragillions, and so on to Quinquagillions; Primo-quinquagillions, Secundo-quinquagillions, and so on to Sexagillions, Primo-sexagillions, Secundo-sexagillions, and so on to Septuagillions; Primo-septuagillions, Secundo-septuagillions, and so on to Octogillions; Primo-octogillions, Secundo-octogillions, and so on to Nonogillions; Primo-nonogillions, Secundo-nonogillions, and so to Centillions.

7. ***Notation*** is the art of writing numbers.

There are three methods of expressing numbers; by words, by letters, called the *Roman* method, and by figures, called the *Arabic* method.

8. The *Roman Notation*, so called from its having originated with the ancient Romans, uses seven capital letters to express numbers; viz., I, V, X, L, C, D, M.

9. The *Arabic Notation*, so called from its having been made known through the Arabs, uses ten characters to express numbers; viz., 1, 2, 3, 4, 5, 6, 7, 8, 9, 0.

10. ***Ordinal Numbers.*** A logical definition of *number* is not easy to give, for the reason that the idea it conveys is a simple notion. The clearest idea of what counting and numbers mean may be gained from the observation of children and of

nations in the childhood of civilization*. When children count or add they use their fingers, or small sticks, or pebbles which they adjoin singly to the things to be counted or otherwise to be ordinally associated with them. History informs us that the Greeks and Romans employed their fingers when they counted or added. The reason why the fingers are so universally used as a means of numeration is, that everyone possesses a definite number, sufficiently large for purposes of computation and that they are always at hand.

Let us consider the row of objects, X X X X X X X X X X X X X X X X . . . . ., with regard to their order, say from left to right, freeing our minds from all notions of magnitude. Beginning with any one object in this row, we speak of the one we begin with as being the *first*, the next in order to it to the right the second, the next in order to the right of the second the third, and so on. The name or mark we thus attach to an object to tell its place in the row is called an ***integer.*** This process, or *operation*, of labeling the objects is called *counting* and it is the ***fundamental operation of mathematics.*** To count objects is to label the objects, not primarily to tell how many there are†. In thus labeling the objects, we may replace the objects by the fingers, by sticks, by pebbles, by marks, or by characters. The method of tallying used at the present time is such a method. In counting objects marks are made until four are made, then these are crossed with a fifth mark and so on. Thus, 卌, 卌, 卌.

Suppose that in counting the objects in the row, we use our fingers, and for each object in the row beginning with a certain one we bring in correspondence with that object the little finger of the right hand, with the next object to the right the next to the little finger of the right hand, and so on until an object and the thumb of the right hand are brought into correspondence. For the group of objects thus counted, let us bring into correspondence the little finger of the left hand. Now continue the counting of the objects of the row as before, and when a second group is reached bring into correspondence with this group the next to the little finger of the left hand. Continue this process until a group of the objects as represented by the fingers of the right hand is brought into correspondence with the thumb of the left hand. Thus the fingers of the left hand represent a group of groups of objects. Bring this group represented by the fingers of the left hand into correspondence with

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*Schubert's *Mathematical Essays and Recreations*.

†My friend, Dr. William Rullkoetter, told me of a case coming under his personal observation, where a farmer, unable to count, but when desirous of knowing if any of his cattle were missing, would have them driven through a gate or past some point where he could see them as they passed singly. He would then say, "You are here," "and you are here," "and you are here," and so on until all had passed by. In this way he was able to tell if any were missing, but not able to tell how many he had.

the little toe of the right foot. Now continue the process of counting the objects and so on as before until the big toe of the right foot is brought into correspondence with a group corresponding to the fingers of the left hand. Thus the toes of the right foot represent a group of a group of a group of objects. In this manner, we could build up the system of numeration called the *Quinary*, a system in which five objects as represented by the fingers of the right hand make a unit or group as represented by a finger of the left hand, five groups of five objects as represented by the fingers of the left hand make a group as represented by the toes of the right foot, and so on.

The decimal system of numeration may be built up in the same way, except that the group of objects corresponding to the fingers of both hands would be represented by a toe. After the fingers and toes have been exhausted in the process of counting the numeration would have to be continued by using small sticks or pebbles. It is very probably due to the fact that we have 10 fingers that the decimal system was invented. There are, however, among the uncivilized nations of the world a number of different systems of numeration*.

At the present time, in labeling objects by the process of counting we use the following characters, viz., 1, 2, 3, 4, 5, 6, 7, 8, 9, etc.

1 2 3 4 5 6 7 8 9 . . . . . labels.
Thus X X X X X X X X X X X X X X X X X . . . . . objects.

In labeling, we could begin with the object marked 3 and re-label it 1, then re-label 4 as 2 and 5 as 3, and so on. This is expressed by writing

$$3-2=1,\ 4-2=2,\ 5-2=3,$$

meaning that if we begin after the object whose old mark was 2, then the object which was third becomes first, the object which was fourth becomes second, and so on. Beginning after an object instead of with it suggests that our original row might begin after an object; this object after which the counting begins is marked 0 and called the origin. If there are objects to the left of the origin, we count them in the same way; except that we prefix the sign, —, to show that they are to the left, and we call the marks so changed *negative* integers, thus distinguishing them from the old marks which we call *positive* integers. The marks are . . . . —4, —3, —2, —1, 0, 1, 2, 3, 4, 5, 6, ...

These marks constitute what is called the *natural integer-system*.

When an object marked a is to the left of another marked a', we say that a comes before a' or is inferior to a', and a'

*See Conant's *Number Concept* for a full treatment of the various systems of notation.

comes after a or is superior to a. These ideas are expressed symbolically thus $a<a'$, $a'>a$. Here a and a' mean integers, positive or negative.

Objects considered as a succession from left to right are *in positive order;* when considered from right to left, *in negative order.*

Addition and its inverse operation, Subtraction, are algorithms of *counting*. Multiplication is an algorithm of Addition, and Division is an algorithm of Subtraction. Addition, Subtraction, Multiplication, and Division are only short methods of counting.

If we operate on any integer of the natural-integer series by any one of the operations of Addition, Subtraction, or Multiplication, no new integer is produced. With reference to these operations the natural integer-series is closed, that is to say, there are no breaks in the integer-series into which other integers arising from these operations may be inserted. If, however, we operate on any one of the integers of the integer-series by the operation of Division, the operations in many cases are impossible. Suppose we wish to divide 17 by 5. This operation is absolutely impossible. $\frac{17}{5}$ is a meaningless symbol with reference to the fundamental operation of mathematics. But in this case, as in the case when negative numbers are introduced by the inverse operation, subtraction, we apply a principle called by Hankel, "The Principle of the Permanence of Formal Laws," and by Schubert, "The Principle of No Exception," viz., *That every time a newly introduced concept depends upon operations previously employed, the propositions holding for these operations are assumed to be valid still when they are applied to the new concepts.* In accordance with this principle, we invest the symbol, $\frac{17}{5}$, which has the form of a quotient without its dividend being the product of the divisor and any number yet defined, with a meaning such that we shall be able to reckon with such apparent quotient as with ordinary quotients. This is done by agreeing always to put the product of such a quotient form with its divisor equal to its dividend. Thus, $(\frac{17}{5})\times5=17$. We thus reach the definition of *fractions*. The concept of fractions may also be established as in the next article.

11. ***Fractions.*** Let us now again assume the row of

0 1 2 3 4 5 6 7 8 9 . . . .
objects, X X X X X X X X X X X X . . . . attending to only

zero, the object from which we begin, and the objects on the right of it. Suppose we re-label the alternate objects 2, 4, 6, 8, ... marking them 1, 2, 3, 4, ... We must then invent marks for the objects previously marked 1, 3, 5, 7 . . . . . The marks invented are shown in Figure 1, above the objects, the old marks being below the objects.

$$\begin{array}{cccccccccccccc} 0 & \frac{1}{2} & 1 & \frac{3}{2} & 2 & \frac{5}{2} & 3 & \frac{7}{2} & 4 & \frac{9}{2} & 5 & \ldots\ldots \\ X & X & X & X & X & X & X & X & X & X & X & X & \ldots\ldots \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \ldots\ldots \end{array}$$

Fig. 1.

From this it is clear that instead of re-labeling the alternate objects in a row of objects, 0, 1, 2, 3, 4, 5, 6, 7, ... we can interpolate alternate objects in the row and then mark them $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, and so on.

In the same way we can interpolate two objects between every consecutive two of the given row 0, 1, 2, 3, 4, 5, ... marking the new objects in order $\frac{1}{3}$, $\frac{2}{3}$; $\frac{4}{3}$, $\frac{5}{3}$; $\frac{7}{3}$, $\frac{8}{3}$; and so on.

$$\text{Thus,}\quad \begin{array}{ccccccccccc} 0 & \frac{1}{3} & \frac{2}{3} & 1 & \frac{4}{3} & \frac{5}{3} & 2 & \frac{7}{3} & \frac{8}{3} & 3 & \\ X & X & X & X & X & X & X & X & X & X & \ldots\ldots \\ 0 & & & 1 & & & 2 & & & 3 & \end{array}$$

Fig. 2.

In this way we account for the symbols $\frac{1}{p}$, $\frac{2}{p}$, $\frac{3}{p}$, . . . where p is any positive integer. These we call positive fractional numbers.

By interpolating single objects in the row 0, $\frac{1}{2}$, 1, $\frac{3}{2}$, 2, $\frac{5}{2}$, . . . we have the same sequence of objects as if we interpolate objects by threes in the row

0, 1, 2, 3, 4, 5, 6, ...

and the objects are therefore marked

0, $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$, 1, ...

$$\text{Thus,}\quad \begin{array}{cccccccccc} 0 & \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & 1 & \frac{5}{4} & \frac{6}{4} & \frac{7}{4} & 2 & \ldots\ldots\ldots \\ X & X & X & X & X & X & X & X & X & \ldots\ldots \\ 0 & & \frac{1}{2} & & 1 & & \frac{3}{2} & & 2 & \end{array}$$

Fig. 3.

From this we see that $\frac{1}{2}$ and $\frac{2}{4}$ are marks for the same object. Also $\frac{6}{4}$ and $\frac{3}{2}$. Hence, $\frac{2}{4}=\frac{1}{2}$ and $\frac{6}{4}=\frac{3}{2}$.

A row marked 0, $\frac{1}{6}$, $\frac{5}{6}$, $\frac{1}{3}$, $\frac{1}{2}$, $\frac{2}{3}$, 1, ... is to be understood as arising from the interpolation of objects by fives; that is, by introducing the objects $\frac{1}{6}$, $\frac{2}{6}$, $\frac{3}{6}$, $\frac{4}{6}$, $\frac{5}{6}$, ..., or $\frac{1}{6}$, $\frac{1}{3}$, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{5}{6}$, ...

As $\frac{2}{6}$ comes before $\frac{3}{6}$, we say $\frac{2}{6}<\frac{3}{6}$, or $\frac{1}{3}<\frac{1}{2}$.

We may interpolate as many objects as we please in the natural row, and, by the principle of the least common denominator, we can interpolate so as to explain any assigned positive fractional marks, f_1, f_2, f_3, . . . Also, given any positive rational mark, r, other than zero, we can interpolate rational marks between 0 and r. When no object can be made to fall between an assigned object and 0, that assigned object must be 0 itself.

In the same way we may treat the negative numbers.

We can think of an infinity of objects as being interpolated in the natural row, so that each shall bear a distinct rational number and so that we can say which of any two objects comes first. It is to be noticed that as we approach any of the natural objects there is no last fractional mark; that is, whatever object we take there are always others between it and the natural object.

Thus, if an infinitude of objects be interpolated in the natural row, 0, 1, 2, 3, ...

0 $\frac{1}{7}$ $\frac{1}{2}$ $\frac{20}{21}$ 1
XXXX..to infinityXXX..to infinityXXXXX..to infinityXX..to infinityX
0 1

Fig. 4.

then it is clear that whatever object we take there is an infinity of objects between it and the natural object, thus rendering it evident that there are no *last fractional marks* in this case.

12. *Irrational Numbers.* In considering square numbers from the ordinal point of view, we re-label our natural row as in Fig. 5.

0 1 $\frac{1}{2}$ $\frac{3}{2}$ 2 3
X X X X X X X X X X X X X X X . . . . .
0 1 $\frac{1}{4}$ 2 $\frac{9}{4}$ 3 4 5 6 7 8 9

Fig. 5.

where the old names are below and the new above. We have now to consider how to bring the omitted objects into the scheme of ordinal numbers. Every object whose new name is fractional had a fractional name, so that the object whose old name was 2 cannot now have a rational name. We give it a name which we call irrational. We call it the positive or chief square root of 2 and mark it $\sqrt{2}$ or $2^{1/2}$. As an ordinal number it is perfectly satisfactory, for we know where it comes, whether left or right of any proposed rational number, by means of the old marking. Hence, it separates *all* the rational numbers into two classes, viz., those on its right and those on its left. A rational number separates all *other* rational numbers into two classes; we put it into one of the classes and say it closes that class.

Take, for example, $\frac{9}{7}$. Now, there is no last fractional mark as we approach $\frac{9}{7}$ from the left or from the right. Hence, without $\frac{9}{7}$ neither the class to the left of $\frac{9}{7}$ nor the class to the right of $\frac{9}{7}$ is closed. With $\frac{9}{7}$, either class is closed.

Any process which serves to separate rational numbers into two classes,—those on the left and those on the right, such that the left-hand class is not closed on the right and the right-hand class not closed on the left,—leads to the introduction of a new object named by an irrational number.

For example, $\sqrt{2}$ separates *all* rational numbers into two classes, viz., those on the left of it and those on the right. Now if we take any rational object however near to the $\sqrt{2}$ as we please we can always interpolate new rational objects between it and $\sqrt{2}$. Thus, it is clear that the class on the left of $\sqrt{2}$ is not closed at the right nor the class on the right closed on the left.

Two rational or irrational numbers,—for simplicity take them both irrational and equal to s and s',—are equal if the rational objects to the left of s are the same as the rational objects to the left of s', and the rational objects to the right of s are the same as those to the right of s' Thus, $4^{1/4}$ and $2^{1/2}$ effect the same separation of the rational numbers. Hence, $4^{1/4}=2^{1/2}$.

An equivalent condition for the equality of s and s' is that every rational number to the left of s shall be to the left of s', and every rational number to the left of s' shall be to the left of s.

Between two unequal irrational objects, s and s', there must lie rational objects; for, since s and s' are not equal, there must be a rational number which is before one and not before the other.

It is very important to notice that we have now a closed number-system. When we seek to separate the irrational objects as lying left or right of an object, either the object is rational or if not it separates rational objects and is irrational; in any case it must have for its mark a rational or irrational number, and there is no loop-hole left for the introduction of new real numbers which separate existing numbers. This is often briefly expressed by saying that the whole system of positive and negative integral, fractional, and irrational numbers is *continuous*, or is a *continuum**.

In the way indicated above, the number-concept of Arithmetic is put on a basis consistent with Geometry. If we select any point on a straight line and call it the zero-point, and also a fixed length, measured on this line, be chosen as the unit of length, any real number, a, can be represented by a point on this line at a distance from the zero-point equal to a units of length. Conversely, each point on the line is at a distance from the origin equal to a units of length, when a is a real number. That is, there is a one to one correspondence between the points of line and the numbers of the real number-system. For every point of the line, there corresponds a number of the real number-system and for every number of the real number-system there corresponds a point of the line.

EXAMPLES.

1. Write three hundred seventy quadrillion, one hundred one thousand one hundred thirty-four trillion, seven hundred eighty-

*See Harkness and Morley's *Introduction to the Theory of Functions*, Chapter I., from which, Arts. 10, 11, and 12 have been chiefly adapted nine thousand six hundred thirty-two billion, two hundred ninety-eight thousand seven hundred sixty-five million, four hundred thirty-seven thousand one hundred fifty-six.

2. Read by the *English* method, 78943278102345789328903-24678.

3. Write three thousand one hundred forty-one quintillion, five hundred ninety-two billion six hundred fifty-three million five hundred eighty-nine thousand seven hundred ninety-three quadrillion, two hundred thirty-eight billion four hundred sixty-two million six hundred forty-three thousand three hundred eighty-three trillion, two hundred seventy-nine billion five hundred two million, eight hundred eighty-four thousand one hundred ninety-seven.

4. Read 141421356237309504880168872420969807856971437-89132.

5. Is a billion, a million million? Explain.

6. Write 19 billion billion billion.

7. Write 19 trillion billion million million.

8. Write 19 hundred 56 thousand.

9. Write 457 thousand 341 million.

10. Write 19 trillion trillion billion billion million million.

CHAPTER III.

ADDITION.

1. ***Addition*** is the process of uniting two or more numbers of the same kind into one sum or amount.

2. Add the following, beginning at the right, and prove the result by casting out the 9's:

7845	excess of 9=6	
6780	" " 9=3	
8768	" " 9=2	Excess of 9's=8.
5343	" " 9=6	
3987	" " 9=0	
32723	excess of 9=8	

Explanation.—Adding the digits in the first number, we have 24. Dividing by 9, we have 6 for a remainder, which is the excess of the 9's. Treating the remaining numbers in the same manner, we obtain the excesses 3, 2, 6, 0. Adding the excesses and taking the excess of their sum, we have 8; this being equal to the excess of the sum the work is correct.

3. Add the following, beginning at the left:

```
   8456
   9799
   4363
   5809
   5432
  -----
 31
   26
    23
     29
  -----
 33859
```

From this operation, we see that it is more convenient to begin at the right

Remark.—We can not add 8 apples and 5 peaches because we can not express the result in either denomination. Only numbers of the same name can be added.

EXAMPLES.

1. Add the numbers comprised between 20980189 and 20980197.

2. 6095054+900703+90300420+9890655+37699+29753= what?

3. Add the following, beginning at the left: 97674; 347-893; 789356; 98935679; 123456789.

4. Add all the prime numbers between 1 and 107 inclusive.

5. Add 31989, 63060, 132991, 1280340, 987654321, 78903, and prove the result by casting out the 9's.

6. Add the consecutive numbers from 100 to 130.

7. Add the numbers from 9897 to 9910 inclusive.

8. Add MDCCCLXXVI, MDCXCVIII, DCCCCXLIX, DCCCLXII.

CHAPTER IV.

SUBTRACTION.

1. ***Subtraction*** is the process of finding the difference between two numbers.

2. Subtract the following and prove the result by casting out the 9's:

984895	excess of 9's=7	
795943	" " 9's=1	Excess of 9's=7.
188952	" " 9's=6	

Explanation.—Adding the digits in the first number, we have 43. Dividing by 9 the remainder is 7, which is the excess of the 9's. Treating the subtrahend and remainder in the same manner, we have the excesses 1 and 6. But subtraction is the opposite of addition and since the minuend is equal to the sum of the subtrahend and remainder, the excess of the sum of the excesses in the subtrahend and remainder is equal to the excess in the minuend. This is the same proof as that required if we were to add the subtrahend and remainder.

3. We begin at the right to subtract, so that if a figure of the subtrahend is greater than that corresponding to it in the minuend, we can borrow one from the next higher denomination and reduce it to the required denomination and then subtract.

4. Subtract the following and illustrate the process:

1=9 9 9 9 9 9 9+1	1=9 9 9 9 9 9+1	1=9 9 9+1
90000000	9856342 } Add.	4326546 } Add.
85784895	8978567	3214957
4215105	877775	1111589

EXAMPLES.

1. From 9347893987 take 8968935789. Prove the result by casting out the 9's.

2. 7847893578—6759984699=what?

Which is the nearer number to 920864; 1816090 or 27497?

4. 34567—34518+3—2+3—4+7+18—567+43812—1326+678=what.?

5. 5+6+7—12—13+14—2—3+7—8—6+5+12—8=what?

6. 3+4—(6+7)—8+27—(1+3—2—3)—(7—8+5) 3+7= what?

CHAPTER V.

MULTIPLICATION.

1. ***Multiplication*** is the process of taking one number as many times as there are units in another; or it is a short method of addition when the numbers to be added are equal.

2. Multiply the following and prove the result by casting out the 9's:

```
     7855  excess of 9's=7
      435    "     " 9's=3
    -----            ---
    39275            21 excess of 9's=3.
   23565
  31420
  -------
  3416925=excess of 9's=3.
```

Explanation.—Adding the digits in the multiplicand and dividing the sum by 9, the remainder is 7 which is the excess of the 9's. Adding the digits in the multiplier and dividing the sum by 9, we have the remainder 3 which is the excess of the 9's. Now, since multiplication is a short method of addition when the numbers to be added are equal, we multiply the excess in the multiplicand by the excess in the multiplier and find the excess, and this being equal to the excess in the product, the work is correct.

3. Multiply the following, beginning at the left :

75645
765

1st 49
35
42
2d 4228
3035
36
3d 3524
2530
30
20
25

57868425

3. From this operation, we see that it is more convenient to begin at the right to multiply.

5. In multiplication, the multiplicand may be abstract, or concrete; but the multiplier is always abstract.

6. The sign of multiplication is $\times$, and is read, *multiplied by*, or *times*. When this sign is placed between two numbers it denotes that one is to be multiplied by the other. In this case, it has not been established which shall be the multiplicand and which the multiplier. Thus $8\times5=40$, either may be considered the multiplicand and the other the multiplier. If 8 is the multiplicand, we say, 8 multiplied by 5 equals 40, but if 5 is the multiplicand we say, 8 *times* 5 equals 40.

EXAMPLES.

1. $562402\times345728=$what?

2. 1 mile = 63360 inches; how many inches from the earth to the moon the distance being 239000 miles?

3. Multiply 789627 by 834, beginning at the left to multiply.

4. 1 acre = 43560 sq. in.; how many square inches in a field containing 427 acres?

5. Multiply 6934789643 by 34789. Prove the result by casting out the 9's.

6. $2778588\times34678=$ what?

7. $2\times3\times4-3\times7+3-2\times2+4+8\times2+4-3\times5+27=$ what?

8. $5\times7+6\times7+8\times7-4\times6+6\times6+7\times6=$ what?

9. $356789\times4876=$ what?

10. $395076\times576426=$ what?

11. $7733447\times998800=$ what?

12. $5654321\times999880=$ what?

CHAPTER VI.

DIVISION.

1. ***Division*** is the process of finding how many times one number is contained in another; or, it is a short method of subtraction when the numbers to be subtracted are equal.

2. Divide the following and prove the result by casting out the 9's:

67)5484888(81864
536
———
124
67
———
578
536
———
428
402
———
268
268.
———

Dividend
5484888 excess of 9's=0.

Quotient
81864 excess of 9's=0 }
Divisor
67 excess of 9's=4 } Excess of 9's in this product equals 0.

Explanation.—Adding the digits in the dividend and dividing the sum by 9, we have the remainder 0, which is the excess of the 9's. Adding the digits in the quotient and dividing the sum by 9, we have the remainder 0, which is the excess of the 9's in the quotient. Adding the digits in the divisor and dividing the sum by 9, we have the remainder 4, which is the excess of the 9's in the divisor. Since division is the reverse of multiplication, the quotient corresponding to the multiplicand, the divisor to the multiplier, and the dividend to the product, we multiply the excess in the quotient by the excess in the divisor. The excess of this product is 0. This excess being equal to the excess of the 9's in the dividend, the work is correct.

If there be a remainder after dividing, find its excess and add it to the excess of the product of the excesses of the quotient and divisor. Take the excess of the sum and if it is equal to the excess of the dividend the work is correct.

3. The sign of division is $\div$, and is read *divided by.*

4. When the divisor and dividend are of the same denomination the quotient is abstract; but when of different denominations, the divisor is abstract and the quotient is the same as the dividend. Thus, 24 ct. $\div$ 4ct. $=$ 6, and 24 ct. $\div 4 =$ 6 ct.

Remark.—We begin at the left to divide, that after finding how many times the divisor is contained in the fewest left-hand figures of the dividend, if there be a remainder we can reduce it to the next lower denomination and find how many times the divisor is contained in it, and so on.

Note.—The proof by casting out the 9's will not rectify errors caused by inserting or omitting a 9 or a 0, or by interchanging digits.

EXAMPLES.

1. $4326422 \div 961 =$ what? Prove the result by casting out the 9's.

2. $245379633477 \div 1263 =$ what? Prove the result by casting out the 9's.

3. What number multiplied by 109 with 98 added to the product, will give 106700?

4. The product of two numbers is 212492745; one of the numbers is 1035; what is the other number?

5. $27 \div 9 \times 3 \div 9 - 1 + 3 \div 3 \times 9 - 8 \div 4 + 5 \times 2 - 3 \times 2 \div 2 \div 3 - (3 \times 4 \div 6 + 5 - 2) + 81 \div 27 \times 3 \div 9 \times 18 \div 6 =$ what? [Hint.— Perform the operations indicated by the multiplication and division signs in the exact order of their occurrence.]

6. $(64 \div 32 \times 96 \div 12 - 7 - 5 + 3) \times \{[(27 \div 3) \div 9 - 1 + 2] + 91 \div 13 \times 7 - 45\} \times 9 + 45 \div 9 + 3 - 1 =$ what?

7. $2 \times 2 \div 2 \div 2 \div 2 \times 2 \times 2 \div 2 \div 2 \div 2 =$ what? *Ans.* $\frac{1}{4}$.

8. $3 \div 3 \div 3 \times 3 \times 3 \div 3 \div 0 \times 4 \times 4 \times 5 \times 5 =$ what? *Ans.* ∞.

9. $2 \times 2 \times 2 \div 2 \times 2 \div 2 \div 2 \times 2 \times 2 \times 0 \times 2 \times 2 =$ what? *Ans.* 0.

CHAPTER VII.

COMPOUND NUMBERS.

1. *A Compound Number* is a number which expresses several different units of the same kind of quantity.

2. *A Denominate Number* is a concrete number in which the unit is a measure; as, 5 *feet*, 7 *pints*.

3. *The Terms* of a compound number are the numbers of its different units. Thus, in 4 bu. 3 pk. 7 qt. 1 pt., the terms are 4 bu. and 3 pk. and 7 qt. and 1 pt.

4. *Reduction of Compound Numbers* is the process of changing a compound number from one denomination to another. There are Two Cases, *Reduction Descending* and *Reduction Ascending*.

5. *Reduction Descending* is the process of reducing a number from a higher to a lower denomination.

6. *Reduction Ascending* is the process of reducing a number from a lower to a higher denomination.

I. THE METRIC SYSTEM OF WEIGHTS AND MEASURES.

1. *The Metric System* is a system of weights and measures based upon the decimal system.

In 1795, France adopted a system of weights and measures based upon the decimal scale. A careful measure of an arc of the meridian passing through Paris was made and the $\frac{1}{10000000}$ part of the distance from earth's north pole to the equator measured on this meridian was taken as the unit of length. It was called the ***meter*** Upon it is based the entire system, all other units being *derived units,* that is, units derived from the meter.

It is greatly to be regretted that this system did not come into general use in the United States and England a half century ago. It is the very acme of simplicity and elegance. Its use would greatly simplify international commercial relations and promote educational interests in all countries using it.

The argument commonly advanced against its introduction is that it would entail a great loss of money now invested in expensive machinery; — practically all machinery used in the arts and the various machine constructions would become worthless.

This is true, but the reason that it is not adopted irrespective of the sacrifice of a large sum of money, is the ignorance and prejudice of the common people. If the common people clearly understood how simple and easy this system is, the popular demand would overpower every opposition and its general use would be established in a short time.

In order that the common people understand its merits and appreciate them, every teacher in our public schools should devote considerable time in presenting this system to his pupils, and he should advocate its use whenever an opportunity presents itself. Teachers, why should our old brain-racking system be used, when such a labor-saving and excellent system has been devised to take its place? Transfer its place in the arithmetics from the latter part of the book to that part just preceding our own cumbersome system of weights and measures, disseminate a healthy knowledge of this system among your pupils, and when the time does come that the Metric System shall supercede our present clumsy, awkward, and unscientific system, remember you will have done your full share in contributing to future generations the greatest heritage to which civilization is heir.

2. The following prefixes are used in the Metric System:

Deca signifies 10 times the unit.
Hecto signifies 100 times the unit.
Kilo signifies 1,000 times the unit.
Myria signifies 10,000 times the unit.
Deci signifies the 10th part, or .1 part of the unit.
Centi signifies the 100th part, or .01 part of the unit.
Milli signifies the 1,000th part, or .001 part of the unit.

UNITS OF LENGTH.

1. ***The Prime Unit of Length*** is the *meter.*

LONG MEASURE.

10 millimeters (mm.)	=	1 centimeter (cm.)
10 centimeters	=	1 decimeter (dm.)
10 decimeters	=	1 meter (m.)
10 meters	=	1 dekameter (Dm.)
10 dekameters	=	1 hectometer (Hm.)
10 hectometers	=	1 kilometer (Km.)
10 kilometers	=	1 myriameter (Mm.)

UNITS OF AREA.

1. ***The Prime Unit of Square Measure*** is the *square meter,*$=m^2$.

SQUARE MEASURE.

100 square millimeters (sq. mm. = mm^2.) = 1 square centimeter (sq. cm. = cm^2.)
100 square centimeters = 1 square decimeter (sq. dm. = dm^2.)
100 square decimeters = 1 square meter (sq. m. = m^2.)
100 square meters = 1 square dekameter (sq. Dm. = Dm^2.)
100 square dekameters = 1 square hectometer (sq. Hm. = Hm^2.)
100 square hectometers = 1 square kilometer (sq. Km. = Km^2.)

Note.— A square decameter is called an *are* when used in measuring land.

Remark.— Observe that the abbreviations, sq. mm. and $mm.^2$, sq. cm. and $cm.^2$, sq. m. and $m.^2$, etc., are synonymous. The exponential abbreviation, being more quickly written, is generally used in Physics and other scientific works.

Likewise, cu. mm. and $mm.^3$, cu. cm. and $cm.^3$, cu. m. and m^3. are used synonymously.

LAND MEASURE.

10 centares (ca.) = 1 deciare (da.)
10 deciares = 1 are (a.)
10 ares = 1 decares (Da.)
10 decares = 1 hectare (Ha.)

UNITS OF VOLUME.

1. ***The Standard Unit of Volume*** is the *cubic meter*, = m^3

CUBIC MEASURE.

1000 cubic millimeters (cu. mm. = mm^3.) = 1 cubic centimeter (cu. cm. = cm^3.)
1000 cubic centimeters = 1 cubic decimeter (cu. dm. = dm^3.)
1000 cubic decimeters = 1 cubic meter (cu. m. = m^3.)
1000 cubic meters = 1 cubic dekameter (cu. Dm. = Dm^3.)
1000 cubic dekameters = 1 cubic hectometer (cu. Hm. = Hm^3.)
1000 cubic hectometers = 1 cubic kilometer (cu. Km. = Km^3.)

Note.— A cubic meter is called a *stere* when used in measuring wood.

WOOD MEASURE.

10 decisteres (ds.) = 1 stere (s.)
10 steres = 1 dekastere (Ds.)

UNITS OF CAPACITY.

1. ***The Unit of Capacity*** is the *liter*. The liter is a cubical vessel whose edge is 1 decimeter. Hence, it equals a cubic decimeter. It is used in measuring liquids and dry substances.

MEASURE OF CAPACITY.

10 milliliters (ml.)	= 1 centiliter (cl.)
10 centiliters	= 1 deciliter (dl.)
10 deciliters	= 1 liter (l.)
10 liters	= 1 dekaliter (Dl.)
10 dekaliters	= 1 hectoliter (Hl.)
10 hectoliters	= 1 kiloliter (Kl.)
10 kiloliters	= 1 myrialiter (Ml.)

UNITS OF WEIGHT.

1. ***The Unit of Weight*** is the *gram*. It is the weight of a cubic centimeter of distilled water at 4°C.

MEASURE OF WEIGHT.

10 milligrams (mg.)	= 1 centigram (cg.)
10 centigrams	= 1 decigram (dg.)
10 decigrams	= 1 gram (g.)
10 grams	= 1 dekagram (Dg.)
10 dekagrams	= 1 hectogram (Hg.)
10 hectograms	= 1 kilogram (Kg.)
10 kilograms	= 1 myriagram (Mg.)
10 myriagrams	= 1 quintal (Quin.)
10 quintals	= 1 metric ton (M. T.)

I. Find the weight of a cubic meter of gold, specific gravity being $19\frac{1}{4}$.

II.
1. 1 m.^{3}=100^3 cm.^{3}=1,000,000 cm.^{3}
2. 1 g. =weight of 1 cm.^{3} of water.
3. $19\frac{1}{4}$ g.=weight of 1 cm.^{3} of gold, its sp. gr. being $19\frac{1}{4}$.
4. 19,250,000 g.=1,000,000$\times19\frac{1}{4}$ g.=weight of 100^3 cm.^{3}, or 1 cubic meter of gold.

III. ∴ 1 m.^{3} of gold weighs 19,250,000 g.

I. How many liters in a tank 6 meters long, 5 meters wide, and 2 meters deep?

II.
1. 6=number of meters in length.
2. 5=number of meters in width.
3. 2=number of meters in depth.
4. ∴ $6\times5\times2=60$=number of cubic meters in the tank.
5. 1 cu. m.=100^3 cm.^{3}=1,000,000 cm.^{3}
6. 60 cu. m.=60$\times$1,000,000 cm.^{3}=60,000,000 cm.^{3}
7. 1 cm.^{3}=1 l.
8. 60,000,000 cm.^{3}=60,000,000 l.

III. ∴ The tank contains 60,000,000 l.

Compare the above problems with similar ones in the common system of weights and measures.

II. ENGLISH SYSTEM OF WEIGHTS AND MEASURES.

UNITS OF VALUE.

1. ***The Prime Unit*** of United States money is the *dollar*.

UNITED STATES MONEY.

10 mills (m.)	=	1 cent (ct. or ¢)
10 cents	=	1 dime (d.)
10 dimes	=	1 dollar ($.)
10 dollars	=	1 eagle (E.)

Note.— The coins of the United States are made of *bronze*, *nickel*, *silver*, and *gold*. The *gold coins* are the *double eagle*, *eagle*, *half-eagle*, *quarter-eagle*, *three-dollar*, and *one-dollar*. The *silver coins* are the *trade dollar*, *half-dollar*, *quarter-dollar*, *twenty-cent piece*, *dime*, *half-dime*, and *three-cent piece*. The *nickel coin* is the *five-cent piece*. The *bronze coins* are the *two-cent* piece and the *one cent*. The *three-dollar* and *one dollar* gold pieces are no longer coined; neither are the silver *trade dollar*, *twenty-cent piece*, *half dime*, and *three-cent piece*, and the nickel cent, the old copper cent, the bronze two cent piece. Many of these coins are still in circulation, however.

For many interesting notes under this and the following table the student is referred to *Brooks' Higher Arithmetic*, one of the very best arithmetics published in America.

ENGLISH MONEY.

1. ***English, or Sterling Money,*** is the money of Great Britain and Ireland.

2. ***The Prime Unit*** is the *pound* (£), whose value in United States money is $4.8665. The pound, when coined, is called the *sovereign*.

ENGLISH MONEY.

4 farthings (far.)	=	1 penny (d.)
12 pence	=	1 shilling (s.)
20 shillings	=	1 pound (£.)
5 shillings	=	1 crown.
21 shillings	=	1 guinea.

Note.— The Unit of *French Money* is the franc, which is worth 19.3¢. The Unit of *German Money* is the mark, which is worth 23.85¢.

I. Express 2 guineas in sovereigns and shillings.

II.
1. 1 guinea =21 shillings.
2. 2 guineas =2 times 21 shillings=42 shillings.
3. 20 shillings=1 sovereign.
4. 42 shillings=42÷20=2 sovereigns+2 shillings.

III. ∴ 2 guineas =2 sovereigns and 2 shillings.

I. What is the value of 80 marks in United States money?

II. 1. 1 mark = 23.85 cents = $0.2385.
2. 80 marks = 80 times $0.2385 = $19.08.

III. ∴ 80 marks = $19.08.

UNITS OF WEIGHT.

1. *Avoirdupois Weight* is used for weighing everything except precious metals, and medicines when dispensed.

2. *The Prime Unit of Weight* is the *pound Avoirdupois*.

AVOIRDUPOIS WEIGHT.

16 ounces (oz.)	= 1 pound (lb.)
100 pounds	= 1 hundredweight (cwt.)
20 hundredweight	= 1 ton (T.)

Note.— In the United States Custom House, and in weighing iron and coal at the mines, the long hundredweight, containing 112 pounds, and the long ton, containing 2240 pounds, are used. One pound avoirdupois contains 7000 grains.

3. *Troy Weight* is chiefly used for weighing gold, silver, and jewels.

TROY WEIGHT.

24 grains (gr.)	= 1 pennyweight (pwt.)
20 pennyweights	= 1 ounce (oz.)
12 ounces	= 1 pound (lb.)

Note.— A Troy pound contains 5760 grains.

4. *Apothecaries' Weight* is used in mixing and selling medicines. Druggists, buy their medicines by Avoirdupois weight, but mix and sell them by Apothecaries' weight.

APOTHECARIES' WEIGHT.

20 grains (gr.)	= 1 scruple (℈)
3 scruples	= 1 dram (ʒ)
8 drams	= 1 ounce (℥)
12 ounces	= 1 pound (lb.)

I. Reduce 2 lbs. 16 gr., Troy weight, to grains.

II. 1. 1 lb. = 12 oz.
2. 2 lbs. = 2 times 12 oz. = 24 oz.
3. 1 oz. = 20 pwt.
4. 24 oz. = 24 times 20 pwt. = 480 pwt.
5. 1 pwt. = 24 grains.
6. 480 pwt. = 480 times 24 grains = 11,520 gr.
7. 11,520 gr. + 16 gr. = 11,536 gr.

III. ∴ 2 lbs. 16 gr. = 11,536 gr.

UNITS OF LENGTH.

1. ***The Standard Unit*** of length is the *yard.*

LONG MEASURE.

12 inches (in.)	=	1 foot (ft.)
3 feet	=	1 yard (yd.)
$5\frac{1}{2}$ yards or $16\frac{1}{2}$ ft.	=	1 rod (rd.)
40 rods	=	1 furlong (fur.)
8 furlongs	=	1 mile (mi.)

Note.— **A hand, used in measuring the height of horses, = 4 inches; a knot, used in navigation, = 6086 feet = 1.15 mi.; and a fathom, used in measuring depth at sea, = 6 feet.**

I. Reduce 63,244 feet to miles.

II.
1. 3 feet =1 yard.
2. 63,244 feet =63,244÷3=21,081 yards+1 foot.
3. $5\frac{1}{2}$ yards=1 rod.
4. 21,081 yards=21,081÷$5\frac{1}{2}$=3,832 rods+5 yards.
5. 320 rods =1 mile.
6. 3,832 rods =3,832÷320=11 miles+312 rods.

III. ∴ 63,244 feet=11 mi. 312 rd. 5 yd. 1 ft.

Explanation.—In step 4, in order to divide by $5\frac{1}{2}$, we reduce 21,081 to halves obtaining 42,162 halves. $5\frac{1}{2}$=11 halves. Dividing 42,162 halves by 11 halves, gives 3,832 and a remainder of 10 halves or 5. So the remainder being 10 half yards, gives 5 yards.

CLOTH MEASURE.

2. This is an obsolete system formerly used in measuring cloth.

I. Reduce 2 E. Fr. 1 E. En. 2 E. Fl. 3 yd. 2 na. to nails.

6	5	$4+1\frac{1}{5}$ in.	3	4	4	
E. Fr.	E. En.	E. Sc.	E. Fl.	yd.	qr.	na.
2	1		2	3		2
6	5		3	4		
12qr.	5qr.		6qr.	12 qr.		

12 qr.
6 "
5 "
12 "
35 qr.
4
140 na.
2 "
142 na.

3. ***Surveyors' Linear Measure*** is used by surveyors in measuring land.

4. ***The Standard, or Prime, Unit*** is the *chain*, called *Gunter's chain*.

SURVEYOR'S LINEAR MEASURE.

100 links (l.) = 1 chain (ch.)
80 chains = 1 mi.

Note.—1 chain = 4 rods = 66 feet = 792 inches. 1 link = 7.92 inches.

I. Reduce 3 chains to inches.

II.
1. 1 chain = 100 links.
2. 3 chains = 3×100 links = 300 links.
3. 1 link = 7.92 inches.
4. 300 links = 300×7.92 inches = 2,376 inches.

III. ∴ 3 chains = 2,376 inches.

UNITS OF SURFACE, OR SQUARE MEASURE.

1. ***The Standard Unit*** of square measure is the *square yard*. It is a square each side of which is 1 yard in length. It is derived from the corresponding unit of linear measure and is, therefore, called a *derived unit*.

SQUARE MEASURE.

144 square inches (sq. in.)	= 1 square foot (sq. ft.)
9 square feet	= 1 square yard (sq. yd.)
$30\frac{1}{4}$ square yards	= 1 square rod (sq. rd.)
160 square rods	= 1 acre (A.)
640 acres	= 1 square mile (sq. mi.)

Remark.—Observe that the table of square measure is derived from the table of linear measure by squaring the corresponding units in linear measure.

2. 10 square chains = 1 acre; 1 acre = 4,840 sq. yds. = 43,560 sq. ft.

I. What is the difference between 10 feet square and 10 square feet?

II.
1. 10 feet square is a square each side of which is 10 feet in length.
2. ∴ 100 square feet = 10×10 square feet, = 10×10×1 sq. ft. = the surface of the square.
3. 10 square feet = 10 units of which one unit is a square foot.
4. 100 square feet—10 square feet = 90 square feet, the numerical difference.

III. ∴ 90 square feet is the numerical difference.

Explanation.—In finding the area of a rectangle, we multiply the number expressing its length by the number expressing its width to obtain the number expressing its area. Thus, to find the area of a rectangle 4 feet long and 3 feet wide: 4 is the number expressing its length in the unit of length and 3 is the number expressing its width in the same unit of length. Therefore, 4×3, or 12, is the number expressing its area in the corresponding unit of area, viz., the *square foot.*

It is not true that feet, multiplied by feet gives square feet. If it were, we might have circles multiplied by circles giving square circles, bottles multiplied by bottles giving square bottles, and days multiplied by days giving square days.

For further discussion, see *Mensuration,* page 239.

I. What is the area of a ceiling 20 feet long and 16 feet wide?

II.
1. 20=number of feet in the length.
2. 16=number of feet in the width.
3. ∴ $20\times16=320$=number of square feet in its area.

III. ∴ The ceiling contains 320 square feet.

UNITS OF VOLUME, OR CUBIC MEASURE.

1. *The Standard Unit of Volume* is the *cubic yard* which is a cube each edge of which is one yard in length.

CUBIC MEASURE.

1,728 cubic inches (cu. in)	= 1 cubic foot (cu. ft.)
27 cubic feet	= 1 cubic yard (cu. yd.)

Remarks.—1. Observe that the table of cubic measure is derived from that of linear measure by cubing the corresponding linear units. Thus, 1 cubic foot=$12\times12\times12\times1$ cubic inch= 1,728 cubic inches; 1 cubic yard=$3\times3\times3\times1$ cubic foot=27 cubic feet.

2. Firewood is measured by the cord, a cord being a pile 8 feet long, 4 feet wide, and 4 feet high and containing 128 cubic feet.

In finding the volume, or solid contents, of a rectangular solid, we multiply the numbers expressing its dimensions together to obtain the number expressing its volume.

I. Find the cost of a pile of wood 24 feet long, $5\frac{1}{2}$ feet high, and 4 feet wide at $3 per cord.

II.
1. 24 =number expressing the length of the pile in feet.
2. $5\frac{1}{2}$=number expressing the height of the pile in feet.
3. 4 =number expressing the width of the pile in feet.
4. $24\times5\frac{1}{2}\times4=528$=the number expressing the volume in *cubic feet.*

5. 128 cubic feet=1 cord.
6. 528 cubic feet=528÷128=$4\frac{1}{8}$ cords.
7. \$3=cost of 1 cord.
8. \$$12\frac{3}{8}$=$4\frac{1}{8}$×\$3=cost of $4\frac{1}{8}$ cords.

III. ∴ The pile of wood will cost \$$12\frac{3}{8}$.

UNITS OF CAPACITY.

1. ***The Standard Unit of Capacity*** is the *gallon.*

LIQUID MEASURE.

4 gills (gi.) = 1 pint (pt.)
2 pints = 1 quart (qt.)
4 quarts = 1 gallon (gal.)

Note.—The capacity of cisterns, reservoirs, etc., is often expressed in barrels (bbl.) of $31\frac{1}{2}$ gallons each, or hogsheads (hhd.) of 63 gallons each. In old tables the following were given: a tierce (tr.)=42 gal.; a puncheon (pn.)=84 gal.; and a pipe (p.)=126 gal.
A gallon contains 231 cubic inches.

I.—Reduce 2 p. 3 pn. 1 tr. 1 hhd. 1 gal. 1 qt. to pints.

126
84
42
$31\frac{1}{2}$
4— 63 4 2

p.	pn.	tr.	bbl.	T.	hhd.	gal.	qt.	pt.
2	3	1			1	1	1	
126	84	42			63	63		
252 gal.	252 gal.	42 gal.			63 gal.			

42
252
252
610 gal.
4
2440 qt.
1 "
2441 "
2
4882 pt.

DRY MEASURE.

2 pints (pt.) = 1 quart (qt.)
8 quarts = 1 peck (pk.)
4 pecks = 1 bushel (bu.)

Note.—A bushel is a cylindrical vessel $18\frac{1}{2}$ inches in diameter and 8 inches deep and contains 2150.42 cubic inches.

I. Reduce 2 bu. 3 pk. 2 qt. 1 pt. to pints.

Equation Method.

Solution: II.
1. 1 bu.=4 pk.
2. 2 bu.=2×4 pk.=8 pk.
3. 8 pk.+3 pk.=11 pk.
4. 1 pk.=8 qt.
5. 11 pk.=11×8qt.=88 qt.
6. 88 qt.+2 qt.=90 qt.
7. 1qt.=2 pt.
8. 90 qt.=90×2 pt.=180 pt.
9. 180 pt.+1 pt.=181 pints.

Conclusion: III. ∴ 2 bu. 3 pk. 2 qt. 1 pt.=181 pints.

I. Reduce 529 pints to bushels.

Equation method.

Solution: II.
1. 2 pt.=1 qt.
2. 529 pt.=529÷2=264 qt.+1 pt.
3. 8 qt.=1 pk.
4. 264 qt.=264÷8=33 pk.
5. 4 pk.=1 bu.
6. 33 pk.=33÷4=8 bu.+1 pk.

Conclusion: III. ∴ 529 pints=8 bu. 1 pk. 1 pt.

I. How many gallons will a tank 4 ft. long, 3 ft. wide, and 1 ft. 8 in. deep contain?

Solution: II.
1. 4 ft.=length,
2. 3 ft.=width, and
3. 1 ft. 8 in.=$1\frac{2}{3}$ ft.=depth.
4. $4\times3\times1\frac{2}{3}=20$ cubic ft.=contents of tank.
5. 1 cu. ft.=1728 cu. in.
6. 20 cu. ft.=20×1728 cu. in.=34560 cu. in.
7. 231 cu. in.=1 gal.
8. ∴ 34560 cu. in.=34560÷231=$149\frac{47}{77}$ gal.

Conclusion: III. ∴ The tank will contain $149\frac{47}{77}$ gallons.

(*Fish's Comp. Arith., p. 126, prob. 2.*)

III. ANGULAR OR CIRCULAR MEASURE.

1. ***Angular Measure*** is used to measure angles, directions, latitude, longitude, in navigation, astronomy, etc.

2. ***An Angle*** is the amount of divergence between two lines which meet in a point.

3. ***An Angle*** is measured by the arc of a circle intercepted by the sides, the vertex of the angle being the center of the circle.

It is proved in Geometry that, *In the same circle, or equal circles, two angles at the center have the same ratio as their intercepted arcs.* The two intersecting lines are called the *sides* of the angle.

This being true, we may divide the circumference of a circle into any number of equal parts and adopt one of these equal parts as the ***unit*** of *circular measure.* Then, the angle subtended by this unit of circular measure may be adopted as the ***unit*** of *angular measure.*

In Geometry, the circumference of a circle is divided into four equal parts and one of these equal parts is taken as the *unit* of circular measure. It is called a *quadrant.* The angle subtended by the quadrant is the *unit* of angular measure and it is called a *right angle.* In Geometry, therefore, the ***unit*** of *angular measure* is the ***right angle.***

In Trigonometry, the circumference of the circle is divided into 360 equal parts and one of these equal parts is taken as the unit of *circular measure.* It is called a ***degree*** of arc, or *arc-degree.* The angle subtended by an arc-degree is taken as the unit of angular measure and is called a ***degree,*** (°).

However, the most common *unit* of angular measure used in trigonometry is the angle subtended by an arc of the circumference equal in length to the radius of the circle. This angle is called a ***radian.*** Since the radius is contained in a semicircumference π times, it follows that π radians=180°. From this it follows that 1 radian$=\frac{180°}{\pi}=57°.3$ nearly. Also, $1°=\frac{1}{180}\pi$ radians, marked $\frac{1}{180}\pi^{(r)}$.

ANGULAR OR CIRCULAR MEASURE.

60 seconds (″)	=	1 minute (′).
60 minutes	=	1 degree (°).
360 degrees	=	1 circumference.

I. Reduce 72° to radians.

II.
1. $180°=\pi$ radians.
2. $1°=\frac{1}{180}$ of π radians$=\frac{\pi}{180}$ radians.
3. $72°=72\times\frac{\pi}{180}$ radians$=\frac{2}{5}\pi$ radians.

III. ∴ $72°=\frac{2}{5}\pi$ radians.

Note.—π represents 3.14159265 . . . ., the ratio of the circumference of a circle to the diameter.

I. Reduce $\frac{2}{3}\pi$ radians to degrees.

II.
1. π radians=180°.
2. $\frac{2}{3}\pi$ radians$=\frac{2}{3}$ of $180°=120°$.

III. ∴ $\frac{2}{3}\pi$ radians=120°.

IV. TIME MEASURE.

1. ***Time*** is a measured portion of duration.

2. The *measures* of time are fixed by the rotation of the earth on its axis and its revolution around the sun.

3. ***A Day*** is the time of one rotation of the earth on its axis.

4. ***A Year*** is the time of one revolution of the earth around the sun.

TABLE.

60 seconds (sec.)	make	1 minute (min.)
60 minutes	"	1 hour (hr.)
24 hours	"	1 day (da.)
7 days	"	1 week (wk.)
4 weeks	"	1 lunar month (mo.)
13 lunar months, 1 da. 6 hr.	"	1 year (yr.)
12 calender months	"	1 year.
365 days	"	1 common year.
365 da. 5 hr. 48 min. 46.05 sec.	"	1 solar year.
365 da. 6 hr. 9 min. 9 sec.	"	1 sidereal year.
365 da. 6 hr. 13 min. 45.6 sec.	"	1 Anomalistic year.
366 days	"	1 leap year, or bissextile year.
354 days	"	1 lunar year.
19 years	"	1 Metonic cycle.
28 years	"	1 solar cycle.
15 years	"	1 Cycle of Indiction.
532 years	"	1 Dionysian Period.

5. The *unit of time* is the day.

6. ***The Sidereal Day*** is the exact time of one rotation of the earth on its axis. It equals 23 hr. 56 min. 4.09 sec.

7. ***The Solar Day*** is the time between two successive appearances of the sun on a given meridian.

8. ***The Astronomical Day*** is the solar day, beginning and ending at noon.

9. ***The Civil Day,*** or ***Mean Solar Day,*** is the average of all the solar days of the year. It equals 24 hr. 3 min. 56.556 sec.

10. ***The Solar Year,*** or ***Tropical Year,*** is the time between two successive passages of the sun through the vernal equinox.

11. ***The Sidereal Year*** is the time of a complete revolution of the earth about the sun, measured by a fixed star.

12. ***The Anomalistic Year*** is the time of two successive passages of the earth through its perihelion.

13. ***A Lunar Year*** is 12 lunar months and consists of 354 day.

14. ***A Metonic Cycle*** is a period of 19 solar years, after which the new moons again happen on the same days of the year.

15. ***A Solar Cycle*** is a period of 28 solar years, after which the first day of the year is restored to the same day of the week. To find the year of the cycle, we have the following rule:

Add nine to the date, divide the sum by twenty-eight; the quotient is the number of cycles, and the remainder is the year of the cycle. Should there be no remainder the proposed year is the twenty-eighth, or last of the cycle. The formula for the above rule is $\left\{\frac{x+9}{28}\right\}_r$ in which x denotes the date, and r the remainder which arises by dividing $x+9$ by 28, is the number required.

Thus, for 1892, we have $(1892+9)\div28=67\frac{25}{28}$. . . 1892 is the 25th year of the 68 cycle.

16. ***The Lunar Cycle*** is a period of 19 years, after which the new moons are restored to the same day of the civil month.

The new moon will fall on the same days in any two years which occupy the same place in the cycle; hence, a table of the moon's phases for 19 years will serve for any year whatever when we know its number in the cycle. This number is called the *Golden Number.*

To find the Golden Number: *Add 1 to the date, divide the sum by 19; the quotient is the number of the cycle elapsed and the remainder is the Golden Number.*

The formula for the same is $\left\{\frac{x+1}{19}\right\}_r$ in which r is the remainder after dividing the date+1 by 19. It is the Golden Number.

17. ***A Dionysian*** or ***Paschal Period*** is a period of 532 year, after which the new moons again occur on the same day of the month and the same day of the week. It is obtained by multiplying a *Lunar Cycle* by a *Solar Cycle.*

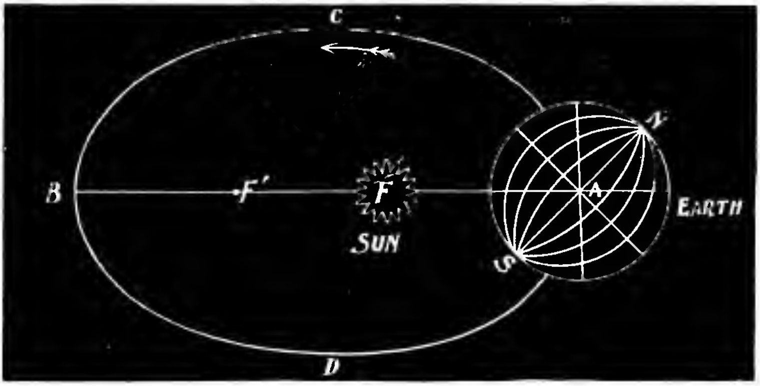
18. ***A Cycle of Indiction*** is a period of 15 years, at the end of which certain judicial acts took place under the Greek emperors.

19. ***Epact*** is a word employed in the calender to signify the moon's age at the beginning of the year.

The common solar year, containing 365 days, and the lunar year only 354, the difference is 11 days; whence, if a new moon fall on the first of January in any year, the moon will be 11 days old on the first day of the following year, and 22 days on the first of the third year. The *epact* of these years are, *therefore,* eleven and twenty-two respectively. Another addition of eleven days would give thirty-three for the *epact* of the fourth year; but in consequence of the insertion of the intercalary month in each third year of the lunar cycle, this epact is reduced to three. In like manner the epacts of all the following years of the cycle are obtained by successively adding eleven to the epact of the former year, and rejecting thirty as often as the sum exceeds that number.

V. LONGITUDE AND TIME.

In the diagram, the curve $ACBD$ represents the path of the earth in its journey around the sun. This curve is called an *ellipse*. The ellipse may be drawn by taking any two points F and F' and fastening in them the extremities of a thread whose



length is greater than the distance $F'F$. Place the point of a pencil against the thread and slide it so as to keep the thread constantly stretched; the point of the pencil in its motion will describe the ellipse.

The points F and F' are called the *foci*, the plural of *focus*.

The sun occupies one of these foci. The plane of the earth's orbit, or path, is called the *ecliptic*. When the earth is at A it is nearer the sun than when it is at B. When the earth is nearest the sun it is said to be in *perihelion* (Gr. περί=*peri*, near, and ἥλιος =*helios*, sun); when farthest from the sun, it is said to be in *aphelion* (Gr. ἀπό =*apo*, from, and ἥλιος =*helios*, sun). The points A and B, in the diagram, represent the perihelion and aphelion distances, respectively. The earth is nearest the sun about the first of January and farthest from the sun about the first of July. It takes the earth 365 da. 6 hr. 13 min. 45.6 sec. to travel from A, west around through C, B, and D back to A.

This period of time is called the *anomalistic year*. The west point as here spoken of, may be thought of thus: Suppose you were located at some point on the surface of the sun in a position enabling you to see the North Star. Then if you should face that star you would be facing north, your right hand would be to the east, and your left hand to the west, and south to your back.

While the earth makes one revolution around the sun, it rotates 366 times on one of its diameters. The diameter upon which it rotates is called its axis. The axis of the earth is inclined from a perpendicular to the plane of the earth's orbit at an angle of $23\frac{1}{2}°$. If the axis of the earth were extended indefinitely, it would pass very near, $1\frac{1}{4}°$, from the North Star.

The earth's axis and the sun determine a plane, and this plane is of great importance in gaining a thorough understanding of *Longitude and Time*. Suppose you are on the earth's surface, facing the sun and in this plane. Then you will have noon, while just to the east of you it will be after noon and just to the west it will be before noon. The intersection of this plane with the earth's surface is called the *trace* of the plane on the earth's surface. This trace is called a *meridian*.

If you could travel in such a way as to remain in this plane for a whole day, that is, 24 hours, you would have noon during the whole time. But if you remain stationary on the earth's surface, you will be carried out of this plane eastward by the earth's rotation. You may conceive that you are at Greenwich, formerly a small suburban town of London, Eng., but now incorporated in that city, with the sun visible and having such a position that you are in the plane formed by the earth's axis and the sun. It will then be noon to you at that place.

Suppose we take the trace of the plane, in this position, on the earth's surface as our standard meridian. Then all places east of this line will have had noon and all places west of it are yet to have noon. As the earth continues to rotate, rotating as it does from west to east, it will bring points west of the plane into coincidence with the plane and thus these points will have noon successively as they come into the plane. Suppose we start when Greenwich is in this plane, and mark the trace of the plane on the earth's surface and every four minutes we mark the trace of the plane; in this way, in a complete rotation of the earth, we will have drawn 360 of these traces, which we have agreed to call meridians.

The distance of these lines apart, measured on the equator, is called a degree of longitude, better an arc-degree of longitude. Instead of measuring longitude from Greenwich entirely around the earth through the west, we generally measure it east and west to 180°.

Thus, a place located on the 70th meridian, west, is said to be 70° west longitude, and a place situated on the 195th meridian, counting from Greenwich around through the west, is said to be 185° east longitude.

From the above discussion, we see that, since the earth turns on its axis once in 24 hours,

24 hrs. ≎ 360° of long., or 360° of long. ≎ 24 hrs.
1 hr. ≎ $\frac{1}{24}$ of 360°=15° of long., or 15° of long. ≎ 1 hr.
1 min. ≎ $\frac{1}{60}$ of 15°=15′ of long., or 15′ of long. ≎ 1 min.
1 sec. ≎ $\frac{1}{60}$ of 15′=15″ of long., or 15″ of long. ≎ 1 sec.

Hence, if we have the difference of longitude of two places, we can readily find the difference of time between these two places.

For example, the longitude of St. Petersburg is 30° 16′ E., and the longitude of Washington is 77° 0′ 36″ W. Now the difference of longitude between these two places is 77° 0′ 36″ + 30° 16′=107° 16′ 36″. Hence, since 15° ≎ 1 hour, 107° 16′ 36″ ≎ (107° 16′ 36″)÷15, or 7 hrs. 9 min. 6.4 sec., which is the difference of time between Washington and St. Petersburg.

Conversely, if we know the difference of time between two places, we can easily find the difference of longitude.

For example, the difference of time between New York City and St. Louis is 1 hr. 4 min. $47\frac{1}{3}$ sec. Find the difference of longitude.

II.
1. 1 hr. ≎ 15°. (For ≎, read "corresponds to.")
2. 1 min. ≎ 15′.
3. 4 min. ≎ 60′, or 1°.
4. 1 sec. ≎ 15″.
5. $47\frac{1}{3}$ sec. ≎ $47\frac{1}{3}\times15''=710''=11'\ 50''$.

III. Hence, the difference of longitude is 16° 11′ 50″.

In some cases, problems are so proposed that we are to find the longitude or time of one place, having given the longitude or time of another place and the difference of time or difference of longitude of the two places. Such problems require no principles beyond those already established.

For example, the difference of time between two places is 2 hr. 30 min. The longitude of the eastern place is 56° W. Find the longitude of the western place.

II.
1. 1 hr. ≎ 15°.
2. 2 hr. ≎ 30°
3. 1 min. ≎ 15′.
4. 30 min. ≎ 30×15′=450′=7° 30′.
5. ∴ 37° 30′=difference of longitude.
6. ∴ 56°+37° 30′=93° 30′, the longitude of the western place.

Had the place whose longitude is given been in east longitude, we would have subtracted the difference of longitude to find the longitude of the western place.

The following suggestions may prove helpful in the solution of problems in Longitude and Time:

1. *When the longitude of a place is required, having given the longitude of some other place and the difference of longitude between the two places.*

Conceive yourself located at the place whose longitude is given. Then ask yourself this question, Is the place whose longitude is required, east or west? If west, add the difference of longitude when the given longitude is west, and subtract if the given longitude is east. If the answer to your question is east, subtract the difference of longitude when the given longitude is west and add the difference of longitude when the given longitude is east.

If the places are on opposite sides of the standard meridian, subtract the given longitude from the difference of longitude and the difference will be the longitude required, and will be opposite in name from the given longitude. That is to say, if the given longitude is east, the required will be west, and *vice versa.*

2. *When the time of place is required, having given the time at some other place and the difference of time between the two places.*

Conceive yourself located at the place whose time is given. Then ask yourself this question, Is the place whose time is required, east or west of me? If the answer to your question is west, subtract the difference of time for the required time. If the answer to your question is east, add the difference of time for the required time.

STANDARD TIME.

In 1883, the railroad officials of the United States and Canada adopted what is called *standard time.* These officials agreed to adopt the solar time of some standard meridian as the local time of an extended area. The standard meridians thus adopted are 75th, 90th, 105th, and 120th. All stations in the belt of country $7\frac{1}{2}°$ wide on either side of these standard meridians have as local time the solar time of the respective meridian. For example, all points or stations in the belt of country $7\frac{1}{2}°$ wide on either side of 90th meridian, i. e., the belt of country lying between $82\frac{1}{2}°$ and $97\frac{1}{2}°$ west longitude have as local time the solar time, or sun time, of the 90th meridian. In other words, all time-pieces of the various stations in this belt indicate the same time of day as clocks in the depots situated on the 90th meridian. For example, when the clock in the Union Depot at St. Louis indicates noon, 12 o'clock M., the clocks in the Union Depots at Indianapolis and Kansas City also indicate noon, though at Indianapolis it is a little more than 16 min. past noon and at Kansas City it is a little more than 17 min. till noon, sun time. That is, the standard time and local time at Indianapolis differ by a little more than 16 min., standard time being about 16 min. slower than local time, and at Kansas City standard time and local time differ by about 17 min., standard time being about 17 min. faster than local time.

If one were to set his watch with the railroad clock in the depot at Columbus, Ohio, then take the train for Springfield, Mo., on arriving at Springfield, Mo., one would find that his watch agrees with the clock in the Frisco depot. This is because Columbus, Ohio, and Springfield, Mo., are located in the belt of country having *central time*, i. e., having the sun time of the 90th meridian.

How about the local time of these two places? The local time at Columbus, O., is about 28 min. faster than *standard time*. The sun comes to the meridian of Columbus before it comes to the 90th meridian. When the sun comes to the meridian at Columbus it is noon, local time, but it will not be noon, *standard time*, until the sun comes to the 90th meridian, which will be about 28 min. later. Hence, local time at Columbus, O., is about 28 min. faster than standard time. A passenger going from Columbus, Ohio, to Springfield, Mo., and carrying standard time of Columbus, would have standard time at St. Louis, standard time and local time at St. Louis being very nearly the same. On arriving at Springfield, Mo., his watch would still indicate standard time and would agree with the regulator in the depot at Springfield, but his time would be about 8 min. faster than the local time at Springfield.

The time in the belt of country between $67\frac{1}{2}°$ west longitude and $82\frac{1}{2}°$ west longitude is called *Eastern Time;* between $82\frac{1}{2}°$ W. and $97\frac{1}{2}°$ W., *Central Time;* between $97\frac{1}{2}°$ W. and $112\frac{1}{2}°$ W., *Mountain Time;* and between $112\frac{1}{2}°$ W. and $127\frac{1}{2}°$ W., *Pacific Time*. We might call the time in the belt between $7\frac{1}{2}°$ E. and $7\frac{1}{2}°$ W,. *Greenwich Time;* between $7\frac{1}{2}°$ W. and $22\frac{1}{2}°$ W., *East Atlantic Time;* between $22\frac{1}{2}°$ W. and $37\frac{1}{2}°$ W., *Central Atlantic Time;* between $37\frac{1}{2}°$ W. and $52\frac{1}{2}°$ W., *West Atlantic Time;* and between $52\frac{1}{2}°$ W. and $67\frac{1}{2}°$ W., *Colonial Time*.

By some appropriate system of nomenclature, the naming of the time in the belt beginning with $127\frac{1}{2}°$ W. longitude might be extended. However, these names would have a very limited use and are therefore not worth coining.

THE INTERNATIONAL DATE LINE.

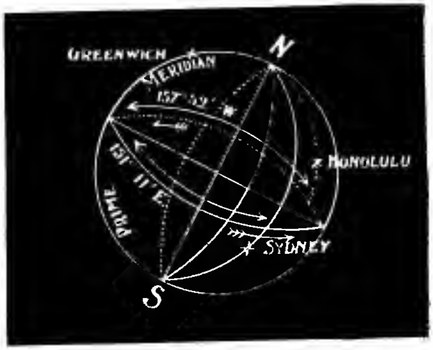
The International Date Line is an irregular line passing through Bering Strait, along the coast of Asia to near Borneo and Philippine Islands, and thence along the northern limits of the East Indian Islands, New Zealand, and New Guinea. It is the line from which every date on the earth is reckoned. At present, however, the 180th meridian is very generally used in its stead.

Suppose one were standing on the 180th meridian at the time it is noon, Wednesday (say), at Greenwich, and facing north. Then, just to right, or east of this line, Wednesday is beginning, i. e., Wednesday 12 o'clock, A. M., while just to left, or west of the line, Wednesday is ending, i. e., Wednesday 12 o'clock, P. M. The difference of time between places immediately east and immediately west of the line is therefore 24 hours, and just west of the line it is one day later than just east of it. This is made still clearer by considering what takes place as the earth rotates on its axis; the places just west of the line will be carried eastward, and since these places had Wednesday *ending*, they must now have Thursday beginning. But these places are west of the line, the places east of the line still having Wednesday. Hence, it is clear that it is one day *later* just west of the line than just east of the line. In crossing this line, therefore, from the east one day must be added, while in crossing it from the west one day must be subtracted.

Professor C. A. Young, in his General Astronomy, in answering the question, "Where does the day begin?" says, "If we imagine a traveler starting from Greenwich on Monday noon and traveling westward as swiftly as the earth turns to the east under his feet, he would, of course, keep the sun exactly on the meridian all day long and have continual noon. But what noon? It was Monday when he started, and when he gets back to London, twenty-four hours later, it is Tuesday noon there, and there has been no intervening sunset. When does Monday noon become Tuesday noon? The convention is that the change of date occurs at the 180th meridian from Greenwich. A ship crossing this line from the east skips one day in so doing. If it is Monday forenoon when the ship reaches the line, it becomes Tuesday forenoon the moment it passes it, the intervening twenty-four hours being dropped from the reckoning on the log-book. *Vice versa*, when a vessel crosses the line from the western side, it counts the same day twice, passing from Tuesday forenoon back to Monday, and having to do its Tuesday over again."

The consideration of this line in the solution of problems in longitude and time should add no serious difficulty. Solve the problem completely, leaving out of account the date line. Then, if the time of the given place is west of the line, while the place whose time is required is east, we simply subtract a day, and if the conditions are reversed, we add a day.

1. When it is five minutes after four o'clock on Sunday morning at Honolulu, what is the hour and day of the week at Sydney, Australia? (*Ray's Higher Arithmetic, p. 171, prob. 7.*)



II.

1. 157° 52′ W.= longitude of Honolulu.
2. 151° 11′ E.= longitude of Sydney.
3. 309° 3′=difference of longitude measured from Honolulu through Greenwich to Sydney.
4. 360°— 309° 3′ = 50° 57′= difference of longitude measured directly on the equator from the meridian through Honolulu to the meridian through Sydney.
5. 15° $\backsimeq$ 1 hr.
6. 1° $\backsimeq$ $\frac{1}{15}$ hr.=4 min.
7. 50° 57′=$50\frac{57}{60}$°=$50\frac{19}{20}$° $\backsimeq$ $50\frac{19}{20}\times 4$ min.=3 hr. 23 min. 48 sec., difference of time.
8. 4 hr. 5 min., Sunday—3 hr. 23 min. 48 sec.=41 min. 12 sec., Sunday.
9. Regarding the date line, Sunday is changed to Monday, since Honolulu is east of the line, while Sydney is west of it.

EXAMPLES.

1. When it is 5 o'clock Monday morning at Paris, France, longitude 2° 20′ E., what is the hour and day of the week at Honolulu, Hawaiian Islands, longitude 157° 52′ W.?

Ans. 19 min. 12 sec. past 6 o'clock P. M., Sunday.

2. When it is five minutes after 3 o'clock on Sunday morning at Honolulu, Hawaiian Islands, longitude 157° 52′ W., what is the hour and day of the week at Sydney, Australia, longitude 151° 11′ E.?

Ans. 41 min. and 12 sec. before 12 o'clock P. M., Sunday.

3. When it is 20 minutes past 12 o'clock on Saturday morning at Chicago, Ill., longitude 87° 35′, what is the hour and day of the week at Pekin, China, longitude 116° 26′ E.?

Ans. 56 min. 4 sec. past 1 o'clock P. M., Saturday.

4. When it is ten minutes until 12 o'clock, Friday, midnight, at Constantinople, Turkey, longitude 28° 59′ E., what is the hour and day of the week at Honolulu, Hawaiian Islands, longitude 157° 52′ W.?

Ans. 22 min. 36 sec. past 11 o'clock A. M., Friday.

5. At what hour must a man start, and how fast must he travel, at the equator, so that it would be noon for him for twenty-four hours?

Ans. Noon; 1037.4 statute miles per hr.

6. What is the difference of time between Constantinople, Turkey, and Sydney, Australia? *Ans.* 8 hr. 10 min. 48 sec.

7. A traveler sets his watch with the time of the sun at New York. He then travels from there and on arriving at his destination finds that his watch is 1 hr. 20 min. 30 sec. fast. What is the longitude of his destination if the longitude of New York is 74° 0′ 24″ W.? *Ans.* 94° 7′ 54″ W.

8. When it is 1 o'clock P. M. at Rome, Italy, longitude 12° 28′ E., what is the hour at New York, longitude 74° 0′ 24″ W.?

Ans. 14 min. $6\frac{2}{15}$ sec. past 7 A. M.

9. When it is 1:20 A. M. at St. Louis, longitude 90° 15′ 15″ W., it is 8 hr. 35 min. $1\frac{2}{5}$ sec. A. M. at the Cape of Good Hope. What is the longitude of the Cape of Good Hope? *Ans.* 18° 30′ 6″ E.

10. When it is 3:55 A. M. at Constantinople, longitude 28° 58′ 40″ E., it is 6 hr. 50 min. 38 sec. A. M. at Bombay. What is the longitude of Bombay? *Ans.* 72° 53′ 10″ E.

11. When it is 6:33 P. M. at Jerusalem, longitude 35° 30′ 48″ E., it is 11 hr. 17 min. 13 sec. A. M. at Montreal. What is the longitude of Montreal? *Ans.* 73° 25′ 57″ W.

12. When it is 1 o'clock P. M. at Rome, it is 54 min. 34 sec. after 6 o'clock A. M. at Buffalo, N. Y. What is the longitude of Buffalo? *Ans.* 78° 53′ 30″ W.

MISCELLANEOUS PROBLEMS.

1. How many links in 46 mi. 3 fur. 5 ch. 25 links?

2. How many acres in a field containing 1377 square chains?

3. How many cubic inches in 29 cords of wood?

4. In 1436 nails how many Ell English?

5. How many miles in 3136320 inches?

6. In 47 ℔. 2 ℥ 3 ʒ 1 ℈ 19 gr. how many grains?

7. Change 16 lb. 3 oz. 1 gr., Troy weight to Avoirdupois weight.

8. An apothecary bought by Avoirdupois weight, 2 ℔. 8 oz. of quinine at $2.40 per ounce, which he retailed at 20 ct. a scruple. What was his gain on the whole?

9. How many seconds in a Dionysian Period?

10. How many seconds in the month of February, 1892.

11. How many seconds in the circumference of a wagon wheel?

12. How long would it take a body to move from the earth to the moon, moving at the rate of 30 miles per day.

13. If a man travels 4 miles per hour, how far can he travel in 2 weeks and 3 days?

14. How much may be gained by buying 2 hogsheads of molasses, at 40 ct. per gallon, and selling it at 12 cents per quart? *Ans.* \$10.08

15. In 74726807872 seconds, how many solar years? *Ans.* 2368 years.

16. At \$4 per quintal, how many pounds of fish may be bought for \$50.24? *Ans.* 1256 pounds.

17. How many bottles of 3 pints each will it take to fill a hogshead? *Ans.* 168.

18. What will 73 bushels of meal cost, at 2 cents per quart? *Ans.* \$46.72.

19. How many ounces of gold are equal in weight to 6 ℔. of lead? *Ans.* $87\frac{1}{2}$ oz.

20. What is the difference between the weight of $42\frac{3}{8}$ ℔. of iron and 42.375 ℔. of gold? *Ans.* 52545 gr.

21. How many bushels of corn will a vat hold that holds 5000 gallons of water. *Ans.* $537\frac{7}{64}$ bu.

22. A cellar 40 ft. long, 20 ft. wide and 8 ft. deep is half full of water. What will it cost to pump it out, at 6 cents a hogshead? *Ans.* \$22.797+.

23. If a man buys 10 bu. of chestnuts at \$5 a bushel, dry measure, and sells the same at 25 cents a quart, liquid measure, how much does he gain? *Ans.* \$43.09+ gain.

24. How many steps, 2 ft. 8 in. each, will a man take in walking a distance of 15 miles? *Ans.* 29700.

25. How many hair's width in a 40 ft. pole, if 48 hair's width equals 1 line?

26. How many chests of tea, weighing 24 pounds each, at 43 cents a pound, can be bought for \$1548? *Ans.* 150 chests.

27. How long will it take to count 6 million, at the rate of 80 a minute, counting 10 hours a day? *Ans.* 125 days.

28. How long will it take to count a billion, at the rate of 80 a minute, counting 12 hours a day? *Ans.*——

29. What will 15 hogsheads of beer cost, at 3 cents a pint. *Ans.* \$194.40.

30. How many shingles will it take to cover the roof of a building 60 ft. long and 56 ft. wide, allowing each shingle to be 4 inches wide and 18 inches long, and to lie $\frac{1}{3}$ to the weather? *Ans.* 20160.

31. There are 9 oz. of iron in the blood of 1 man. How many men would furnish iron enough in their veins to make a plowshare weighing $22\frac{1}{2}$ lbs.? *Ans.* 40.

CHAPTER VIII.

PROPERTIES OF NUMBERS.

DEFINITIONS.

1. ***The Properties of Numbers*** are those qualities which belong to them.

2. ***Number***

- 1_1 Real
  - 1_2 ....
    - 1_3 Integer
      - 1_4 ....
        - 1_5 Prime
        - 2_5 Composite
      - 2_4 ....
        - 1_5 Perfect
        - 2_5 Imperfect
      - 3_4 ....
        - 1_5 Odd
        - 2_5 Even
    - 2_3 Fraction
    - 3_3 Mixed
  - 2_2 ....
    - 1_3 Rational
    - 2_3 Irrational
- 2_1 Imaginary
  - 1_2 Pure
  - 2_2 Complex

3. ***An Integer*** is a whole number; as, 1, 2, 3, etc.

4. ***A Prime Number*** is one that cannot be divided by any other integers except itself and unity.

Thus, 1, 2, 3, 5, 7, 11, 13, etc., are prime numbers.

Note.—There is no general expression known for the representation of primes. The numbers, 2^p-1, when p is prime, are known as Mersenne's numbers and are prime when p is 1, 2, 3, 5, 7, 13, 17, 19, 31, 61. It is believed that these and 67, 127, and 257 are the only values of p less than 257 for which 2^p-1 is prime. All values of p less than 257 have been tested, except the following twenty-three: 67, 71, 89, 101, 103, 107, 109, 127, 137, 139, 149, 157, 163, 167, 173, 181, 193, 197, 199, 227, 229, 241, and 257.

The expression, x^2+x+41, is prime for all values of x less than 41. The number, $2^{61}-1=2{,}305{,}843{,}009{,}213{,}693{,}951$ is the largest number at present known to be prime. Euclid (circ. 330 B. C.) showed that the number of primes is infinite.

5. ***A Composite Number*** is a number that can be exactly divided by some other whole number besides itself and unity; as, 4, 10, 12, etc.

6. Two numbers are *prime to each other*, when unity is the only number that will exactly divide both; as, 6 and 25.

7. ***An Even Number*** is a number which is divisible by 2; as, 2, 4, 6, 8, etc.

8. ***An Odd Number*** is a number which cannot be divided by 2 without a remainder; as, 1, 3, 5, 7, 9, 25, etc.

9. ***A Perfect Number*** is one which is equal to the sum of all its divisors; as, $6=1+2+3$; $28=1+2+4+7+14$.

Note.—It is probable that all perfect numbers are included in the formula $2^{p-1}(2^p-1)$, where 2^p-1 is prime. Euclid proved that all numbers of this form are perfect, and Euler that the formula includes all even perfect numbers. W. W. Rouse Ball says, in his *History of Mathematics*, "There is reason to believe—though a rigid demonstration is wanting—that an odd number cannot be perfect."

If we assume this statement to be true, then every perfect number is of the above form. Then, if $p=$ 2, 3, 5, 7, 13, 17, 19, 31, 61, the corresponding perfect numbers are $2^{p-1}(2^p-1)=6$, 28, 496, 8128, 33550336, 8589869056, 137438691328, 2305843008139952128, 2658455991569831744654692615953842176.

10. ***An Imperfect Number*** is a number not equal to the sum of all its divisors; and is either *Abundant* or *Defective* according as the number is less or greater than the sum of its divisors; as, $12<1+2+3+4+6$ and $10>1+2+5$.

11. ***An Imaginary Number*** is a number arising from the extraction of an even root of a negative number; as, $\sqrt{-4}$, $\sqrt[6]{-3}$, etc.

The *unit* of pure imaginary numbers is $\sqrt{-1}$. For $\sqrt{-1}$, we write i, the initial of the word, imaginary, a notation due to Gauss.

Remark.—The limits of this work forbid any discussion of imaginary numbers. Such discussion will have to be looked for in treatises on Algebra, of which among the very best in the English language is Chrystal's, 2 vols.

12. *A Divisor* of a number is a number that exactly divides it. Thus, 3 is a divisor of 12.

13. *A Multiple* of a number is a number that exactly contains that number. Thus, 30 is a multiple of 6.

I. FACTORING.

1. *Factoring* is resolving composite numbers into factors; and depends upon the following principles:

Principle 1. *A factor of a number is a factor of any multiple of that number.*

Principle 2. *A factor of any two numbers is also a factor of their sum or their difference.*

From these principles are derived the following six propositions:

Proposition 1. *Every number ending with* 0, 2, 4, 6, or 8 *is divisible by* 2.

Proposition 2. *A number is divisible by* 4, *when the number denoted by its two right-hand digits is divisible by* 4.

Proposition 3. *A number ending in* 0 *or* 5 *is divisible by* 5.

Proposition 4. *Every number ending in* 0, 00, 000, *etc. is divisible by* 10, 100, 1,000, *etc.*

Proposition 5. *A composite number is divisible by the product of any two or more of its prime factors.*

Proposition 6. *Every prime number except* 2 *and* 5, *ends with* 1, 3, 7, *or* 9.

Remark.—The proofs of the above propositions are easily made and are left for the student as exercises.

Proposition 7. *Any integer is divisible by* 9 *if the sum of its digits be divisible by* 9.

Proof.

1. Every number is of the form $a10^n+b10^{n-1}+c10^{n-2}+\ldots\ldots+l$.
2. But $a10^n+b10^{n-1}+c10^{n-2}+\ldots+l=a(9+1)^n+b(9+1)^{n-1}+c(9+1)^{n-2}+\ldots+l=a\text{M}(9)+a+b\text{M}(9)+b+c\text{M}(9)+c+\ldots\ldots+l$, where $\text{M}(9)$ means a multiple of 9, $=\text{M}(9)+a+b+c+\ldots+l$.
3. ∴ The number is divisible by 9 when the sum of its digits is divisible by 9.

Proposition 8. *Any integer is divisible by* 11 *if the difference of the sums of the digits in the odd places and even places is divisible by* 11.

The proof of this proposition is the same as that of the last, except that, for 10, we write 11—1.

II. GREATEST COMMON DIVISOR.

1. ***A Divisor*** of a number is a number that will exactly divide it.

2. ***A Common Divisor*** of two or more numbers is a number that will exactly divide each of them.

3. ***The Greatest Common Divisor,*** or ***Highest Common Factor,*** of two or more numbers is the greatest number that will exactly divide each of them.

I. Find the G. C. D. of 60, 120, 150, 180.

II.
1. $60=2\times2\times3\times5$.
2. $120=2\times2\times2\times3\times5$.
3. $150=2\times3\times5\times5$.
4. $180=2\times2\times3\times3\times5$.
5. G. C. D.$=2\times3\times5=30$.

III. ∴ G. C. D.$=30$.

Explanation.—By inspecting the factors of each number we observe that 2 is found in each set of factors; hence, each of the numbers can be divided by 2. But only once, since it is found only once in the factors of 150. We also observe that 3 will divide the numbers only once, since it occurs only once in the factors of 60 and 120. Also, 5 will divide them but once, since 60, 120 and 180 contain it but once. Hence, the numbers, 60, 120, 150, 180, being divisible by 2, 3 and 5, are divisible by their product, $2\times3\times5=30$.

I. Find the G. C. D. of 180, 1260, 1980.

II.
1. $180=2\times2\times3\times3\times5$.
2. $1260=2\times2\times3\times3\times5\times7$.
3. $1980=2\times2\times3\times3\times5\times11$.
4. G. C. D.$=2\times2\times3\times3\times5=180$.

III. ∴ G. C. D. of 180, 1260, 1980$=180$.

Explanation.—2 being found twice in each number, they are each divisible by 2×2 or 4; also 3 being found twice in each number, they are each divisible by 3×3 or 9. 5 being found in each number, they are each divisible by 5. Hence, they are divisible by the product of these factors, $2\times2\times3\times3\times5=180$.

EXAMPLES.

1. Find the G. C. D. of 78, 234, and 468.

2. What is the G. C. D. of 36, 66, 198, 264, 600 and 720?

3. I have three fields: the first containing 16 acres, the second 20 acres, and the third 24 acres. What is the largest sized lots

containing each an exact number of acres, into which the whole can be divided? *Ans.* 4 A. lots.

4. A farmer has 12 bu. of oats, 18 bu. of rye, 24 bu. of corn and 30 bu. of wheat. What are the largest bins of uniform size, and containing an exact number of bushels, into which the whole can be put, each kind by itself, and all the bins be full? *Ans.* 6 bu. bins.

5. A has a four-sided field whose sides are 256, 292, 384, and 400 feet respectively; what is the length of the rails used to fence it, if they are all of equal length and the longest that can be used? *Ans.* 4 ft.

6. In a triangular field whose sides are 288, 450, and 390 feet respectively, how many rails will it require to fence it, if the fence is 5 rails high, and what must be the length of the rails if they lap over one foot? *Ans.* Length of rail, 7 ft. No. 940.

III. LEAST COMMON MULTIPLE.

1. ***A Multiple*** of a number is a number that will exactly contain it; thus, 24 is a multiple of 6.

2. ***A Common Multiple*** of two or more numbers is a number that will exactly contain each of them.

3. ***The Least Common Multiple*** of two or more numbers is the least number that will exactly contain each of them.

I. Find the L. C. M. of 30, 40, 50.

II.
1. $30=2\times3\times5$.
2. $40=2\times2\times2\times5$.
3. $50=2\times5\times5$.
4. L. C. M.$=2\times2\times2\times3\times5\times5=600$.

III. ∴ L. C. M. of 30, 40, 50=600.

Explanation.—The L. C. M. must contain 2 three times, or it would not contain 40; it must contain 5 twice, or it would not contain 50; it must contain 3 once, or it would not contain 30. Since all the factors of the numbers, 30, 40, 50, are contained in the L. C. M., it will contain each of them without a remainder.

I. Find the L. C. M. of 2310, 210, 30, 6.

II.
1. $2310=2\times3\times5\times7\times11$.
2. $210=2\times3\times5\times7$.
3. $30=2\times3\times5$.
4. $6=2\times3$.
5. L. C. M.$=2\times3\times5\times7\times11=2310$.

III. ∴ L. C. M. of 2310, 210, 30, 6=2310.

Explanation.—2 and 3 must be used, else the L. C. M. would not contain 6. 2, 3, and 5 must be used, else the L. C. M. would not contain 30. Hence 5 must be taken with the factors of 6. In like manner 7 must be taken with the factors already taken, else the L. C. M. would not contain 210. The factor 11 must be taken with those already taken, else the L. C. M. would not contain 2310. Hence 2, 3, 5, 7, and 11 are the factors to be taken and their product 2310 is the L. C. M.

I. The product of the L. C. M. by the G. C. D. of three numbers between 1 and 100 is 6,804; and the quotient of the L. C. M. divided by the G. C. D. is 84. What are the numbers?

II.
1. L. C. M. $\times$ G. C. D. $=6804$, and
2. $\dfrac{\text{L. C. M.}}{\text{G. C. D.}}=84$.
3. $\therefore$ L. C. M. $\times$ G. C. D. $\div\dfrac{\text{L. C. M.}}{\text{G. C. D.}}=$ L. C. M. $\times$ G. C. D. $\times\dfrac{\text{G. C. D.}}{\text{L. C. M.}}=(\text{G. C. D.})^2=6804\div84=81$.
4. G. C. D. $=\sqrt{81}=9$, by extracting the square root.
5. $\therefore$ L. C. M. $=6804\div9=756$.
6. $9=3\times3$.
7. $756=2\times2\times3\times3\times3\times7$.
8. $3\times3\times2\times2=36$.
9. $3\times3\times3\times2=54$.
10. $3\times3\times7\quad=63$.

III. $\therefore$ 36, 54, and 63 = the numbers.

Explanation.—Since 9 is the G. C. D., each of the numbers contains the factors of 9. Since there are two 2's in the L. C. M., one of the numbers must contain these factors. In like manner one of the numbers must contain three 3's; one of them must also contain 7. $\therefore$ We write two 3's for each of the numbers, two 2's to any set of these 3's, and 3 and 7 with either of the remaining sets, observing that the product of the factors in any set does not exceed 100. If we omit 2 in step 9, the product of the factors is 27. Hence 27, 36, 63 are numbers also satisfying the conditions of the problem.

EXAMPLES.

1. What is the L. C. M. of 13, 14, 28, 39, and 42?

2. What is the L. C. M. of 6, 8, 10, 18, 20, 36, and 48?

3. What is the L. C. M. of 18, 24, 36, 126, 20, 48, 96, 720, and 84?

4. What is the smallest sum of money with which I can purchase a number of oxen at \$50 each, cows at \$40 each, or horses at \$75 each? *Ans.* \$600.

5. Find three numbers whose L. C. M. is 840 and G. C. D. 42. *Ans.* 84, 210, and 420.

6. What three numbers between 30 and 140 having 12 for their G. C. D. and 2772 for their L. C. M.? *Ans.* 36, 84, and 132.

7. At noon the second, minute, and hour hands of a clock are together; how long after will they be together again for the first time?

8. J. S. H. has 5 pieces of land; the first containing 3 A. 2 rd. 1 p.; the second, 5 A. 3 rd. 15 p.; the third 8 A. 29 p., the fourth, 12 A. 3 rd. 17 p.; and the fifth, 15 A. 31 p. Required the largest sized house-lots, containing each an exact number of square rods, into which the whole may be divided. *Ans.* 1 A. 21 p.

9. The product of the L. C. M. of three numbers by their G. C. D.=864, and the L. C. M. divided by the G. C. D.=24; find the numbers. *Ans.* 12, 18, and 48.

CHAPTER IX.

FRACTIONS.

1. ***A Fraction*** is a number of the equal parts of a unit.

2. Fraction.
- 1. Common, or Vulgar.
  - 1. As to Form.
    - 1. Simple.
    - 2. Complex.
    - 3. Compound.
  - 2. As to Value.
    - 1. Proper.
    - 2. Improper.
    - 3. Mixed.
- 2. Decimal.
  - 1. Pure.
  - 2. Mixed.
  - 3. Circulating.
    - 1. Pure.
    - 2. Mixed.
- 3. Continued Fractions.

3. ***A Common Fraction,*** or ***Vulgar Fraction,*** is one in which the unit is divided into *any number* of equal parts; and is expressed by two numbers, one written above the other, with a horizontal line between them. Thus, $\frac{5}{6}$ expresses five-sixths.

4. ***A Simple Fraction*** is a fraction having a single integral numerator and denominator; as, $\frac{3}{5}$.

5. ***A Complex Fraction*** is a fraction whose numerator, or denominator, or both, are fractional; as, $\frac{4}{3\frac{1}{2}}$, $\frac{2\frac{1}{2}}{3\frac{1}{3}}$, $\frac{2\frac{1}{3}}{5}$.

6. ***A Compound Fraction*** is a fraction of a fraction; as, $\frac{2}{3}$ of $\frac{3}{4}$.

7. ***A Proper Fraction*** is a simple fraction whose numerator is less than its denominator; as, $\frac{4}{5}$.

8. ***An Improper Fraction*** is a simple fraction whose numerator is greater than its denominator; as, $\frac{5}{4}$.

9. ***A Mixed Number*** is a whole number and a fraction; as, $3\frac{3}{4}$.

10. ***A Decimal Fraction*** is a fraction whose denominator is ten, or some power of ten; as, $\frac{3}{10}$, $\frac{4}{100}$, $\frac{27}{1000}$. The denominator of a decimal is usually omitted and the point (.) is used to determine the value of the decimal expression. Thus, $\frac{3}{10}=.3$, $\frac{27}{1000}=.027$.

11. ***A Pure Decimal*** is one which consists of decimal figures only; as, .375.

12. ***A Mixed Decimal*** is one which consists of an integer and a decimal; as, 5.25.

13. ***A Circulating Decimal***, or a ***Circulate***, is a decimal in which one or more figures are repeated in the same order; as, .2121 etc. When a common fraction is in its lowest terms and the denominator contains factors other than 2 or powers of 2, and 5 or powers of 5, the equivalent decimal fraction will be circulating. Thus, $\frac{7}{1500}=\frac{7}{2^2\times3\times5^3}$ will, when reduced to a decimal, be circulating because the denominator contains the factor 3.

The repeating figure or set of figures is called a *Repetend*, and is indicated by placing a dot over the first and the last figure repeated.

14. ***A Pure Circulate*** is one which contains no figures but those which are repeated; as, $.\dot{2}7\dot{3}$.

15. ***A Mixed Circulate*** is one which contains one or more figures before the repeating part; as, $.45\dot{3}4\dot{2}$.

16. ***A Simple Repetend*** contains but one figure; as, $.\dot{3}$.

17. ***A Compound Repetend*** contains more than one figure; as, $\dot{3}5\dot{4}$.

18. ***Similar Repetends*** are those which begin and end at the same decimal places; as, $.\dot{3}46\dot{7}$, and $.0\dot{3}5\dot{8}$.

19. ***Dissimilar Repetends*** are those which begin or end at different decimal places; as, $.53\dot{6}$, $.8\dot{3}\dot{5}$, and $.3\dot{5}6\dot{7}$.

20. ***A Perfect Repetend*** is one which contains as many decimal places, less 1, as there are units in the denominator of the equivalent common fraction; thus, $\frac{1}{7}=.\dot{1}4285\dot{7}$.

21. ***Conterminous Repetends*** are those which end at the same decimal place; as, $.42\dot{6}\dot{7}$, $.\dot{3}27\dot{5}$, and $.03\dot{2}\dot{1}$.

22. ***Co-originous Repetends*** are those which begin at the same decimal place; as, $.\dot{3}7\dot{8}$, $.\dot{5}62\dot{4}$, and $3.\dot{6}2\dot{3}$.

23. ***A Continued Fraction*** is a fraction whose numerator is an integer and whose denominator is an integer plus a fraction whose denominator is also a fraction, and so on.

Thus, $\cfrac{2}{3+\cfrac{1}{4+\cfrac{2}{5+\cfrac{3}{7}}}}$ is a continued fraction, and is equal to the simple fraction $\frac{83}{134}$.

For convenience in printing such fractions, modern writers have adopted the following method of expressing them: $\frac{2}{3+}\frac{1}{4+}\frac{2}{5+}\frac{3}{7}$. The plus sign being placed between the denominators of the several fractions readily distinguishes the continued fraction from the expression $\frac{2}{3}+\frac{1}{4}+\frac{2}{5}+\frac{3}{7}$, where we wish to indicate that the several simple fractions are to be added together.

24. ***Discussion of Definitions.***

From the definition of a fraction, viz., that it is a number of the equal parts of a unit, it follows that $\frac{2}{3\frac{1}{2}}$, $\frac{2\frac{1}{2}}{4\frac{1}{5}}$, $\frac{1.2}{.3}$, and $\frac{7}{3}$ are not fractions.

When the idea of a fraction originated the above definition incorporated that idea. But this idea being new became the starting point of numerous other new but related ideas, and in the course of time the new ideas diverged so far from the original idea that the original description of the original idea became inadequate to accurately comprise all the new ideas which have thus sprung from it. In all such cases in mathematics it is the custom to enlarge the original description of an idea so as to include new ideas. Thus while *fraction* originally meant a number of the equal parts of a unit, its meaning has now been enlarged so as to include such expressions as $\frac{2}{3\frac{1}{2}}$, $\frac{3\frac{1}{4}}{5\frac{1}{2}}$, $\cfrac{2}{3+\cfrac{1}{4+\cfrac{1}{5}}}$, $\frac{\sqrt{2}}{\sqrt{5}}$, etc.

So too the idea of number has been enlarged until it comprises the ideas represented by such expressions as

$$\frac{3}{4},\ 5,\ \frac{\sqrt{2}}{\sqrt{3}},\ \sqrt{7},\ 3^{-2},\ \sqrt{-3},\ 2+\sqrt{-4},\ \text{etc.}$$

Is $\$\frac{5}{4}$ a fraction? Yes, for it is a number of fourths. From the primary idea of a fraction, viz., that it is a number of the equal parts of a unit, it, of course, follows that there cannot be a greater number of parts taken in a unit than the parts into which it has been divided, and, therefore, $\$\frac{5}{4}$ would not be a fraction.

How is $\$\frac{5}{4}$ read? $\frac{5}{4}$ of a dollar. For a valuable discussion of such questions the reader is referred to Brooks' *Philosophy of Arithmetic,* a book that ought to be read by every teacher of arithmetic.

I. REDUCTION OF FRACTIONS.

1. *Reduction of Fractions* is the process of changing their form without altering their value.

There are six cases of reduction, viz.,

1st. Integers or mixed numbers to fractions.	4th. Fractions to higher terms.
2d. Fractions to integers or to mixed numbers.	5th. Compound fractions to simple fractions.
3d. Fractions to lower terms.	6th. Complex fractions to simple fractions.

CASE I.

To reduce integers or mixed numbers to improper fractions.

I. Reduce $9\frac{3}{5}$ to an improper traction.

II. $\begin{cases} 1. & 1=\frac{5}{5}, =5\text{-fifths.} \\ 2. & 9=9 \text{ times } \frac{5}{5}=\frac{45}{5}, =9 \text{ times 5-fifths}=45\text{-fifths.} \\ 3. & 9\frac{3}{5}=9+\frac{3}{5}=\frac{45}{5}+\frac{3}{5}=\frac{48}{5}, =45\text{-fifths}+3\text{-fifths}=48\text{-fifths.} \end{cases}$

III. $\therefore$ $9\frac{3}{5}=\frac{48}{5}$, =48-fifths.

Remark 1.—Steps 1 and 2 show how to reduce an integer to an improper fraction.

Remark 2.—In the above solution, we have indicated two ways of writing a solution. Thus, for example, in step 2, we may say at once that 9=9 times 5-fifths=45-fifths. Then 45-fifths+3-fifths=48-fifths. In this way, pupils see a similarity between adding $\frac{45}{5}$ and $\frac{3}{5}$, and 45 apples and 3 apples.

CASE II.

To reduce an improper fraction to an integer or a mixed number.

I. Reduce $\frac{49}{11}$ to a mixed number.

II. $\begin{cases} 1. & \frac{11}{11}=11\text{-elevenths}=1. \\ 2. & \frac{49}{11}=49\text{-elevenths}=49\text{-elevenths}\div 11\text{-elevenths}=4\frac{5}{11}. \end{cases}$

III. $\therefore$ $\frac{49}{11}=4\frac{5}{11}$.

Explanation.—If we consider $\frac{49}{11}$ as 49-elevenths and not as 49 divided by eleven, we can not divide the numerator, 49, by the denominator, 11, in order to reduce it to a mixed number, because the denominator denominates, or names, the parts. From this point of view, 11 in $\frac{49}{11}$, bear the same relation to 49, that pecks do to 25, in 25 pecks. 25 pecks consist of two parts, viz., (1) a *unit* of measure, the peck; (2) 25, called the *numeric.* 25, the numeric, shows how many times the unit of measure is contained in the quantity measured with the peck. So too, 49 in $\frac{49}{11}$ is the numeric showing how many times the unit of measure, one-eleventh, is contained in the quantity measured with the one-eleventh.

By convention, however, a fraction may be considered as an unexecuted division, the numerator being considered the dividend and the denominator the divisor. From this point of view, we may simply divide the numerator by the denominator.

From the first point of view, 49-elevenths equals as many times 1 as 11-elevenths is contained in 49-elevenths which is $4\frac{5}{11}$ times 1, or $4\frac{5}{11}$.

In presenting this principle to classes, every member of a class should have a thorough comprehension of it, but it should not be made a hobby by the teacher so that the other method be excluded entirely.

CASE III.

To reduce a fraction to lower terms.

1. ***Reducing a fraction to lower terms*** is the process of reducing it to an equivalent fraction having a smaller numerator and denominator.

I. Reduce $\frac{9}{12}$ to its lowest terms.

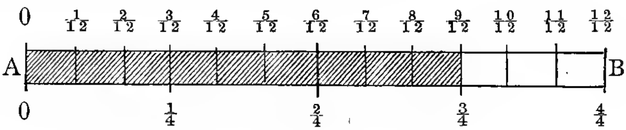


FIG. 1.

Let AB be a line of unit length, and let it be divided into 12 equal parts. Counting 9 of these parts, we get $\frac{9}{12}$ of AB, or $\frac{9}{12}$ of 1, or $\frac{9}{12}$. Now if we consider the line divided into 4 equal parts and count 3 of these we get $\frac{3}{4}$ of AB, or $\frac{3}{4}$ of 1, or $\frac{3}{4}$. It is thus seen that $\frac{9}{12}$ of AB$=\frac{3}{4}$ of AB.

Hence, $\frac{9}{12}=\frac{3}{4}$.

In the same way it may be shown that $\frac{6}{12}=\frac{1}{2}$; $\frac{8}{12}=\frac{2}{3}$; $\frac{2}{4}=\frac{1}{2}$, etc.

On comparing $\frac{9}{12}$ and $\frac{3}{4}$, it is seen that $\frac{3}{4}$ may be obtained from $\frac{9}{12}$ by dividing both numerator and denominator of $\frac{9}{12}$ by 3. In like manner, $\frac{2}{3}$ may be obtained from $\frac{8}{12}$ by dividing both numerator and denominator of $\frac{8}{12}$ by 4; and that $\frac{1}{2}$ may be obtained from $\frac{2}{4}$ by dividing both numerator and denominator of $\frac{2}{4}$ by 2.

Hence, in general, to reduce a fraction to lower terms, *divide both numerator and denominator by any common divisor,* and to reduce a fraction to its lowest terms, *divide both numerator and denominator by their greatest common divisor,* it being in its ***Lowest Terms*** when its numerator and denominator have no common factor.

I. Reduce $\frac{6}{8}$ to its lowest terms.

1. $\frac{6}{8}=\frac{6\div2}{8\div2}=\frac{3}{4}$.

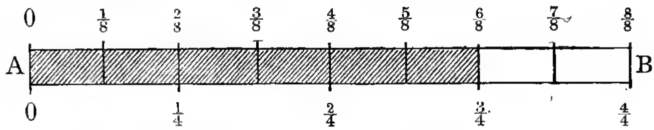


FIG. 2.

Let the line AB represent a unit; and let it be divided into 8 equal parts as indicated by the figure. The row of characters above the line numbers the parts beginning at A. The characters below the line number the parts of the line when it is considered as divided into 4 equal parts. It is thus seen that $\frac{6}{8}$ of AB$=\frac{3}{4}$ of AB, or $\frac{6}{8}$ of a unit$=\frac{3}{4}$ of the same unit. Now the fraction $\frac{3}{4}$ may be obtained from $\frac{6}{8}$ by dividing both numerator and denominator of $\frac{6}{8}$ by 2.

Hence, if both numerator and denominator of $\frac{6}{8}$ be divided by 2, the value of the fraction is not changed.

In the same way, we can show that $\frac{4}{8}=\frac{2}{4}=\frac{1}{2}$; $\frac{5}{10}=\frac{1}{2}$; $\frac{15}{20}=\frac{3}{4}$, etc., it being understood by these equalities that we mean, for example, $\frac{15}{20}$ of a unit$=\frac{3}{4}$ of the same unit. We thus arrive at the general

Principle: *If both numerator and denominator of a fraction be divided by the same number, the value of the fraction remains unchanged.*

CASE IV.

To reduce a fraction to higher terms.

I. Reduce $\frac{2}{3}$ to ninths.

1. $\frac{2}{3}=\frac{2\times3}{3\times3}=\frac{6}{9}$.

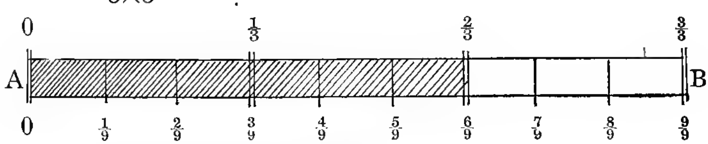


FIG. 3.

Let AB represent a unit and let it be divided into three equal parts of which two parts are taken. The shaded part of the line represents $\frac{2}{3}$ of the line AB. Now if we divide each one of these three equal parts into three equal parts, the whole line AB will be broken into 9 equal parts of which the shaded part contains 6 of the 9 parts.

Hence, $\frac{2}{3}$ of AB$=\frac{6}{9}$ of AB, or $\frac{2}{3}$ of a unit$=\frac{6}{9}$ of the same unit.

Now $\frac{6}{9}$ may be obtained from $\frac{2}{3}$, by multiplying both numerator and denominator of $\frac{2}{3}$ by 3. From the figure, we also see that $\frac{1}{3}=\frac{3}{9}$. In the same way, we can show that $\frac{3}{4}=\frac{9}{12}$; $\frac{2}{5}=\frac{6}{15}$; $\frac{1}{4}=\frac{3}{12}$, etc.

We thus arrive at the general

Principle.—*If both numerator and denominator of a fraction be multiplied by the same number, the value of the fraction remains unchanged.*

CASE V.

To reduce compound fractions to simple fractions.

I. What is $\frac{2}{3}$ of $\frac{4}{5}$?

II.
1. $\frac{1}{3}$ of $\frac{1}{5}=\frac{1}{3}$ of one of the 5 equal parts into which a unit has been divided, =, therefore, one of the 15 equal parts into which a unit is divided, that is, $\frac{1}{3}$ of $\frac{1}{5}$ $=\frac{1}{15}$.
2. $\frac{1}{3}$ of $\frac{4}{5}=4$ times $\frac{1}{3}$ of $\frac{1}{5}=4$ times $\frac{1}{15}=\frac{4}{15}$.
3. $\frac{2}{3}$ of $\frac{4}{5}=2$ times $\frac{1}{3}$ of $\frac{4}{5}=2$ times $\frac{4}{15}=\frac{8}{15}$.

III. $\therefore$ $\frac{2}{3}$ of $\frac{4}{5}=\frac{8}{15}$.

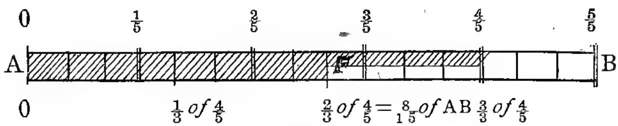


FIG. 4.

Explanation.—Let AB represent a unit, and let it be divided into 5 equal parts of which 4 are taken as represented by the shaded part of the line. To take $\frac{2}{3}$ of $\frac{4}{5}$ requires us to divide the shaded part of the line AB into three equal parts and take 2 of them. Thus $\frac{2}{3}$ of the shaded part of AB=AF. This division may be made by dividing each of the fifths of AB into three equal parts. This is equivalent to dividing AB into 15 equal parts. $\frac{4}{5}$, therefore, contains 12 of the 15 equal parts, and $\frac{2}{3}$ of $\frac{4}{5}$ or AF contains 8 of the 15 equal parts. Hence $\frac{2}{3}$ of $\frac{4}{5}=\frac{8}{15}$.

Now $\frac{8}{15}$ is obtained from $\frac{2}{3}$ and $\frac{4}{5}$, by multiplying their numerators together for the numerator of the result of the operation and their denominators together for the denominator of the result.

CASE VI.

To reduce complex fractions to simple fractions.

I. Reduce $\frac{\frac{2}{3}}{\frac{1}{3}}$ to a simple fraction.

This means that $\frac{2}{3}$ is to be divided by $\frac{1}{3}$. Now it is clear that $\frac{2}{3}$ is 2 times $\frac{1}{3}$ or that $\frac{2}{3}$ contains $\frac{1}{3}$, twice.

I. Reduce $\frac{\frac{2}{7}}{\frac{2}{9}}$ to a simple fraction.

This means that $\frac{2}{7}$ is to be divided by $\frac{2}{9}$. By division of fractions it is found that $\frac{2}{7}$ contains $\frac{2}{9}$, $1\frac{2}{7}$ times.

Remark.—It must not be forgotten that a complex fraction is not a fraction according to the primary idea of a fraction. It is impossible to interpret $\frac{\frac{2}{3}}{\frac{4}{5}}$ as a fraction, if we take as our definition of a fraction, that it is one or more of the equal parts into which a unit has been divided. But ideas of fractions are extended to include such expressions under the term, fraction.

Reduction of complex fractions to simple fractions properly comes under *Division of Fractions.* In the complex fraction above, 3 and 4 are called the *means* and 2 and 5 the *extremes.* By the principle of division of fractions, it will be seen that the reduction is effected by multiplying the means together for the denominator of the resulting fraction and the extremes together for the numerator. This operation may often be shortened by *cancellation.* Thus,

$$\frac{\frac{5}{6}}{\frac{10}{9}}=\frac{5}{6}\div\frac{10}{9}=\frac{\overset{1}{\cancel{5}}}{\underset{2}{\cancel{6}}}\times\frac{\overset{3}{\cancel{9}}}{\underset{2}{\cancel{10}}}=\frac{3}{4}.$$

Observe that in the process of cancellation, we say that 5 divides 5 and 10, going in 5 once and into 10 twice. The neglect of this often leads pupils to conclude that the result of such cancellations as the following

$$\frac{2\times5\times7\times11}{11\times2\times5\times7}\text{ is 0, instead of 1.}$$

It may not be out of place, to remark here, that with a certain class of teachers overcome with the desire to make everything in education easy, the complex fraction has fallen into disrepute. These teachers strongly advocate the omission of complex fractions in our arithmetics and omit the presentation of them in their classes. This is all done, it is claimed, for the benefit of the student. As a matter of fact it results in the student's eternal injury. The argument is that since such fractions are seldom encountered in business, it is a loss of time and energy to give them any attention in the schoolroom. Such a doctrine loses sight of the fundamental object of an education, viz., the liberation of the powers of the mind. Education should be directed to the development of the mental powers and not to specialization for business. As a result of such bad teaching and bad pedagogy, it is often a pitiable sight to see average students in Analytical Geometry or Calculus struggling with arithmetical operations and complex fractions, simply because they did not receive the proper training in Arithmetic. In the higher branches of mathematics complex fractions occur frequently and the student who takes up these subjects needs all his energy to develop the principles of these subjects and should not be forced, by necessity, to dissipate any of it in struggling with arithmetical principles.

II. ADDITION OF FRACTIONS.

I. Add $\frac{3}{4}$, $\frac{5}{6}$, $\frac{7}{8}$.

FIRST SOLUTION.

II.
1. L. C. D.=24-twenty-fourths.
2. 1=24-twenty-fourths.
3. $\frac{3}{4}=\frac{3}{4}$ times 24-twenty-fourths=18-twenty-fourths.
4. $\frac{5}{6}=\frac{5}{6}$ times 24-twenty-fourths=20-twenty-fourths.
5. $\frac{7}{8}=\frac{7}{8}$ times 24-twenty-fourths=21-twenty-fourths.
6. $\frac{3}{4}+\frac{5}{6}+\frac{7}{8}$=18-twenty-fourths + 20-twenty-fourths + 21-twenty-fourths =59-twenty-fourths=$2\frac{11}{24}$.

III. $\therefore \frac{3}{4}+\frac{5}{6}+\frac{7}{8}=2\frac{11}{24}$.

SECOND SOLUTION.

II.
1. L. C. D.=24.
2. $1=\frac{24}{24}$.
3. $\frac{3}{4}=\frac{3}{4}\times\frac{24}{24}=\frac{18}{24}$.
4. $\frac{5}{6}=\frac{5}{6}\times\frac{24}{24}=\frac{20}{24}$.
5. $\frac{7}{8}=\frac{7}{8}\times\frac{24}{24}=\frac{21}{24}$.
6. $\therefore \frac{3}{4}+\frac{5}{6}+\frac{7}{8}=\frac{18}{24}+\frac{20}{24}+\frac{21}{24}=\frac{59}{24}=2\frac{11}{24}$.

III. $\therefore \frac{3}{4}+\frac{5}{6}+\frac{7}{8}=2\frac{11}{24}$.

III. SUBTRACTION OF FRACTIONS.

I. Subtract $\frac{5}{8}$ from $\frac{9}{10}$.

1. L. C. D.=40.
2. $1=\frac{40}{40}$, =40-fortieths.
3. $\frac{5}{8}=\frac{5}{8}\times\frac{\overset{5}{40}}{40}=\frac{25}{40}$, $=\frac{5}{8}\times$40-fortieths=25-fortieths.

II. 4. $\frac{9}{10}=\frac{9}{10}\times\frac{40}{40}=\frac{36}{40}$, $=\frac{9}{10}\times$40-fortieths=36-fortieths.

5. $\frac{9}{10}-\frac{5}{8}=\frac{36}{40}-\frac{25}{40}=\frac{11}{40}$, =36-fortieths—25-fortieths=11-fortieths.

III. $\therefore$ $\frac{9}{10}-\frac{5}{8}=\frac{11}{40}$, =11-fortieths.

IV. MULTIPLICATION OF FRACTIONS.

To multiply a fraction by an integer.

I. Multiply $\frac{2}{5}$ by 2.

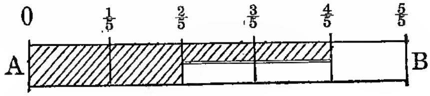


FIG. 5.

Let the line AB represent a unit and let it be divided into 5 equal parts.

Then AC=$\frac{2}{5}$ of AB and AD=$\frac{4}{5}$ of AB. But AD is twice AC. Hence, the fraction $\frac{4}{5}$ is twice the fraction $\frac{2}{5}$. Now $\frac{4}{5}$ may be obtained from $\frac{2}{5}$ by multiplying the numerator, 2, of $\frac{2}{5}$ by 2. In the same way we can show that $\frac{6}{7}$ is three times $\frac{2}{7}$; that $\frac{10}{3}$ is five times $\frac{2}{3}$, and so on. Hence, the

Principle.—*Multiplying the numerator of a fraction by any integer multiplies the fraction by that integer.*

I. Multiply $\frac{3}{8}$ by 4.

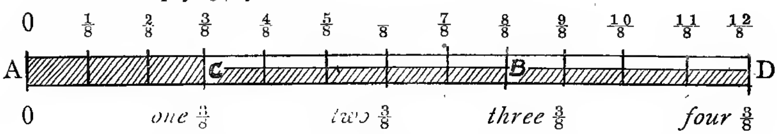


FIG. 6.

Let the line AB represent a unit and let AC represent $\frac{3}{8}$ of AB, or $\frac{3}{8}$ of a unit. Now if AC be used as a measure and applied 4 times along the line AD, beginning at A, the second application carries the extremity of AC to $\frac{6}{8}$, the third to $\frac{9}{8}$, and the fourth to $\frac{12}{8}$. Hence, $\frac{12}{8}$=4 times $\frac{3}{8}$. But we know by Case III. of Reduction of Fractions, that $\frac{12}{8}=\frac{3}{2}$. Hence, $\frac{3}{2}$=4 times $\frac{3}{8}$. But $\frac{3}{2}$ may be obtained from $\frac{3}{8}$ by dividing the denominator, 8, of $\frac{3}{8}$ by 4. In like manner, we can show that $\frac{5}{3}$=2 times $\frac{5}{6}$, that $\frac{2}{5}$=3 times $\frac{2}{15}$, that $\frac{5}{7}$=4 times $\frac{5}{28}$, and so on. Hence, the

Principle.—*Dividing the denominator of a fraction by any integer, multiplies the fraction by that integer.*

Remark.—To multiply a whole number by a fraction is the same as multiplying the fraction by the whole number, so far as the result is concerned. Thus 3 times $\frac{2}{5}=\frac{2}{5}$ times 3.

To multiply a fraction by a fraction.

I. Multiply $\frac{2}{3}$ by $\frac{4}{5}$.

To multiply $\frac{2}{3}$ by $\frac{4}{5}$ is to take $\frac{4}{5}$ of $\frac{2}{3}$.

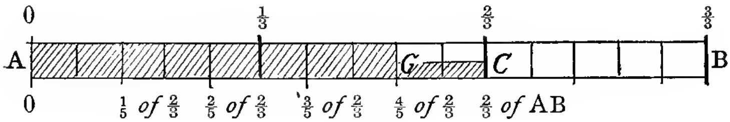


FIG. 7.

Let the line AB represent a unit and let AC represent $\frac{2}{3}$ of AB, or $\frac{2}{3}$ of a unit. Now if AB be divided into 15 equal parts, $\frac{2}{3}$ of AB, or AC, will contain 10 of the 15 equal parts. Hence, $\frac{2}{3}$ of AB$=\frac{10}{15}$ of AB. By this division, we may consider AC divided into 5 equal parts, and AG will represent $\frac{4}{5}$ of $\frac{2}{3}$. But AG also represents $\frac{8}{15}$.

Hence, $\frac{4}{5}$ of $\frac{2}{3}=\frac{8}{15}$. Now $\frac{8}{15}$ may be obtained from $\frac{2}{3}$ by multiplying the numerator by 4 and the denominator by 5, that is, $\frac{8}{15}$ is obtained by multiplying the numerators of $\frac{2}{3}$ and $\frac{4}{5}$ for the numerator and their denominators together for the denominator of $\frac{8}{15}$. Hence, to multiply a fraction by a fraction, we have the

Rule.—*Multiply the numerators together for the numerator of the product and the denominators together for its denominator.*

I. Multiply $\frac{7}{8}$ by $\frac{5}{6}$.

II.
1. $\frac{6}{6}$ times $\frac{7}{8}=\frac{7}{8}$.
2. $\frac{1}{6}$ times $\frac{7}{8}=\frac{1}{6}$ of $\frac{7}{8}=\frac{7}{48}$.
3. $\frac{5}{6}$ times $\frac{7}{8}=5$ times $\frac{7}{48}=\frac{35}{48}$.

$\therefore$ $\frac{7}{8}\times\frac{5}{6}=\frac{35}{48}$.

I. Multiply $\frac{6}{7}$ by 3.

II.
1. 1 times $\frac{6}{7}=\frac{6}{7}$.
2. 3 times $\frac{6}{7}=\frac{18}{7}$.

$\therefore$ $\frac{6}{7}\times3=\frac{18}{7}$.

I. Multiply 7 by $\frac{3}{4}$.

II.
1. $\frac{4}{4}$ times $7=7$.
2. $\frac{1}{4}$ times $7=\frac{1}{4}$ of $7=\frac{7}{4}$.
3. $\frac{3}{4}$ times $7=3$ times $\frac{7}{4}=\frac{21}{4}$.

$\therefore$ $7\times\frac{3}{4}=\frac{21}{4}$.

V. DIVISION OF FRACTIONS.

To divide a fraction by an integer.

I. Divide $\frac{6}{7}$ by 3.

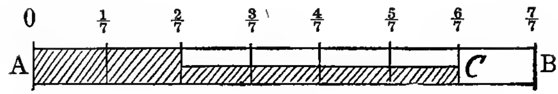


FIG. 8.

Let the line, AB, represent a unit and let AC represent $\frac{6}{7}$ of AB. Now $\frac{6}{7}$ is 3 times $\frac{2}{7}$. Hence, $\frac{2}{7}$ is $\frac{6}{7}\div3$. But $\frac{2}{7}$ is obtained from $\frac{6}{7}$ by dividing the numerator, 6, of $\frac{6}{7}$ by 3. In like manner, it may be shown that $\frac{4}{11}=\frac{8}{11}\div2$, that $\frac{5}{9}=\frac{25}{9}\div5$, and so on. Hence, the

Principle.—*Dividing the numerator of a fraction by any integer divides the fraction by that integer.*

I. Divide $\frac{3}{4}$ by 2.

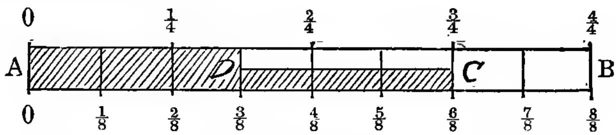


FIG. 9.

Let the line, AB, represent a unit and let AC represent $\frac{3}{4}$ of AB, or $\frac{3}{4}$ of a unit. AC=2 times AD, or $\frac{6}{8}=2\times\frac{3}{8}$, or $\frac{3}{4}=2\times\frac{3}{8}$. $\therefore \frac{3}{8}=\frac{3}{4}\div2$. But $\frac{3}{8}$ may be obtained from $\frac{3}{4}$, by multiplying the denominator of $\frac{3}{4}$ by 2. In like manner, we can show that $\frac{3}{10}=\frac{3}{5}\div2$, that $\frac{2}{9}=\frac{2}{3}\div3$, that $\frac{7}{15}=\frac{7}{3}\div5$, and so on. Hence, the

Principle.—*Multiplying the denominator of a fraction by any integer, divides the fraction by that integer.*

To divide an integer or a fraction by a fraction.

I. Divide 2 by $\frac{3}{4}$.

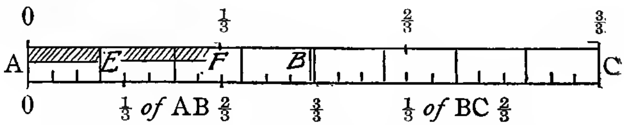


FIG. 10.

Let the line, AC, represent two units, AB and BC each representing a unit. Then AB contains 4-fourths and BC contains 4-fourths. Hence, AC contains 4-fourths+4-fourths or 8-fourths. That is, AC or 2 contains AE or $\frac{1}{4}$, 8 times, or 2 contains $\frac{1}{4}$ 8 times, or 2 divided by $\frac{1}{4}$ is 8. AF is $\frac{1}{3}$ of AC, that is, AF is $\frac{1}{3}$ of the number of times $\frac{1}{4}$ is contained in AC, that is $\frac{8}{3}$ times. Hence, $\frac{3}{4}$ is contained in 2, $\frac{8}{3}$ times. Now $\frac{8}{3}$ is obtained from $\frac{3}{4}$, by writing it $\frac{4}{3}$, that is by inverting it, and multiplying the numerator of $\frac{3}{4}$ thus written by 2. In like manner, we can show that $\frac{3}{4}$ is contained in 4, $\frac{16}{3}$ times, that $\frac{2}{5}$ is contained in 3, $\frac{15}{2}$ times, etc. Hence, the

Rule.—*To divide any number by a fraction, invert the divisor and multiply by the fraction thus inverted.*

Remark.—In presenting the subject of fractions to a primary class, great care should be taken to make the fundamental principles stand out prominently. Thus it should be made clear that in a unit there are 2-halves, 3-thirds, 4-fourths, 5-fifths, and so on. How many times is $\frac{1}{8}$ contained in 1, how many times is $\frac{1}{6}$, how many times $\frac{1}{7}$? How many eighths in 1, is the same question, so far as result is concerned, as how many times is $\frac{1}{8}$ contained in 1? By appropriate object-teaching these fundamental facts can be strongly and firmly impressed upon the mind of the pupil. Thus, take a foot-rule and apply it 3 times to a straight line drawn on the blackboard, how many feet have been measured? Take a 6-inch rule, or $\frac{1}{2}$-foot rule, and apply it to the same line, beginning and ending at the same points as in the previous measurement, how many times will you have to apply the $\frac{1}{2}$-foot rule? 6 times. Then you have measured 6 ($\frac{1}{2}$ ft.), or 3 feet. Hence, 6 times $\frac{1}{2}$ ft.=3 feet. In 3 feet there are 6 half-feet. Use a $\frac{1}{6}$ ft. rule, and measure the same segment of a line as before. How many times do you apply the unit of measure? 18 times. The result of the measurement or the length of the segment of the line measured is 18 ($\frac{1}{6}$ ft.) The unit of measure is $\frac{1}{6}$ ft. In 1 ft., how many ($\frac{1}{6}$ ft.)? Ans. 6. 6 what? 6 ($\frac{1}{6}$ ft.) By continuing such examples older students can be made to understand thoroughly the meaning of such limiting processes as $2 \div 0$. Thus measure a line segment 2 feet long with a foot-rule. It is applied twice. Measure the same line segment with an inch rule. It must be applied 24 times. Measure the same line segment with a $\frac{1}{10}$ in. rule. It must be applied 240 times. Measure the same line segment which is 24 inches long with a $\frac{1}{10000}$ in. rule. It must be applied 240,000 times. In this way, the pupil sees that if any quantity is divided or measured by a quantity which is taken as a unit and made very small, the number of times this measuring unit is contained in the quantity becomes very large. Of course, we can never take as a measure *no part* of something. But we can take as measuring unit a quantity as small as we please and thus its application to the quantity to be measured will be correspondingly large. The limit towards which such small measuring units tend is 0, and when we take one smaller than any assignable or expressible quantity, the number of times it is contained into the finite quantity to be measured is larger than any assignable or expressible quantity. A quantity larger than any assignable quantity is called an *infinite quantity* and is designated by the symbol ∞. Now we cannot divide $2 \div 0$, because we have no unit of measure. But such operations often arise as limiting cases in mathematics and must be properly interpreted. The real meaning is accurately expressed as follows: $2\div h$, as h approaches 0, approaches ∞. When h becomes inexpressibly small $2\div h$ becomes inexpressibly large, that is, ∞. The symbolic statement of the above expression is

$$\left.\frac{2}{h}\right]_{h\doteq 0} \doteq \infty .$$

In the same way $0\div 2$, may be made intelligible; also $2\div\infty$; $\infty\div 3$; $5\times\infty$; 0×6; and many other expressions commonly met with in Higher Mathematics, but of which the student has not the slightest comprehension until made intelligible by the teacher. The expression $0\div 0$ should also receive attention. When the student for the first time comes across this in Algebra, his first answer is 1. He reasons thus $2\div 2=1$; $5\div 5=1$; $7\div 7=1$; $10\div 10=1$, so then must $0\div 0=1$. While $0\div 0$ and 0×0 are impossible operations, the student might at this stage be told that these operations should be made to obey the same laws as quantitive symbols. Hence, $0\div 0=1$, 3, 7, $\frac{5}{3}$, 27, or any other number, since in the case of quantitive symbols, the dividend = the quotient $\times$ the divisor, so $0=0\times 1$, 0×3, or 0 times any other number.

VI. THE GREATEST COMMON DIVISOR OF FRACTIONS.

1. ***The Greatest Common Divisor*** of two or more fractions is the greatest fraction that will exactly divide each of them.

2. One fraction is divisible by another fraction when the numerator of the divisor is a factor of the numerator of the dividend, and the denominator of the divisor is a multiple of the denominator of the dividend.

Thus, $\frac{6}{7}$ is divisible by $\frac{3}{35}$; for $\frac{6}{7}=\frac{30}{35}$; $\frac{30}{35}\div\frac{3}{35}=10$.

3. The greatest common divisor of two or more fractions is that fraction whose numerator is the G. C. D. of the numerators and whose denominator is the L. C. M. of the denominators.

Hence, for finding the G. C. D. of two or more fractions, we have the following

Rule.—*Find the G. C. D. of the numerators of the fractions, and divide it by the L. C. M. of their denominators.*

Remark.—The fractions should be in their lowest terms before the rule is applied.

I. Find the G. C. D. of $\frac{3}{4}$, $\frac{6}{7}$, $\frac{9}{10}$.

1. 3=G. C. D. of the numerators, 3, 6, and 9.
2. 140=L. C. M. of the denominators, 4, 7, and 10.

II. 3. $\frac{3}{140}=3\div140$, the G. C. D. of the numerators divided by the L. C. M. of the denominators,=the G. C. D. of the fractions.

III. ∴ $\frac{3}{140}$ is the **G.** C. D. of $\frac{3}{4}$, $\frac{6}{7}$, and $\frac{9}{10}$.

I. A farmer sells $173\frac{1}{2}$ bushels of yellow corn, $478\frac{1}{8}$ bushels of white corn, $2{,}093\frac{3}{4}$ bushels of mixed corn: required the size of the largest sacks that can be used in shipping, so as to keep the corn from being mixed; also the number of sacks for each kind.

(R. H. A., page 93, problem 8.)

II.
1. $137\frac{1}{2}=\frac{275}{2}, =\frac{5\times5\times11}{2}$
2. $478\frac{1}{8}=\frac{3825}{8}, =\frac{3\times3\times5\times5\times17}{8}$.
3. $2093\frac{3}{4}=\frac{8375}{4}, =\frac{3\times3\times5\times5\times37}{4}$.
4. 25=G. C. D. of the numerators.
5. 8=L. C. M. of the denominators.
6. $3\frac{1}{8}=25\div8$,=the G. C. D. of the fractions.
7. $44=137\frac{1}{2}\div3\frac{1}{8}$,=the number of sacks required for the yellow corn.
8. $153=478\frac{1}{8}\div3\frac{1}{8}$,=the number of sacks required for the white corn.
9. $670=2{,}093\frac{3}{4}\div3\frac{1}{8}$,=the number of sacks required for the mixed corn.

III. ∴
1. The capacity of the largest sacks required is $3\frac{1}{8}$ bushels; and,
2. the number of sacks required for each kind of corn is 44 for the yellow, 153 for the white, and 670 for the mixed.

VII. LEAST COMMON MULTIPLE OF FRACTIONS.

1. ***The Least Common Multiple*** of two or more fractions is the least number, integer or fraction, that each of them will divide without a remainder.

2. A fraction is a *multiple* of a given fraction when its numerator is a multiple, and its denominator is a divisor, of the numerator and denominator, respectively, of the given fraction.

Thus, $\frac{8}{11}$ is a multiple of $\frac{2}{33}$; for the numerator, 8, is a multiple of the numerator, 2, and the denominator, 11, is a divisor of the denominator, 33.

3. A fraction is the *least common multiple* of two or more fractions when its numerator is the least common multiple of the given numerators, and its denominator is the greatest common divisor of the given denominators.

Hence, for finding the L. C. M. of two or more fractions, we have the following

Rule.— *Divide the L. C. M. of the numerators by the G. C. D. of the denominators.*

I. Find the least common multiple of $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, and $\frac{6}{7}$.

II.
1. 12=L. C. M. of the numerators, 2, 3, 4, 6.
2. 1=G. C. D. of the denominators, 3, 4, 5, 7.
3. 12=12÷1,=the L. C. M. of the fractions.

III. ∴ 12=the L. C. M. of $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, and $\frac{6}{7}$.

I. A can walk around an island in $14\frac{2}{7}$ hours; B in $9\frac{1}{11}$ hours; C, in $16\frac{2}{3}$ hours; and D, in 25 hours. If they start from the same point, and at the same time, how many hours after starting till they are all together again?

II.
1. $14\frac{2}{7}=\frac{100}{7},=\frac{2\times2\times5\times5}{7}$
2. $9\frac{1}{11}=\frac{100}{11},=\frac{2\times2\times5\times5}{11}$.
3. $16\frac{2}{3}=\frac{50}{3},=\frac{2\times5\times5}{3}$
4. $25=\frac{25}{1},=\frac{5\times5}{1}$
5. 100=L. C. M. of the numerators, 100, 100, 50, and 25.
6. 1=G. C. D. of the denominators, 7, 11, 3, and 1.
7. 100=100÷1,=the L. C. M. of the fractions.

III. ∴ In 100 hours after starting, the four men will be together again at the point of starting.

MISCELLANEOUS PROBLEMS.

I. Reduce $9\frac{7}{8}$ to an improper fraction.

Solution: II.
1. $9\frac{7}{8}=9+\frac{7}{8}$.
2. $1=\frac{8}{8}$=8-eighths.
3. $9=9\times\frac{8}{8}=\frac{72}{8}$=9×8-eighths=72-eighths.
4. $\frac{72}{8}+\frac{7}{8}=\frac{79}{8}$=79-eighths.

Conclusion: III. ∴ $9\frac{7}{8}=\frac{79}{8}$=79-eighths.

I. Reduce $\frac{5}{8}$ to 24ths.

II.
1. $\frac{8}{8}=\frac{24}{24}$, or 8-eighths=24-twenty-fourths.
2. $\frac{1}{8}=\frac{1}{8}$ of $\frac{\overset{3}{24}}{24}=\frac{3}{24}$, or 1-eighth=$\frac{1}{8}$ of 24-twenty-fourths=3-twenty-fourths.
3. $\frac{5}{8}$=5 times $\frac{3}{24}=\frac{15}{24}$, or 5-eighths=5 times 3-twenty-fourths=15-twenty-fourths.

III. ∴ $\frac{5}{8}=\frac{15}{24}$=15-twenty-fourths.

I. Reduce $\frac{5}{6}$ to 8ths.

II.
1. $\frac{6}{6}=\frac{8}{8}$, or 6-sixths=8-eighths.
2. $\frac{1}{6}=\frac{1}{6}$ of $\frac{8}{8}=\frac{\frac{8}{6}}{8}=\frac{\frac{4}{3}}{8}=\frac{1\frac{1}{3}}{8}$, or 1-sixth=$\frac{1}{6}$ of 8-eighths=$1\frac{1}{3}$-eighths.
3. $\frac{5}{6}=5$ times $\frac{1\frac{1}{3}}{8}=\frac{6\frac{2}{3}}{8}$, or 5-sixths=5 times $1\frac{1}{3}$-eighths=$6\frac{2}{3}$-eighths.

III. $\therefore \frac{5}{6}=\frac{6\frac{2}{3}}{8}=6\frac{2}{3}$-eighths.

I. Reduce $\frac{1}{2}$ to 3rds.

II.
1. $\frac{2}{2}=\frac{3}{3}=3$-thirds.
2. $\frac{1}{2}=\frac{1}{2}$ of $\frac{3}{3}=\frac{\frac{3}{2}}{3}=\frac{1\frac{1}{2}}{3}=1\frac{1}{2}$-thirds=$\frac{1}{2}$ of 3-thirds.

III. $\therefore \frac{1}{2}=\frac{1\frac{1}{2}}{3}=1\frac{1}{2}$-thirds.

Explanation.— In taking $\frac{1}{2}$ of $\frac{3}{3}$, we must divide the numerator by 2. The denominator must be left unchanged; for that is the denomination to which the given fraction is to be reduced.

I. Reduce $\frac{3}{5}$ to 11ths.

II.
1. $\frac{5}{5}=\frac{11}{11}=11$-elevenths.
2. $\frac{1}{5}=\frac{1}{5}$ of $\frac{11}{11}=\frac{\frac{11}{5}}{11}=\frac{2\frac{1}{5}}{11}=\frac{1}{5}$ of 11-elevenths $=2\frac{1}{5}$-elevenths.
3. $\frac{3}{5}=3$ times $\frac{2\frac{1}{5}}{11}=\frac{6\frac{3}{5}}{11}=3$ times $2\frac{1}{5}$-elevenths $=6\frac{3}{5}$-elevenths.

III. $\therefore \frac{3}{5}=\frac{6\frac{3}{5}}{11}=6\frac{3}{5}$-elevenths.

I. Reduce $\frac{29}{3}$ to a mixed number.

II.
1. $\frac{3}{3}=1$.
2. $\frac{29}{3}=29\div3=9\frac{2}{3}$.

Or,
1. 3-thirds=1.
2. 29-thirds=as many times 1 as 3-thirds is contained in 29-thirds, which is $9\frac{2}{3}$.

III. $\therefore \frac{29}{3}=9\frac{2}{3}$

I. Reduce $\frac{2}{3}$, $\frac{3}{4}$, $\frac{5}{6}$ to their L. C. Denominator.

II.
1. L. C. D.=12.
2. $\frac{3}{3}=\frac{12}{12}$.
3. $\frac{1}{3}=\frac{1}{3}$ of $\frac{12}{12}=\frac{4}{12}$.
4. $\frac{2}{3}=2\times\frac{4}{12}=\frac{8}{12}$.
5. $\frac{4}{4}=\frac{12}{12}$.
6. $\frac{1}{4}=\frac{1}{4}$ of $\frac{12}{12}=\frac{3}{12}$.
7. $\frac{3}{4}=3\times\frac{3}{12}=\frac{9}{12}$.
8. $\frac{6}{6}=\frac{12}{12}$.
9. $\frac{1}{6}=\frac{1}{6}$ of $\frac{12}{12}=\frac{2}{12}$.
10. $\frac{5}{6}=5\times\frac{2}{11}=\frac{10}{12}$.

III. $\therefore \frac{2}{3}, \frac{3}{4}, \frac{5}{6}=\frac{8}{12}, \frac{9}{12}, \frac{10}{12}$.

I. Reduce $\frac{1}{2}$, $\frac{4}{5}$, $\frac{5}{8}$ to their L. C. Denominator.

II.
1. L. C. D.=40.
2. $1=\frac{40}{40}$.
3. $\frac{1}{2}=\frac{1}{2}\times\frac{40}{40}=\frac{20}{40}$.
4. $\frac{4}{5}=\frac{4}{5}\times\frac{40}{40}=\frac{32}{40}$.
5. $\frac{5}{8}=\frac{5}{8}\times\frac{40}{40}=\frac{25}{40}$.

III. ∴ $\frac{1}{2}$, $\frac{4}{5}$, $\frac{5}{8}=\frac{20}{40}$, $\frac{32}{40}$, $\frac{25}{40}$.

I. Reduce $\frac{2}{3}$ to a fraction whose numerator is 12.

II.
1. $1=\frac{12}{12}$.
2. $\frac{2}{3}=\frac{2}{3}\times\frac{12}{\underset{6}{\cancel{12}}}=\frac{12}{18}$.

III. ∴ $\frac{2}{3}=\frac{12}{18}$.

Explanation.—Since the numerator of the required fraction is to be 12, we must not, in step 2, cancel 3 in the denominator of $\frac{2}{3}$ and 12 in the numerator of $\frac{12}{12}$. Were we to do so and then multiply the resulting factors in the numerators for the numerator and the resulting factors in the denominators for the denominator, we would get $\frac{4}{6}$ for a result. $\frac{4}{6}$ is the same as $\frac{2}{3}$, but $\frac{4}{6}$ is not the required fraction; for its numerator is not 12 as was required in the problem.

I. Reduce $\frac{1}{6}$ to fourths.

II.
1. $1=\frac{4}{4}$, =4-fourths.
2. $\frac{1}{6}=\frac{1}{\underset{3}{\cancel{6}}}\times\frac{\overset{2}{\cancel{4}}}{4}=\frac{\frac{2}{3}}{4}=\frac{1}{6}$ of 4-fourths, $=\frac{2}{3}$-fourth.

III. ∴ $\frac{1}{6}=\frac{2}{3}$-fourth.

I. Reduce $\frac{3}{4}$ to a fraction whose numerator is 15.

II.
1. $1=\frac{15}{15}$.
2. $\frac{3}{4}=\frac{\cancel{3}}{4}\times\frac{15}{\underset{5}{\cancel{15}}}=\frac{15}{20}$.

III. ∴ $\frac{3}{4}=\frac{15}{20}$.

I. Reduce $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$ to equivalent fractions having least common numerators.

II.
1. L. C. N.=12.
2. $1=\frac{12}{12}$.
3. $\frac{2}{3}=\frac{\cancel{2}}{3}\times\frac{12}{\underset{6}{\cancel{12}}}=\frac{12}{18}$.
4. $\frac{3}{4}=\frac{\cancel{3}}{4}\times\frac{12}{\underset{4}{\cancel{12}}}=\frac{12}{16}$.
5. $\frac{4}{5}=\frac{\cancel{4}}{5}\times\frac{12}{\underset{3}{\cancel{12}}}=\frac{12}{15}$.

III. ∴ $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}=\frac{12}{18}$, $\frac{12}{16}$, $\frac{12}{15}$.

I. Divide 2 by $\frac{3}{4}$.

II.
1. 1 contains $\frac{1}{4}$, 4 times; or $1\div\frac{1}{4}=4$.
2. 1 contains $\frac{3}{4}$, $\frac{1}{3}$ of 4 times$=\frac{4}{3}$ times; or $1\div\frac{3}{4}=\frac{4}{3}$.
3. 2 contains $\frac{3}{4}$, 2 times $\frac{4}{3}$ times$=\frac{8}{3}$ times; or $2\div\frac{3}{4}=\frac{8}{3}$.

III. ∴ 2 divided by $\frac{3}{4}=\frac{8}{3}$.

I. Divide $\frac{5}{8}$ by $\frac{3}{7}$.

II.
1. $\frac{5}{8} \div \frac{7}{7} = \frac{5}{8}$.
2. $\frac{5}{8} \div \frac{1}{7} = 7$ times $\frac{5}{8} = \frac{35}{8}$.
3. $\frac{5}{8} \div \frac{3}{7} = \frac{1}{3}$ of $\frac{35}{8} = \frac{35}{24} = 1\frac{11}{24}$.

III. $\therefore \frac{5}{8} \div \frac{3}{7} = 1\frac{11}{24}$.

Analysis to the last example:

II.
1. $\frac{1}{7}$ is contained in 1, or $\frac{8}{8}$, 7 times.
2. $\frac{3}{7}$ is contained in 1, or $\frac{8}{8}$, $\frac{1}{3}$ of 7 times $= \frac{7}{3}$ times.
3. $\frac{3}{7}$ is contained in $\frac{1}{8}$, $\frac{1}{8}$ of $\frac{7}{3}$ times $= \frac{7}{24}$ times.
4. $\frac{3}{7}$ is contained in $\frac{5}{8}$, 5 times $\frac{7}{24}$ times, or $\frac{35}{24}$ times.

Note.—By inverting the divisor, we find how many times it is contained in 1.

EXAMPLES.

1. One-fifth equals how many twelfths?

2. Reduce $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$, and $\frac{6}{7}$ to fractions having a common denominator.

3. Reduce $\frac{5}{8}$ to a fraction whose numerator is 13. *Ans.* $\dfrac{13}{20\frac{4}{5}}$.

4. Reduce $\frac{7}{8}$ to a fraction whose denominator is 11. *Ans.* $\dfrac{9\frac{5}{8}}{11}$.

5. Reduce $\frac{5}{6}$, $\frac{4}{7}$, $\frac{3}{8}$, to fractions having common numerators.

6. Add $\frac{1}{4}$, $\frac{7}{8}$, $\frac{11}{12}$, $\frac{5}{9}$, and $\frac{7}{11}$.

7. $\frac{3}{5}$ of $8\frac{3}{4} - \frac{2}{3}$ of $5 =$ what?

8. Multiply $\frac{3}{7}$ by $8\frac{3}{4}$.

9. Multiply $\frac{3}{4}$ of $9\frac{1}{4}$ of $\frac{8}{9}$ by $\frac{3}{4}$ of 17.

10. $\dfrac{12\frac{7}{8}}{15\frac{3}{4}} \div \dfrac{5\frac{2}{3}}{3\frac{2}{5}} =$ what?

11. $\dfrac{11\frac{7}{8} - 6\frac{3}{5}}{10\frac{9}{11} - 9\frac{11}{12}} \div \dfrac{7\frac{5}{6} - 5\frac{3}{4}}{8\frac{9}{10} - 9\frac{4}{5}} =$ what?

12. $\frac{25}{57} \times \frac{42}{65} \times \frac{19}{20} \div \frac{7}{13} =$ what? *Ans.* $\frac{1}{2}$.

13. $\begin{array}{c} \frac{1}{2} \\ \hline \frac{1}{3} \\ \hline \frac{1}{4} \\ \hline \frac{1}{5} \\ \hline \frac{1}{6} \\ \hline \frac{1}{7} \\ \hline \frac{1}{8} \\ \hline \frac{1}{9} \\ \hline \frac{1}{10} \end{array} =$ what? *Ans.* 907200.

14. $\begin{array}{c} 1\frac{1}{2} \\ \hline 2\frac{1}{3} \\ \hline 3\frac{1}{4} \\ \hline 4\frac{1}{5} \\ \hline 5\frac{1}{6} \\ \hline 6\frac{1}{7} \\ \hline 2\frac{1}{2} \\ \hline 3\frac{1}{3} \\ \hline 4\frac{1}{4} \\ \hline 5\frac{1}{5} \\ \hline 6\frac{1}{6} \end{array} =$ what? *Ans.* $\frac{2516}{9881}$.

15. $(2\frac{1}{2}\times2\frac{1}{3}+\frac{5}{9}$ of $\frac{7}{16})\times(\frac{8}{9})^{3}\div(7\frac{7}{9}-3\frac{1}{2}\times\frac{49}{50})=$what?

Ans. $\dfrac{400000}{407511}$.

16. $\left\{\dfrac{2\frac{7}{11}}{3\frac{17}{33}}+\dfrac{4\frac{1}{5}}{7\frac{14}{25}}-\dfrac{5\frac{11}{20}}{62\frac{7}{16}}\right\}\times4\dfrac{3\frac{11}{17}}{\frac{4\frac{3}{7}}{8}}\div\dfrac{\frac{1}{20}+\frac{28}{49}-\frac{1}{10}}{4\frac{3}{7}\times5\frac{1}{2}\div200\frac{10}{17}}=$what?

Ans. 3.

17. $2\div2\div2\div2\div2\div2\div2\div2\div\frac{1}{2}\div\frac{1}{2}\div\frac{1}{2}\div\frac{1}{2}\div\frac{1}{2}\div\frac{1}{2}=$what?

Ans. 1.

18. Reduce $\frac{4}{5}$ to thirds. *Ans.* $2\frac{2}{5}$ thirds.

19. What fraction is as much larger than $\frac{5}{6}$ as $\frac{4}{5}$ is less than $\frac{5}{6}$?

Ans. $\frac{13}{15}$.

20. What is the value in the 13th example if a heavy mark be drawn between $\frac{1}{6}$ and $\frac{1}{7}$. *Ans.* $1\frac{3}{4}$.

21. $1\dfrac{5\dfrac{3\frac{1\frac{2}{3}}{2\frac{1}{2}}}{4\frac{2\frac{1}{2}}{1\frac{2}{3}}}}{3\dfrac{2\frac{2\frac{1}{2}}{3\frac{1}{2}}}{3\frac{1\frac{2}{3}}{1\frac{1}{3}}}}=$what? *Ans.* $2\frac{724}{1299}$.

22. Subtract $\frac{1}{2}$ of $\dfrac{4\frac{1}{11}}{17\frac{1}{7}}$ from $\frac{3}{10}$ of $\dfrac{19\frac{3}{7}}{5\frac{2}{3}}$. *Ans.* $\frac{5601}{6160}$.

23. What is the relation of 11 to 3?

Solution:
1. $1=\frac{1}{3}$ of 3.
2. $11=11$ times $\frac{1}{3}$ of $3=\frac{11}{3}$ of $3=3\frac{2}{3}$ times 3.

Conclusion: ∴ 11 is $3\frac{2}{3}$ times 3.

24. What is the relation of 19 to 5? *Ans.* $3\frac{4}{5}$.

25. What is the relation of $\frac{6}{11}$ to 24? *Ans.* $\frac{1}{44}$

26. What part of 3 is 2?

Solution:
1. $1=\frac{1}{3}$ of 3.
2. $2=2$ times $\frac{1}{3}$ of $3=\frac{2}{3}$ of 3.

Conclusion: ∴ 2 is $\frac{2}{3}$ of 3.

27. What part of 6 is 7? *Ans.* $\frac{7}{6}$.

28. What part of 3 is $\frac{1}{6}$? *Ans.* $\frac{1}{18}$.

29. What part of $\frac{1}{6}$ is 3? *Ans.* $\frac{18}{1}$.

30. What part of $\frac{3}{4}$ of $\frac{4}{5}$ is $\frac{3}{5}$ of $\frac{7}{12}$?

Solution:
1. $\frac{3}{4}$ of $\frac{4}{5}=\frac{3}{5}$.
2. $\frac{3}{5}$ of $\frac{7}{12}=\frac{7}{20}$.
3. 1 is $\frac{5}{3}$ of $\frac{3}{5}$.
4. $\frac{7}{20}$ is $\frac{7}{20}$ times $\frac{5}{3}$ of $\frac{3}{5}=\frac{7}{12}$ of $\frac{3}{5}$.

Conclusion: ∴ $\frac{3}{5}$ of $\frac{7}{12}$, or $\frac{7}{20}$, is $\frac{7}{12}$ of $\frac{3}{4}$ of $\frac{4}{5}$.

31. $\cfrac{1}{4+\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{5}}}}$=what? *Ans.* $\frac{38}{163}$.

[*Note.*—This is a continued fraction.]

32. Find the number of which 75 is $\frac{5}{7}$.

33. Find the number of which 180 is $\frac{3}{4}$.

34. $\frac{12}{15}$ is $\frac{3}{4}$ of what number?

Solution:
1. $\frac{3}{4}$ of some number $=\frac{12}{15}$.
2. $\frac{1}{4}$ of that number $=\frac{1}{3}$ of $\frac{12}{15}=\frac{4}{15}$.
3. $\frac{4}{4}$ of that number, or the number required, $=4$ times $\frac{4}{15}=\frac{16}{15}$.

Conclusion: ∴ $\frac{12}{15}$ is $\frac{3}{4}$ of $\frac{16}{15}$.

35. 27 is .3 of what number? *Ans.* 90.

36. $\frac{2}{3}$ of $\frac{15}{16}$ is $\frac{1}{2}$ of $\frac{2}{3}$ of what number?

Solution:
1. $\frac{2}{3}$ of $\frac{15}{16}=\frac{5}{8}$.
2. $\frac{1}{2}$ of $\frac{2}{3}=\frac{1}{3}$.
3. ∴ $\frac{5}{8}=\frac{1}{3}$ of some number, or
4. $\frac{1}{3}$ of some number $=\frac{5}{8}$.
5. $\frac{3}{3}$ of that number, or that number, $=3\times\frac{5}{8}=\frac{15}{8}$.

Conclusion: ∴ $\frac{2}{3}$ of $\frac{15}{16}$ is $\frac{1}{2}$ of $\frac{2}{3}$ of $\frac{15}{8}$.

37. A watch cost $30, and this is $\frac{5}{6}$ of $\frac{6}{7}$ of the cost of the watch and chain together. What did the chain cost. *Ans.* $12

38. A lost $\frac{4}{5}$ of his money and then found $\frac{3}{4}$ as much as he lost and then had $120; how much money had he at first?

39. A sum of money diminished by $\frac{3}{7}$ of itself and $6 equals $12; what is the sum? *Ans.* $31½.

40. If $\frac{5}{12}$ of a ton of hay is worth $8½, how much is 10 tons worth? *Ans.* $204.

41. What number is that $\frac{13}{18}$ of which exceeds $\frac{5}{24}$ of it by 111? *Ans.* 216.

42. What part of $2\frac{1}{4}$ feet is $3\frac{1}{4}$ inches? *Ans.* $\frac{13}{108}$

43. A has $2400; $\frac{5}{8}$ of his money plus $500 is $\frac{5}{4}$ of B's; what sum has B? *Ans.* $1600.

44. What fraction of $\left(\frac{7}{4-\frac{5}{6}}-\frac{5}{6-\frac{3}{8}}\right)\div\left(\frac{4}{7-\frac{4}{7}}+\frac{2}{4-\frac{2}{5}}\right)$ is $\left(14-\frac{1}{\frac{1}{2}-\frac{6}{31}}\right)\div\left(\frac{1}{\frac{1}{2}-\frac{27}{59}}-13\right)$? *Ans.* $\frac{235}{226}$.

45. A pole stands $\frac{2}{7}$ in the mud, $\frac{7}{16}$ in the water, and the remainder, $12\frac{2}{5}$ feet, above water. Find the length of the pole? *Ans.* $44\frac{4}{5}$ feet.

46. If 48 is $\frac{4}{7}$ of some number, what is $\frac{3}{4}$ of the same number? *Ans.* 63.

47. A can do a certain piece of work in 8 days, and B can do the same work in 6 days. In what time can both together do the work? *Ans.* $3\frac{3}{7}$ days.

48. The lesser of two numbers is $\frac{54\frac{3}{5}}{\frac{1}{5}\text{ of }8\frac{2}{3}}$, and their difference is $\frac{1\frac{5}{9}}{\frac{16}{9}}$. What is the greater number? *Ans.* $\frac{259}{8}$.

49. What number multiplied by $\frac{2}{9}$ of $\frac{5}{8}\times3\frac{2}{7}$ will produce $\frac{23}{84}$? *Ans.* $\frac{3}{5}$.

50. What number divided by $1\frac{3}{8}$ will give a quotient of $9\frac{1}{8}$? *Ans.* $\frac{803}{64}$.

51. A post stands $\frac{1}{6}$ in mud, $\frac{1}{4}$ in water, and 21 feet above the water? What is its length? *Ans.* 36 feet.

52. A can do a piece of work in 8 days, A and B can do it in 5 days, and B and C in 6 days. In what time can A, B, and C do the work? *Ans.* $3\frac{3}{19}$ days.

53. If $\frac{3}{4}$ of 6 bushels of wheat cost $\$4\frac{1}{2}$, how much will $\frac{4}{5}$ of 1 bushel cost? *Ans.* 80 cents.

55. What number diminished by the difference between $\frac{2}{7}$ and $\frac{7}{9}$ of itself leaves 1152? *Ans.* 2268.

56. If a piece of gold is $\frac{5}{8}$ pure, how many carats fine is it? *Ans.* 15 carats.

57. The density of the earth is $5\frac{2}{3}$ times that of water, and the sun is $\frac{1}{4}$ as dense as the earth. How many times denser than water is the sun? *Ans.* $\frac{17}{12}$.

CHAPTER X.

CIRCULATING DECIMALS.

I. Change $.\dot{6}\dot{3}$ to a common fraction.

II.
1. $.\dot{6}\dot{3}=.636363+$ etc., *ad infinitum.*
2. $.636363+$etc.,$=.63+.0063+.000063+$etc., ***ad infinitum.***
3. This is a geometrical, infinite, decreasing series whose first term is .63 and ratio $.63\div.0063=\frac{1}{100}$. The sum of such a series is $\frac{a}{1-r}=.63\div(1-\frac{1}{100})=\frac{63}{99}=\frac{7}{11}$.

III. $\therefore .\dot{6}\dot{3}=\frac{7}{11}$.

I. Reduce $\dot{1}.00\dot{1}$ to a common fraction.

II.
1. $\dot{1}.00\dot{1}=1.0011001100110011001\text{1}+$etc., *ad infinitum.*
2. $\dot{1}.00\dot{1}=1.\dot{0}01\dot{1}$.
3. $1.\dot{0}01\dot{1}=1+.0011+00000011+000000000011+$etc., ***ad*** *infinitum.*
4. $.\dot{0}01\dot{1}=$first term.
5. $\frac{1}{10000}=.0011\div00000011=$ratio.
6. $\therefore$ Sum$=\frac{a}{1-r}=.0011\div(1-\frac{1}{10000})=.0011\div\frac{9999}{10000}=\frac{11}{9999}$ $=\frac{1}{909}$.
7. $\therefore 1.\dot{0}01\dot{1}=1+\frac{1}{909}=1\frac{1}{909}$.

(*Ray's H. A., p. 120, ex. 8.*)

III. $\therefore \dot{1}.00\dot{1}=1\frac{1}{909}$.

Remark.—Since the denominator of the ratio is always ten or some power of ten, the numerator of the fraction resulting from subtracting the ratio from 1, will have as many 9's in it as there are ciphers in the denominator of the ratio. By dividing the first term by this fraction, its numerator becomes the denominator of the fraction required. Hence, a *circulate* may be reduced to a common fraction by writing for the denominator of the repetend as many 9's as there are figures in the repetend. Thus, $.6\dot{3}=.6\frac{3}{9}=.6\frac{1}{3}=\frac{6\frac{1}{3}}{10}=\frac{\frac{19}{3}}{10}=\frac{19}{30}$.

I. Reduce $.034\dot{6}3\dot{9}$ to a common fraction.

1. $.034\dot{6}3\dot{9}=.034\frac{639}{999}=\frac{34\frac{639}{999}}{1000}=\frac{34\times999+639}{\frac{999}{1000}}=\frac{34\times999+639}{1000\times999}$, $=\frac{34\times(1000-1)+639}{999000}=\frac{34000-34+639}{999000}=\frac{34000+639-34}{999000}=$ $\frac{34639-34}{999000}=\frac{34605}{999000}=\frac{6921}{199800}=\frac{2307}{66600}=\frac{769}{22200}$.

In case the circulate is mixed, we have the following rule:

1. *For the numerator, subtract that part which precedes the repetend from the whole expression, both quantities being considered as units.*

2. *For the denominator, write as many 9's as there are figures in the repetend, and annex as many ciphers as there are decimal figures before each repetend.*

I. ADDITION OF CIRCULATES.

I. Add $5.0\dot{7}7\dot{0}$, $.\dot{2}\dot{4}$, and $7.\dot{1}2494\dot{3}$.

II.
1. $5.0\dot{7}7\dot{0} = 5.0\dot{7}7\dot{0} = 5.0\dot{7}7\dot{0}77\dot{0}7$ etc.
2. $.\dot{2}\dot{4} = .2\dot{4}\dot{2} = .2\dot{4}242\dot{4}24$ etc.
3. $7.\dot{1}2494\dot{3} = 7.1\dot{2}49431\dot{1} = 7.1\dot{2}49431\dot{2}$ etc.

III. ∴ Sum $=12.4\dot{4}$ $12.4\dot{4}4444\dot{4}$ etc. $=12.4\dot{4}$.

Explanation.—The first thing, in the addition and subtraction of circulates, is to make the circulates *co-originous*, *i. e.*, to make them begin at the same decimal place. That is, if one begins at (say) hundredths, make them all begin at hundredths, providing that each circulate has hundredths repeated. It is best to make them all begin with the circulate whose first repeated figure is farthest from the decimal point, though any order after that may be taken. In the above example we have made them all begin at hundredths. After having made them all begin at hundredths, the next step is to make them conterminous, *i. e.*, to make them all end at the same place. To do this, we find the L. C. M. of the numbers of figures repeated in each circulate, then divide the L. C. M. by the number of figures repeated in each circulate for the number of times the figures as a group must be repeated. Thus, the number of figures in the first repetend is 3; in the second, 2; and in the third, 6.

The L. C. M. of 3, 2, and 6 is 6. $6\div3=2$. ∴ $\dot{7}7\dot{0}$ must be repeated twice. $6\div2=3$. ∴ $\dot{4}\dot{2}$ must be repeated three times. $6\div6=1$. ∴ $\dot{2}4943\dot{1}$ must be taken once.

I. Add $.94\dot{6}$, $.2\dot{4}\dot{8}$, $5.0\dot{7}7\dot{0}$, $3.\dot{4}88\dot{4}$, and $7.\dot{1}2494\dot{3}$.

II.
1. $.94\dot{6} = .94\dot{6} = .94666666666666\dot{6}$ etc.
2. $.2\dot{4}\dot{8} = .24\dot{8}\dot{4} = .24\dot{8}48484848484\dot{8}8$ etc.
3. $5.0\dot{7}7\dot{0} = 5.07\dot{7}0\dot{7} = 5.07\dot{7}07707707707\dot{7}$ etc.
4. $3.\dot{4}88\dot{4} = 3.4884\dot{4}8 = 3.48\dot{8}448844884\dot{4}88$ etc.
5. $7.\dot{1}2494\dot{3} = 7.1\dot{2}49431\dot{2} = 7.1\dot{2}494312494312\dot{4}$ etc.
6. Sum $=16.8856205620562\dot{0}+$,

$=16.88\dot{5}62\dot{0}$.

III. ∴ Sum $=16.88\dot{5}62\dot{0}$.

II. SUBTRACTION OF CIRCULATES

I. Subtract $\dot{1}90.47\dot{6}$ from $199.642857\dot{1}$

II. 1. $199.\dot{6}42857\dot{1} = 199.642857\dot{1}4$

2. $\dot{1}90.47\dot{6} = 190.\dot{4}76190\dot{4}7$

3. Difference $= 9.1\dot{6}6666\dot{6} = 9.1\dot{6}$.

III. ∴ Difference $= 9.1\dot{6}$.

I. Subtract $13.\dot{6}3\dot{7}$ from $10\dot{4}.\dot{1}$.

II. 1. $10\dot{4}.\dot{1} = 104.\dot{1}\dot{4} = 104.\dot{1}41414\dot{1}$ etc.

2. $13.\dot{6}3\dot{7} = 13.\dot{6}3\dot{7} = 13.\dot{6}3763\dot{7}6$ etc.

3. Difference $= 90.\dot{5}0377\dot{6}$

III. ∴ Difference $= 90.\dot{5}0377\dot{6}$.

III. MULTIPLICATION OF CIRCULATES.

I. Multiply $.0\dot{7}06\dot{7}$ by $.9\dot{4}3\dot{2}$.

$.0\dot{7}06\dot{7} = .0\dot{7}06\dot{7}7$

$.9\dot{4}3\dot{2} = .9\frac{16}{37}$

$.06\dot{3}60\dot{9}$

$.00\dot{3}05\dot{6}$

$.06\dot{6}66\dot{5}$ = product.

Multiply by the fraction thus:

$.0\dot{7}06\dot{7}7$

16

$.\dot{4}240\dot{6} = .\dot{4}2406\dot{2}$

$.\dot{7}06\dot{7} = .\dot{7}0677\dot{0}$

37)$1.1\dot{3}083$($\dot{3}05\dot{6}$ =

1 11

208

185

233

222

11

$00\dot{3}05\dot{6}$, because the fraction is $.0\frac{16}{37}$

I. Multiply $1.\dot{2}5678\dot{4}$ by $6.420\dot{8}\dot{1}$.

$1.\dot{2}5678\dot{4} = 1.\dot{2}5678\dot{4}2$

$6.420\dot{8}\dot{1} = 6.420\frac{9}{11}$

$.02\dot{5}1356\dot{8} = .025\dot{1}35685^{1}$

$.5\dot{0}2713\dot{7} = .502\dot{7}13702^{7}$

$7.\dot{5}4070\dot{5} = 7.540\dot{7}05540^{7}$

$.001\dot{0}2827\dot{0} = .001\dot{0}2827\dot{0}^{0}$

$8.069\dot{5}8319\dot{8}$

Multiply by the fraction thus:

$1.\dot{2}5678\dot{4}^{2}$

$.000\frac{9}{11}$

11)$.011\dot{3}1105\dot{7}$

$.001\dot{0}2827\dot{0}$

Remark.—In multiplying by any number, begin sufficiently far beyond the last figure of the repetend, so that if there is any to carry it may be added to the repetends of the partial products, making them complete. Thus in the above example, when multiplying by 4, we begin at 5, the second decimal place beyond 4, the last figure of the repetend; and so when we multiply $\dot{4}$ by 4, the first figure of the repetend in the partial product is $\dot{7}$.

IV. DIVISION OF CIRCULATES.

Rule.—*Change the terms to common fractions; then divide as in division of fractions, and reduce the quotient to a repetend.*

I. Divide $.\dot{7}\dot{5}$ by $.\dot{1}$

II.
1. $.\dot{7}\dot{5}=\frac{75}{99}=\frac{25}{33}$.
2. $.\dot{1}=\frac{1}{9}$.
3. $\frac{25}{33}\div\frac{1}{9}=\frac{25}{33}\times 9=\frac{25}{11}=6.8181$ etc. $=6.\dot{8}\dot{1}$.

III. $\therefore .\dot{7}\dot{5}\div .\dot{1}=6.\dot{8}\dot{1}$.

EXAMPLES.

1. Add $.\dot{8}\dot{7}$, .8, and $\dot{8}7\dot{6}$. *Ans.* $2.\dot{6}4455\dot{3}$.
2. Add $.\dot{3}$, .45, $.\dot{4}\dot{5}$, $.\dot{3}5\dot{1}$, .6468, $.646\dot{8}$, $.6\dot{4}6\dot{8}$, and $6\dot{4}6\dot{8}$. *Ans.* $4.1766\dot{3}4561\dot{8}$.
3. Add $2\dot{7}.5\dot{6}$, 5.632, $6.\dot{7}$, $16.3\dot{5}\dot{6}$, $.7\dot{1}$, and $6.\dot{1}23\dot{4}$. *Ans.* $63.1\dot{6}9067086888\dot{8}$.
4. Add $5.\dot{1}634\dot{5}$, $8.\dot{6}38\dot{1}$, and $3.\dot{7}\dot{5}$. *Ans.* $17.\dot{5}59191208473740903\dot{2}$.
5. From $31\dot{5}.8\dot{7}$ take $78.\dot{0}37\dot{8}$. *Ans.* $237.\dot{8}3807209549\dot{7}$.
6. From $16.13\dot{4}\dot{7}$ take $11.08\dot{8}\dot{4}$. *Ans.* $5.04\dot{6}\dot{2}$.
7. 18 is .6 of what number? *Ans.* 27.
8. From $\frac{9}{17}$ take $\frac{6}{17}$. *Ans.* $.\dot{1}76470588235294\dot{1}$.
9. From $5.\dot{1}234\dot{5}$ take $2.3\dot{5}2345\dot{6}$. *Ans.* $2.7\dot{7}1\dot{1}05582166692777798888859999\dot{4}$.
10. Multiply $87.32\dot{5}8\dot{6}$ by 4.37. *Ans.* $381.6140\dot{3}3\dot{8}$.
11. Multiply 382.347 by $.0\dot{3}$. *Ans.* $13.5\dot{1}6953\dot{3}$.
12. Multiply $.\dot{9}62566844919786\dot{0}$ by $.\dot{7}\dot{5}$. *Ans.* $.\dot{7}\dot{2}$.
13. Divide $234.\dot{6}$ by $.\dot{7}$. *Ans.* $701.\dot{7}1428\dot{5}$.
14. Divide $13.5\dot{1}6953\dot{3}$ by $3.1\dot{4}\dot{5}$. *Ans.* $4.\dot{2}9\dot{7}$.

15. Divide $2.37\dot{0}$ by $4.\dot{9}2307\dot{6}$. *Ans.* $\dot{4}8\dot{1}$.

16. Divide $.\dot{3}\dot{6}$ by $.2\dot{5}$. *Ans.* $1.\dot{4}22924901185770750988\dot{1}$.

17. Divide $.\dot{7}\dot{2}$ by $.7\dot{5}$. *Ans.* $.\dot{9}62566844919786\dot{0}$

18. $54.0\dot{6}7813\dot{2} \div 8.\dot{5}9\dot{4}=$what? *Ans.* $6.2\dot{9}\dot{0}$.

19. $4.9\dot{5}\dot{6} \div .75=$what? *Ans.* $6.6\dot{0}8754\dot{2}$.

20. $7.\dot{7}1428\dot{5} \div .\dot{9}5238\dot{0}=$what? *Ans.* 8.1.

CHAPTER XI.

PERCENTAGE AND ITS VARIOUS APPLICATIONS.

1. ***Percentage*** is a method of computation in which 100 is taken as the basis of comparison.

2. ***Per cent.*** is an abbreviation from the Latin, *per centum*, *per*, by, and *centum*, a hundred.

3. ***The Terms*** used in percentage are the *Base*, the *Rate*, the *Percentage*, and the *Amount* or *Difference.*

4. ***The Base*** is the number on which the percentage is computed.

5. ***The Rate*** is the number of hundredths of the base which is to be taken.

6. ***The Percentage*** is the result obtained by taking a certain per cent. of the base.

7. ***The Amount*** or ***Difference*** is the sum or difference of the base and percentage.

8. ***The sign,*** %, is used instead of the words "*per cent.*" and "one-hundredths," following the number expressing the rate. Thus, for example, for 5 *per cent.*, or 5 one-hundredths, we write 5%.

Hence, we have the following identical expressions:

5 per cent.=5 one-hundredths$=\frac{5}{100}=.05=5\%$. In each of these expressions the *fractional unit* is $\frac{1}{100}$. The fundamental principle of percentage is that our computation shall be made on the basis of hundredths. That this principle be not violated, the denominator of the fraction must always be 100. Thus, since $\frac{10}{100}=\frac{1}{10}$, we can take $\frac{1}{10}$ of a number instead of $\frac{10}{100}$ of it and get the same result; but using fractions whose denominators are numbers other than 100 to express the rate is not the method of percentage, but merely the method of common fractions. However, in teaching percentage the method of common

fractions should also be used, as this method, because of its brevity, is more often used in practice.

As an illustration, find 5% of $600.

II. { 1. 100 one-hundredths, or $\frac{100}{100}$, or 1.00, or 100%=$600,
2. 1 one-hundredth, or $\frac{1}{100}$, or .01, or 1%,=$\frac{1}{100}$ of $600=$6,
3. 5 one-hundredths, or $\frac{5}{100}$, or .05, or 5%=5 times $6=$30.

III. ∴ 5% of $600=$30.

Explanation.—It must be borne in mind that what we mean by saying 100%=$600 is that $\frac{100}{100}$ of $600 is $600, which is certainly a true statement. But for brevity we simply say 100%=$600, with the understanding that we mean 100% of $600=$600, 1% of $600=$6, etc.

I. What is 8% of 150 yards?

FIRST SOLUTION.

II. { 1. $\frac{100}{100}$=150 yards.
2. $\frac{1}{100}=\frac{1}{100}$ of $\frac{100}{100}=\frac{1}{100}$ of 150 yards=1.5 yards.
3. $\frac{8}{100}$=8 times 1.5 yards=12 yards.

III. ∴ 8% of 150 yards=12 yards.

Remark.—This solution is by the method of percentage purely.

SECOND SOLUTION.

II. { 1. 100%=150 yards.
2. 1%=$\frac{1}{100}$ of 100%=$\frac{1}{100}$ of 150 yards=1.5 yards.
3. 8%=8 times 1%=8 times 1.5 yards=12 yards.

III. ∴ 8% of 150 yards=12 yards.

THIRD SOLUTION.

Briefly, by fractions:

8% of 150 yards=$\frac{8}{100}$ of 150 yards=12 yards.

CASE I.

Given { the base and the rate per cent. } to find the percentage.

Formula.—$B\times R=P$, where B is the base, R the rate, and P the percentage.

I. What is 8% of \$500?

II. 1. 100%=\$500,
2. 1%=$\frac{1}{100}$ of \$500=\$5, and
3. 8%=8 times \$5=\$40.

III. ∴ 8% of \$500=\$40.

I. What is $\frac{3}{4}$% of 800 men?

II. 1. 100%=800 men.
2. 1%=$\frac{1}{100}$ of 800 men=8 men, and
3. $\frac{3}{4}$%=$\frac{3}{4}$ times 8 men=6 men.

III. ∴ $\frac{3}{4}$% of 800 men=6 men.

I. What is 10% of 20% of \$13.50?

II. (1.) 1. 100%=\$13.50.
2. 1%=$\frac{1}{100}$ of \$13.50=\$.135, and
3. 20%=20 times \$.135=\$2.70.
(2.) 100%=\$2.70.
(3.) 1%=$\frac{1}{100}$ of \$2.70=\$.027, and
(4.) 10%=10 times \$.027=\$.27=27 cents.

III. ∴ 10% of 20% of \$13.50=27 cents.

I. A. had \$1200; he gave 30% to a son, 20% of the remainder to his daughter, and so divided the rest among four brothers that each after the first had \$12 less than the preceding. How much did the last receive?

II. (1.) 1. 100%=\$1200,
2. 1%=$\frac{1}{100}$ of \$1200=\$12, and
3. 30%=30 times \$12=\$360=son's share.
4. \$1200—\$360=\$840=remainder.
(2.) 1. 100%=\$840,
2. 1%=$\frac{1}{100}$ of \$840=\$8.40, and
3. 20%=20 times \$8.40=\$168=daughter's share.
4. \$840—\$168=\$672=amount divided among four brothers.
(3.) 100%=fourth brother's share;
(4.) 100%+\$12=third brother's share.
(5.) 100%+\$24=second brother's share, and
(6.) 100%+\$36=first brother's share.
(7.) 100%+(100% + \$12)+(100% + \$24)+ (100%+ \$36)=400%+\$72=am't the four brothers rec'd.
(8.) \$672=amount the four brothers received.
(9.) ∴ 400% +\$72=\$672.
(10.) 400%=\$672—\$72=\$600.
(11.) 1%=$\frac{1}{400}$ of \$600=\$1.50.
(12.) 100%=100 times \$1.50=\$150=fourth brother's share.

III. ∴ The last received \$150. (*R. H. A., p. 191, prob. 25.*)

I. What number increased by **20% of 3.5, diminished by** $12\frac{1}{2}\%$ of 9.6, gives $3\frac{1}{2}$?

II.
- (1.) 100%=the number.
- (2.)
  1. $100\%=3.5$,
  2. $1\%=\frac{1}{100}$ of $3.5=.035$, and
  3. $20\%=20$ times $.035=.7$.
- (3.)
  1. $100\%=9.6$,
  2. $1\%=\frac{1}{100}$ of $9.6=.096$, and
  3. $12\frac{1}{2}\%=12\frac{1}{2}$ times $.096=1.2$.
- (4.) $\therefore 100\%+.7-1.2=3\frac{1}{2}$,
- (5.) $100\%-.5=3.5$, and
- (6.) $100\%=4$, the number.

III. $\therefore$ The number=4. (*R. H. A., p. 191, prob. 26.*)

CASE II.

Given $\left\{\begin{matrix}\text{the base and the}\\ \text{percentage}\end{matrix}\right\}$ to find the rate per cent.

Formula.—$P\div B=R$, where B is the base, P the percentage and R the rate per cent.

I. 750 men is what % of 12000 men?

II.
1. 12000 men$=100\%$,
2. 1 man$=\frac{1}{12000}$ of $100\%=\frac{1}{120}\%$, and
3. 750 men$=750$ times $\frac{1}{120}\%=6\frac{1}{4}\%$.

III. $\therefore$ 750 men is $6\frac{1}{4}\%$ of 12000 men.

I. A's money is 50% more than B's; then B's is how **many %** less than A's?

II.
1. 100%=B.'s money. Then,
2. $100\%+50\%=150\%$=A.'s money.
3. $150\%=100\%$ of itself.
4. $1\%=\frac{1}{150}$ of $100\%=\frac{2}{3}\%$, and
5. $50\%=50$ times $\frac{2}{3}\%=33\frac{1}{3}\%$.

III. $\therefore$ B.'s money is $33\frac{1}{3}\%$ less than A.'s

(*R. H. A., p. 192, prob. 11.*)

I. 30% of the whole of an article is how many % of $\frac{2}{3}$ of it?

II.
1. 100%=whole article.
2. $66\frac{2}{3}\%=\frac{2}{3}$ of $100\%=\frac{2}{3}$ of the article.
3. $66\frac{2}{3}\%=100\%$ of itself.
4. $1\%=\dfrac{1}{66\frac{2}{3}}$ of $100\%=1\frac{1}{2}\%$, and
5. $30\%=30$ times $1\frac{1}{2}\%=45\%$.

III $\therefore$ 30% of the whole of an article is 45% of $\frac{2}{3}$ of it.

(*R. H. A., p. 192, prob. 20.*)

I. If a miller takes 4 quarts for toll from every bushel he grinds, what % does he take for toll?

II.
1. 1 bu.=32 qt.
2. 32 qt.=100%,
3. 1 qt.=$\frac{1}{32}$ of 100%=$3\frac{1}{8}$%, and
4. 4 qt.=4 times $3\frac{1}{8}$%=$12\frac{1}{2}$%.

III. ∴ He takes $12\frac{1}{2}$% for toll.

CASE III

Given { the percentage and / the rate per cent. } to find the base.

Formula.—$P \div R = B$, where P is the percentage, R the rate per cent., and B the base.

I. $24 is $\frac{3}{8}$% of what sum?

II.
1. 100%=sum.
2. $\frac{3}{8}$%=$24,
3. $\frac{1}{8}$%=$\frac{1}{3}$ of $24=$8,
4. $\frac{8}{8}$%, or 1%,=8 times $8=$64, and
5. 100%=100 times $64=$6400.

III. ∴ $24 is $\frac{3}{8}$% of $6400.

I. I drew 48% of my funds in bank, to pay a note of $150; how much had I left?

II.
1. 100%=amount in bank.
2. 48%=amount drawn out.
3. 100%—48%=52%=amount left.
4. 48%=$150,
5. 1%=$\frac{1}{48}$ of $150=$3.125, and
6. 52%=52 times $3.125=$162.50=amount left.

III. ∴ $162.50=amount I had left.

I. I pay $13 a month for board, which is 20% of my salary; what is my salary?

II.
1. 100%=my monthly salary.
2. 20%=$13,
3. 1%=$\frac{1}{20}$ of $13=$.65, and
4. 100%=100 times $.65=$65, my monthly salary.
5. ∴ $780=12 times $65=my yearly salary.

III. ∴ My salary=$780. (*R. H. A., p. 194, prob.* **20.**)

CASE IV.

Given $\left\{\begin{array}{c}\text{the amount and the}\\ \text{rate per cent.}\end{array}\right\}$ to find the base.

Formula.—$A\div(1+R)=B$, where A is the amount, that is, the base and the percentage, R the rate per cent., and B the base.

I. A sold a horse for \$150 and gained 25%; what did the horse cost?

II.
1. 100%=cost of horse.
2. 25%=gain.
3. 100%+25%=125%=selling price of horse, and
4. \$150=selling price of horse;
5. ∴ 125%=\$150,
6. 1%=$\frac{1}{125}$ of \$150=\$1.20, and
7. 100%=100 times \$1.20=\$120=cost of horse.

III. ∴ The horse cost \$120.

I. I sold two horses for the same price, \$150; on one I gained 25% and on the other I lost 25%; what was the cost of each?

II.

A.
1. 100%=cost of first horse.
2. 25%=gain.
3. 100%+25%=125%=selling price of first horse,
4. \$150=selling price of first horse;
5. ∴ 125%=\$150,
6. 1%=$\frac{1}{125}$ of \$150=\$1.20, and
7. 100%=100 times \$1.20=\$120=cost of first horse.

B.
1. 100%=cost of second horse.
2. 25%=loss on second horse.
3. 100%—25%=75%=selling price of 2d horse, and
4. \$150=selling price of second horse;
5. ∴ 75%=\$150,
6. 1%=$\frac{1}{75}$ of \$150=\$2, and
7. 100%=100 times \$2=\$200=cost of second horse.

III. ∴ $\left\{\begin{array}{l}\text{\$120=cost of first horse, and}\\ \text{\$200=cost of second horse.}\end{array}\right.$

I. A coat cost \$32; the trimmings cost 70% less, and the making 50% less than the cloth; what did each cost?

II.
1. 100%=cost of cloth. Then
2. $100\%-70\%=30\%$=cost of trimmings, and
3. $100\%-50\%=50\%$=cost of making.
4. $100\%+30\%+50\%=180\%$=cost of coat.
5. $32=cost of coat;
6. ∴ $180\%=\$32$,
7. $1\%=\frac{1}{180}$ of $\$32=\$.1777\frac{7}{9}$.
8. $100\%=100$ times $\$.1777\frac{7}{9}=\$17.77\frac{7}{9}$=cost of cloth.
9. $30\%=30$ times $\$.1777\frac{7}{9}=\$5.33\frac{1}{3}$=cost of trimming.
10. $50\%=50$ times $\$.1777\frac{7}{9}=\$8.88\frac{8}{9}$=cost of making.

III. ∴ $\$17.77\frac{7}{9}$=cost of cloth, $\$5.33\frac{1}{3}$=cost of trimmings, and $\$8.88\frac{8}{9}$=cost of making.

(*R. H. A.*, *p.* 196, *prob.* 12.)

I. In a company of 87, the children are $37\frac{1}{2}\%$ of the women, who are $44\frac{4}{9}\%$ of the men; how many of each?

II.
(1.) 100%=number of men. Then
(2.) $44\frac{4}{9}\%$=number of women.
(3.) 1. $100\%=44\frac{4}{9}\%$,
 2. $1\%=\frac{1}{100}$ of $44\frac{4}{9}\%=.44\frac{4}{9}\%$, and
 3. $37\frac{1}{2}\%=37\frac{1}{2}$ times $.44\frac{4}{9}\%=16\frac{2}{3}\%$=number of children in terms of the number of men.
(4.) $100\%+44\frac{4}{9}\%+16\frac{2}{3}\%=161\frac{1}{9}\%$=number in the company,
(5.) 87=number in the company;
(6.) ∴ $161\frac{1}{9}\%=87$,
(7.) $1\%=\frac{1}{161\frac{1}{9}}$ of $87=.54$,
(8.) $100\%=100$ times $.54=54$=number of men,
(9.) $44\frac{4}{9}\%=44\frac{4}{9}$ times $.54=24$=number of women, and
(10.) $16\frac{2}{3}\%=16\frac{2}{3}$ times $.54=19$=number of children.

III. ∴ 54=number of men, 24=number of women, and 19=number of children.

(*R. H. A.*, *p.* 197, *prob.* 20.)

I. Our stock decreased $33\frac{1}{3}\%$, and again 20%; then it rose 20%, and again $33\frac{1}{3}\%$; we have thus lost $66; what was the stock at first?

II.
- (1.) 100%=original stock.
- (2.) $33\frac{1}{3}$%=decrease.
- (3.) 100%—$33\frac{1}{3}$%=$66\frac{2}{3}$%=stock after first decrease.
- (4.)
  1. 100%=$66\frac{2}{3}$%,
  2. 1%=$\frac{1}{100}$ of $66\frac{2}{3}$%=$\frac{2}{3}$%, and
  3. 20%=20 times $\frac{2}{3}$%=$13\frac{1}{3}$%=second decrease.
  4. $66\frac{2}{3}$%—$13\frac{1}{3}$%=$53\frac{1}{3}$%=stock after second decrease.
- (5.)
  1. 100%=$53\frac{1}{3}$%,
  2. 1%=$\frac{1}{100}$ of $53\frac{1}{3}$%=.$53\frac{1}{3}$%, and
  3. 20%=20 times .$53\frac{1}{3}$%=$10\frac{2}{3}$%=first increase.
  4. $53\frac{1}{3}$%+$10\frac{2}{3}$%=64%=stock after first increase
- (6.)
  1. 100%=64%,
  2. 1%=$\frac{1}{100}$ of 64%=.64%, and
  3. $33\frac{1}{3}$%=$33\frac{1}{3}$ times .64%=$21\frac{1}{3}$%=second increase.
  4. 64%+$21\frac{1}{3}$%=$85\frac{1}{3}$%=stock after second increase.
- (7.) 100%—$85\frac{1}{3}$%=$14\frac{2}{3}$%=whole loss;
- (8.) $66=whole loss;
- (9.) ∴ $14\frac{2}{3}$%=$66;
- (10.) 1%=$\frac{1}{14\frac{2}{3}}$ of $66=$4.50, and
- (11.) 100%=100 times $4.50=$450=original stock.

III. ∴ $450=original stock.

I. A brewery is worth 4% less than a tannery, and the tannery 16% more than the boat; the owner of the boat has traded it for 75% of the brewery, losing thus $103; what is the tannery worth?

FIRST SOLUTION.

II.
- (1.) 100%=value of the tannery. Then
- (2.) 100%—4%=96%=value of the brewery.
- (3.)
  1. 100%=value of the boat. Then
  2. 100%+16%=116%=value of tannery in terms of the boat.
  3. 116%=100%, the value of tannery from step (1),
  4. 1%=$\frac{1}{116}$ of 100%=$\frac{25}{29}$%, and
  5. 100%=100 times $\frac{25}{29}$%=$86\frac{6}{29}$%=value of the boat in terms of the tannery.
- (4.)
  1. 100%=96%,
  2. 1%=$\frac{1}{100}$ of 96%=.96%, and
  3. 75%=75 times .96%=72%=what the owner of the boat received for it.
- (5.) ∴ $86\frac{6}{29}$%—72%=$14\frac{6}{29}$%=what the owner of the boat lost in the trade.
- (6.) $103=what he lost;
- (7.) ∴ $14\frac{6}{29}$%=$103,
- (8.) 1%=$\frac{1}{14\frac{6}{29}}$ of $103=$7.25, and
- (9.) 100%=100 times $7.25=$725=value of tannery.

III. ∴ $725=value of the tannery. (*R. H. A., p. 197, prob. 23.*)

Remark.—The value of the brewery and boat being expressed in terms of the tannery, 75% of the brewery is also expressed in terms of the tannery; hence, it is plain that the owner of the boat has traded $86\frac{6}{29}\%$ for 72% of the same value, losing $86\frac{6}{29}\%-72\%$, or $14\frac{6}{29}\%$.

SECOND SOLUTION.

II.
(1.) 100%=value of the boat. Then
(2.) 100%+16%=116%=value of the tannery.
(3.)
 1. 100%=116%,
 2. $1\%=\frac{1}{100}$ of 116%=1.16%, and
 3. 4%=4 times 1.16%=4.64%.
(4.) 116%—4.64%=111.36%=the value of brewery in terms of the boat.
(5.)
 1. 100%=111.36%,
 2. $1\%=\frac{1}{100}$ of 111.36%=1.1136%, and
 3. 75%=75 times 1.1136%=83.52%=what the owner of the boat received for it.
(6.) ∴ 100%—83.52%=16.48%=what he lost in the trade.
(7.) $103=what he lost.
(8.) ∴ 16.48%=$103,
(9.) $1\%=\frac{1}{16.48}$ of $103=$6.25, and
(10.) 116%=116 times $6.25=$725=value of tannery.

III. ∴ $725=value of the tannery.

THIRD SOLUTION.

II.
(1.) 100%=value of brewery.
(2.)
 1. 100%=value of tannery. Then
 2. 100%—4%=96%=value of the tannery.
 3. ∴ 96%=100%, the value of brewery in step (1),
 4. $1\%=\frac{1}{96}$ of 100%=$1.04\frac{1}{6}\%$, and
 5. 100%=100 times $1.04\frac{1}{6}\%=104\frac{1}{6}\%$=value the tannery in terms of the brewery.
(3.)
 1. 100%=value of boat. Then
 2. 100%+16%=116%=value of the tannery in terms of the boat.
 3. ∴ 116%=$104\frac{1}{6}\%$, the value of the tannery in step 5 of (2),
 4. $1\%=\frac{1}{116}$ of $104\frac{1}{6}\%=.89\frac{139}{174}\%$, and
 5. 100%=100 times $.89\frac{139}{174}\%=89\frac{139}{174}\%$=value of the boat in terms of the tannery, and consequently in terms of the brewery.
(4.) ∴ $89\frac{139}{174}\%-75\%=14\frac{139}{174}\%$=what the owner of the boat lost in the trade.
(5.) $103=what the owner of the boat lost;
(6.) ∴ $14\frac{139}{174}\%$=$103,
(7.) $1\%=\frac{1}{14\frac{139}{174}}$ of $103=$6.96, and
(8.) $104\frac{1}{6}\%=104\frac{1}{6}$ times $6.96=$725=value of tannery.

III. ∴ \$725=value of of the tannery.

Remark.—In step 5 of (3), we have the value of the boat in terms of the tannery; but the value of the tannery is in terms of the brewery: hence, the value of the boat is also in terms of the brewery. The owner of the boat, therefore, traded $89\frac{139}{174}\%$ for 75% of the same value.

MISCELLANEOUS PROBLEMS.

I. A man sold a horse for \$175, which was $12\frac{1}{2}\%$ less than the horse cost; what did the horse cost?

II.
1. 100%=cost of horse.
2. $12\frac{1}{2}\%$=loss.
3. $100\%-12\frac{1}{2}\%=87\frac{1}{2}\%$=selling price.
4. \$175=selling price.
5. ∴ $87\frac{1}{2}\%$=\$175,
6. $1\%=\dfrac{1}{87\frac{1}{2}}$ of \$175=\$2, and
7. 100%=100 times \$2=\$200,

III. ∴ \$200=cost of the horse. (*R. 3d p., p. 204, prob. 5.*)

I. A miller takes for toll 6 quarts from every 5 bushels of wheat ground; what % does he take for toll.?

II.
1. 1 bu.=32 qt.
2. 5 bu.=5 times 32 qt.=160 qt.
4. 160 qt.=100%,
4. 1 qt.=$\frac{1}{160}$ of 100%=$\frac{5}{8}\%$, and
5. 6 qt.=6 times $\frac{5}{8}\%=3\frac{3}{4}\%$.

III. ∴ He takes $3\frac{3}{4}\%$ for toll. (*R. 3d p., p. 204, prob. 11.*)

I. A farmer owning 45% of a tract of land, sold 540 acres, which was 60% of what he owned; how many acres were there in the tract?

II.
(1.) 100%=number of acres in the tract.

(2.)
1. 100%=numbers of acres the farmer owned.
2. 60%=number of acres the farmer sold.
3. 540 acres=what he sold.
4. ∴ 60%=540 acres,
5. 1%=$\frac{1}{60}$ of 540 acres=9 acres, and
6. 100%=100 times 9 acres=900 acres=what he owned.

(3.) 45%=what he owned.
(4.) ∴ 45%=900 acres,
(5.) 1%=$\frac{1}{45}$ of 900 acres=20 acres, and
(6.) 100%=100 times 20 acres=2000=number of acres in the tract.

III. ∴ The tract contained 2000 acres.

(*R. 3d p., p. 204, prob. 12.*)

I. A, wishing to sell a cow and a horse to B, asked 150% more for the horse than for the cow; he then reduced the price of the cow 25%, and the horse $33\frac{1}{3}\%$, at which price B took them, paying \$290; what was the price of each?

II.
(1.) 100%=asking price of the cow. Then
(2.) 100%+150%=250%=asking price of the horse.
(3.) 100%—25%=75%=selling price of cow.
(4.)
1. 100%=250%,
2. $1\%=\frac{1}{100}$ of 250%=2.50%, and
3. $33\frac{1}{3}\%=33\frac{1}{3}$ times 2.5%=$83\frac{1}{3}\%$=reduction on the asking price of the horse.
(5.) $250\%-83\frac{1}{3}\%=166\frac{2}{3}\%$=selling price of the horse.
(6.) $75\%+166\frac{2}{3}\%=241\frac{2}{3}\%$=selling price of both.
(7.) \$290=selling price of both.
(8.) ∴ $241\frac{2}{3}\%$=\$290,
(9.) $1\%=\frac{1}{241\frac{2}{3}}$ of \$290=\$1.20, and
(10.) 75%=75 times\$ 1.20=\$90=selling price of the cow.
(11.) $166\frac{2}{3}\%=166\frac{2}{3}$ times \$1.20=\$200=selling price of the horse.

III. ∴
\$ 90=selling price of the cow, and
\$200=selling price of the horse.

(*Brooks' H. A., p. 243, prob. 18.*)

I. A mechanic contracts to supply dressed stone for a church for \$87560, if the rough stone cost him 18 cents a cubic foot; but if he can get it for 16 cents a cubic foot, he will deduct 5% from his bill; required the number of cubic feet and the charge for dressing the stone.

II.
1. 100%=\$87560.
2. $1\%=\frac{1}{100}$ of \$87560=\$875.60, and
3. 5%=5 times \$875.60=\$4378=the deduction.
4. 18¢—16¢=2¢=the deduction per cubic foot.
5. ∴ \$4378=the deduction of 4378÷.02, or 218900 cubic feet. Then
6. \$87560=cost of 218900 cubic feet.
7. \$.40=\$87560÷218900=cost of one cubic feet.
8. ∴ \$.40—\$.18=\$.22=cost of dressing per cubic foot.

III. ∴
218900=number of cubic feet, and
22 cents=cost of dressing per cubic foot.

(*Brooks'. H. A., p. 241, prob. 21.*)

PROBLEMS.

1. A merchant, having $1728 in the Union Bank, wishes to withdraw 15%; how much will remain? *Ans.* $1468.80.

2. A Colonel whose regiment consisted of 900 men, lost 8% of them in battle, and 50% of the remainder by sickness; how many had he left? *Ans.* 414 men.

3. What % of $150 is 25% of $36? *Ans.* 6%.

4. What % of $\frac{1}{2}$ of $\frac{4}{5}$ of $\frac{8}{3}$ is $\frac{1}{3}$? *Ans.* $31\frac{1}{4}$%.

5. If a man owning 45% of a mill, should sell $33\frac{1}{3}$% of his share for $450; what would be the value of the mill? *Ans.* $3000.

6. A. expends in a week $24, which exceeds by $33\frac{1}{3}$% his earnings in the same time. What were his earnings? *Ans.* $18.

7. Bought a carriage for $123.06, which was 16% less than I paid for a horse; what did I pay for the horse? *Ans.* $146.50.

8. Bought a horse, buggy, and harness for $500. The horse cost $37\frac{1}{2}$% less than the buggy, and the harness cost 70% less than the horse; what was the price of each?
Ans. buggy $$275\frac{25}{29}$, horse $$172\frac{12}{29}$, and harness $$51\frac{21}{29}$.

9. I have 20 yards of yard-wide cloth, which will shrink on sponging 4% in length and 5% in width; how much less than 20 square yards will there be after sponging? *Ans.* $1\frac{19}{25}$ yards.

10. A. found $5; what was his gain %? *Ans.* ∞.

11. The population of a city whose gain of inhabitants in 5 years has been 25%, is 87500; what was it 5 years ago? *Ans.* 70000.

12. The square root of 2 is what % of the square root of 3? *Ans.* $\sqrt{6}\times 33\frac{1}{3}$%.

13. A laborer had his wages twice reduced 10%; what did he receive before the reduction, if he now receives $$2.02\frac{1}{2}$ per day? *Ans.* $2.50.

14. The cube root of 2985984 is what % of the square root of the same number? *Ans.* $8\frac{1}{3}$%.

15. A man sold two horses for $210; on one he gained 25%, and on the other he lost 25%; how much did he gain, supposing the second horse cost him $\frac{2}{3}$ as much as the first? *Ans.* $10.

16. A merchant sold goods at 20% gain, but had it cost him $49 more he would have lost 15% by selling at the same price; what did the goods cost him? *Ans.* $119.

17. If an article had cost 20% more, the gain would have been 25% less; what was the gain %? (See page 147.)
Ans. 50%

I. COMMISSION.

1. ***Commission*** is the percentage paid to an agent for the transaction of business. It is computed on the actual amount of the sale.

2. ***An Agent, Factor,*** or ***Commission Merchant,*** is a person who transacts business for another.

3. ***The Net Proceeds*** is the sum left after the commission and charges have been deducted from the amount of the sales or collections.

4. ***The Entire Cost*** is the sum obtained by adding the commission and charges to the amount of a purchase.

I. An agent received $210 with which to buy goods; after deducting his commission of 5%, what sum must he expend?

II.
1. 100%=what he must expend.
2. 5%=his commission.
3. 100%+5%=105%=what he receives.
4. $210=what he receives.
5. ∴ 105%=$210,
6. 1%=$\frac{1}{105}$ of $210=$2, and
7. 100%=100 times $2=$200=what he expends.

III. ∴ $200=what he must expend.

(R. 3d p., p. 207, prob. 4.)

Note.—Since the agent's commission is in the $210, we must not take 5% of $210; for we would be computing commission on his commission. Thus, 5% of ($200+$10)=$10+$.50. This is $.50 to much

I. An agent sold my corn, and after reserving his commission, invested the proceeds in corn at the same price; his commission, buying and selling was 3%, and his whole charge $12; for what was the corn first sold?

II.
(1.) 100%=cost of the corn.
(2.) 3%=the commission.
(3.) 100%−3%=97%=net proceeds, which he invested in corn.
(4.)
1. 100%=cost of second lot of corn.
2. 3%=the commission.
3. 100%+3%=103%=entire cost of second lot of corn.
4. 97%=entire cost of second lot of corn.
5. ∴ 103%=97%,
6. 1%=$\frac{1}{103}$ of 97%=$\frac{97}{103}$%, and
7. 100%=100 times $\frac{97}{103}$%=$94\frac{18}{103}$%=cost of second lot of corn in terms of the first.
8. 3%=3 times $\frac{97}{103}$%=$2\frac{85}{103}$%=commission on second lot.

(5.) 3%+$2\frac{85}{103}$%=$5\frac{85}{103}$%=whole commission.
(6.) $12=whole commission.
(7.) ∴ $5\frac{85}{103}$%=$12,
(8.) 1%=$\frac{1}{5\frac{85}{103}}$ of $12=$2.06, and
(9.) 100%=100 times $2.06=$206=cost of first lot of corn

III. ∴ $206=cost of first lot of corn. (*R. H. A., p. 219, prob. 10.*)

I. Sold cotton on commission, at 5%; invested the net proceeds in sugar, commission, 2%; my whole commission was $210; what was the value of the cotton and sugar?

II.
(1.) 100%=cost of cotton.
(2.) 5%=commission.
(3.) 100%−5%=95%=net proceeds, which he invested in sugar.
1. 100%=cost of sugar.
2. 2%=commission.
3. 102%=entire cost of sugar.
4. 95%=entire cost of sugar.
5. ∴ 102%=95%,
6. 1%=$\frac{1}{102}$ of 95%=$\frac{95}{102}$%, and
7. 100%=100 times $\frac{95}{102}$%=$93\frac{7}{51}$%=cost of sugar in terms of cotton.
8. 2%=2 times $\frac{95}{102}$%=$1\frac{44}{51}$%=commission on the sugar.

(4.) 5%+$1\frac{44}{51}$%=$6\frac{44}{51}$%=whole commission.
(5.) $210=whole commission.
(6.) ∴ $6\frac{44}{51}$%=$210,
(7.) 1%=$\frac{1}{6\frac{44}{51}}$ of $210=$30.60, and
(8.) 100%=100 times $30.60=$3060=cost of cotton.
(9.) $93\frac{7}{51}$%=$93\frac{7}{51}$ times $30.60=$2850=cost of sugar.

III. ∴ {$3060=cost of cotton, and
$2850=cost of sugar. (*R. H. A., p. 219, prob. 6.*)

I. A lawyer received \$11.25 for collecting a debt; his commission being 5%; what was the amount of the debt?

II.
1. 100%=amount of the debt.
2. 5%=commission.
4. \$11.25=commission.
4. ∴ $5\% = \$11.25$.
5. $1\% = \frac{1}{5}$ of $\$11.25 = \$2.25$, and
6. $100\% = 100$ times $\$2.25 = \$225$=amount of the debt.

III. ∴ \$225=amount of debt.

(R, 3d p., p. 207, prob. 6.)

I. Charge \$52.50 for collecting a debt of \$525; what was the rate of commission?

II.
1. $\$525 = 100\%$.
2. $\$1 = \frac{1}{525}$ of $100\% = \frac{4}{21}\%$, and.
3. $\$52.50 = 52.5$ times $\frac{4}{21}\% = 10\%$=rate of commission.

III. ∴ 10%=rate of commission.

I. My agent sold my flour at 4% commission; increasing the proceeds by \$4.20, I ordered the purchase of wheat at 2% commission; after which, wheat declining $3\frac{1}{3}\%$, my whole loss was \$5; what was the flour worth?

II.

(1.) 100%=cost of flour.

(2.) 4%=commission on flour.

(3.) $100\% - 4\% = 96\%$=net proceeds.

(4.)
1. 100%=cost of wheat.
2. 2%=commission on wheat.
3. $100\% + 2\% = 102\%$=entire cost of wheat.
4. $96\% + \$4.20$=entire cost of wheat.
5. ∴ $102\% = 96\% + \$4.20$,
6. $1\% = \frac{1}{102}$ of $(96\% + \$4.20) = .94\frac{2}{17}\% + \$.0411\frac{13}{17}$,
7. $100\% = 100$ times $(.94\frac{2}{17}\% + \$.0411\frac{13}{17}) = 94\frac{2}{17}\% + \$4.11\frac{13}{17}$=cost of wheat.
8. $2\% = 2$ times $(.94\frac{2}{17}\% + \$.0411\frac{13}{17}) = 1\frac{15}{17}\% + \$.08\frac{4}{17}$ =commission on wheat.

(5.)
1. $100\% = 94\frac{2}{17}\% + \$4.11\frac{13}{17}$,
2. $1\% = \frac{16}{17}\% + \$.04\frac{2}{17}$, and
3. $3\frac{1}{3}\% = 3\frac{1}{3}$ times $(\frac{16}{17}\% + \$.04\frac{2}{17}) = 3\frac{7}{51}\% + \$.13\frac{37}{51}$= loss on wheat.

(6.) $4\% + 1\frac{15}{17}\% + \$.08\frac{4}{17} + 3\frac{7}{51}\% + \$.13\frac{37}{51} = 9\frac{1}{51}\% + \$.21\frac{49}{51}$=whole loss.

(7.) \$5=whole loss.

(8.) ∴ $9\frac{1}{51}\% + \$.21\frac{49}{51} = \$5$, or

(9.) $9\frac{1}{51}\% = \$5 - \$.21\frac{49}{51} = \$4.78\frac{2}{51}$.

(10.) $1\% = \frac{1}{9\frac{1}{51}}$ of $\$4.78\frac{2}{51} = \$.53$, and

(11.) $100\% = 100$ times $\$.53 = \$53$.

III. ∴ \$53=cost of flour. *(R. H. A., p. 219, prob. 11.)*

EXAMPLES.

1. A broker in New York exchanged $4056 on Canal Bank, Portland, at $\frac{5}{8}\%$; what did he receive for his trouble?

Ans. $25.35.

2. A sold on commission for B 230 yards of cloth at $1.25 per yard, for which he received a commission of $3\frac{1}{2}\%$; what was his commission and what sum did he remit?

Ans. Commission $10.06¼, and Remittance $277.43¾.

3. A sold a lot of books on commission of 20%, and remitted $160; for what were the books sold? *Ans.* $200.

4. A lawyer charged $80 for collecting $200; what was his rate of commission? *Ans.* 40%

5. I sent my agent $1364.76 to be invested in pork at $6 per bbl. after deducting his commission of 2%; how many barrels of pork did he buy? *Ans.* 223 bbl.

6. How much money must I send my agent, so that he may purchase 250 bbl. of flour for me at $6.25 per bbl., if I pay him $2\frac{1}{2}\%$ commission?. *Ans.* $1601.5625.

7. If an agent's commission was $200, and his rate of commission 5%; what amount did he invest? *Ans.* $4000

8. My agent sold cattle at 10% commission, and after I increased the proceeds by $18, I ordered him to buy hogs at 20% commission. The hogs had declined $6\frac{2}{3}\%$, when he sold them at $14\frac{2}{7}\%$ commission. I lost in all $86; what did the cattle sell for?

Ans. $200.

9. An agent sells flour on commission of 2%, and purchases goods on true commission of 3%. If he had received 3% for selling and 2% for buying, his whole commission would have been $5 more. Find the value of the goods bought.

Ans. $9996

II. TRADE DISCOUNT.

1. ***Trade Discount*** is the discount allowed in the purchas and sale of merchandise.

2. ***A List,*** or ***Regular Price,*** is an established price, assumed by the seller as a basis upon which to calculate discount.

3. ***A Net Price*** is a fixed price from which no discount is allowed.

4. ***The Discount*** is the deduction from the list, or regular price.

I. Sold 20 doz. feather dusters, giving the purchaser a discount of 10, 10 and 10% off, his discounts amounting to \$325.20; how much was my price per dozen?

II.
(1.) 100%=whosesale price.
(2.) 10% of 100%=10%=first discount.
(3.) 100%—10%=90%=first net proceeds.
(4.)
1. 100%=90%,
2. 1%=$\frac{1}{100}$ of 90%=$\frac{9}{10}$%, and
3. 10%=10 times $\frac{9}{10}$%=9%=second discount.
4. 90%—9%=81%=second net proceeds.
(5.)
1. 100%=81%,
2. 1%=$\frac{1}{100}$ of 81%=$\frac{81}{100}$%, and
3. 10%=10 times $\frac{81}{100}$%=8.1%=third discount.
(6.) 10%+9%+8.1%=27.1%=sum of discounts.
(7.) \$325.20=sum of discounts.
(8.) ∴ 27.1%=\$325.20,
(9.) 1%=$\frac{1}{27.1}$ of \$325.20=\$12, and
(10.) 100%=100 times \$12=\$1200=wholesale price of 20 dozen.
(11.) \$60=\$1200÷20=wholesale price of 1 dozen.

III. ∴ \$60=wholesale price per dozen.

(*R. 3d p., p. 209, prob. 5.*)

I. Bought 100 dozen stay bindings at 60 cents per dozen for 40, 10, and 7½% off; what did I pay for them?

II.
(1.) 60¢=list price of 1 dozen.
(2.) \$60=100 times \$.60=list price of 100 dozen.
(3.)
1. 100%=\$60,
2. 1%=$\frac{1}{100}$ of\$60=\$.60, and
3. 40%=40 times \$.60=\$24=first discount.
4. \$60—\$24=\$36=first net proceeds.
(4.)
1. 100%=\$36,
2. 1%=$\frac{1}{100}$ of \$36=\$.36. and
3. 10%=10 times \$.36=\$3.60=second discount.
4. \$36—\$3.60=\$32.40=second net proceeds.
(5.)
1. 100%=\$32.40,
2. 1%=$\frac{1}{100}$ of \$32.40=\$.324, and
3. 7½%=7½ times \$.324=\$2.43=third discount.
4. \$32.40—\$2.43=\$29.97=cost.

III. ∴ I paid \$29.97. (*R. 3d p., p. 209, prob. 6.*)

I. A retail dealer buys a case of slates containing 10 dozen for \$50 list, and gets 50, 10, and 10% off; paying for them in the usual time, he gets an additional 2%; what did he pay per dozen for the slates?

II. (1.) 1. 100%=\$50.
2. 1%=$\frac{1}{100}$ of \$50=\$.50.
3. 50%=50 times \$.50=\$25=first discount.
4. \$50—\$25=\$25=first net proceeds.

(2.) 1. 100%=\$25.
2. 1%=$\frac{1}{100}$ of \$25=\$.25.
3. 10%=10 times \$.25=\$2.50=second discount
4. \$25—\$2.50=\$22.50=second net proceeds.

(4.) 1. 100%=\$22.50.
2. 1%=$\frac{1}{100}$ of \$22.50=\$.225.
3. 10%=10 times \$.225=\$2.25=third discount.
4. \$22.50—\$2.25=\$20.25=third net proceeds.

(5.) 1. 100%=\$20.25.
2. 1%=$\frac{1}{100}$ of \$20.25=\$.2025.
3. 2%=2 times \$.2025=\$.405=fourth discount.
4. \$20.25—\$.405=\$19.845=cost of 10 dozen slates.
5. \$1.9845=\$19.845÷10=cost of 1 dozen slates.

III. ∴ \$1.9845=cost of 1 dozen slates.

(*R. 3d p., p. 209, prob. 9.*)

I. Sold a case of hats containing 3 dozen, on which I had received a discount of 10% and made a profit of $12\frac{1}{2}$% or $37\frac{1}{2}$¢ on each hat; what was the wholesale merchant's price per case?

II. (1.) $37\frac{1}{2}$¢=profit on one hat.
(2.) \$13.50=36 times \$.$37\frac{1}{2}$=profit on 3 dozen hats.
(3.) 100%=wholesale merchant's price per case.
(4.) 10%=discount.
(5.) 100%—10%=90%=my cost.
(6.) 1. 100%=90%.
2. 1%=$\frac{1}{100}$ of 90%=.9%.
3. $12\frac{1}{2}$%=$12\frac{1}{2}$ times .9%=$11\frac{1}{4}$%=profit in terms of wholesale price.
(7.) ∴ $11\frac{1}{4}$%=\$13.50.
(8.) 1%=$\frac{1}{11\frac{1}{4}}$ of \$13.50=\$1.20.
(9.) 100%=100 times \$1.20=\$120=wholesale merchant's price per case.

III. ∴ \$120=wholesale merchant's price per case.

(*R. 3d p., p. 212, prob. 4.*)

I. A bookseller purchased books from the publishers at 20% off the list; if he retail them at the list what will be his per cent. of profit?

II.
1. 100%=list price.
2. 20%=discount.
3. 100%—20%=80%=cost.
4. 100%=bookseller's selling price, because he sold them at the list price.
5. ∴ 100%—80%=20%=gain.
6. 80%=100% of itself.
7. 1%=$\frac{1}{80}$ of 100%=1¼%, and
8. 20%=20 times 1¼%=25%=his gain %.

III. ∴ 25%=his % of profit. (*R. 3d p., p. 211, prob. 1.*)

Note.—Observe that since his cost is 80%, and his gain 20%, we wish to know what % 20% is of 80%. It will become evident if we suppose the list price to be (say) $400, and then proceed to find the % of gain as in the above solution.

I. Bought 50 gross of rubber buttons for 25, 10, and 5% off; disposed of the lot for $35.91, at a profit of 12%; what was the list price of the buttons per gross?

II.
(1.) 100%=list price.
(2.) 25% of 100%=25%=first discount.
(3.) 100%—25%=75%=first net proceeds.
(4.)
1. 100%=75%,
2. 1%=$\frac{1}{100}$ of 75%=¾%, and
3. 10%=10 times ¾%=7½%.
4. 75%—7½%=67½%=second net proceeds
(5.)
1. 100%=67½%,
2. 1%=$\frac{1}{100}$ of 67½%=.67½%, and
3. 5%=5 times .67½%=3.375%=third discount.
4. 67½%—3.375%=64.125%=cost.
(6.)
1. 100%=64.125%,
2. 1%=.64125%, and
3. 12%=12 times .64125%=7.695%=gain.
4. ∴ 64.125%+7.695%=71.82%=selling price.
(7.) $35.91=selling price.
(8.) ∴ 71.82%=$35.91,
(9.) 1%=$\frac{1}{71.82}$ of $35.91=$.50, and
(10.) 100%=100 times $.50=$50=list price of 50 gross,
(11.) $1.00=$50÷50=list price of one gross.

III. ∴ $1.00=list price of one gross.
(*R. 3d p., p. 212, prob. 10.*)

I A dealer in notions buys 60 gross shoestrings at 70¢ per gross, list, 50, 10, and 5% off; if he sell them at 20, 10, and 5% off list, what will be his profit?

II.
- (1.) 70¢=list price of one gross.
- (2.) $42=60 times $.70=list price of 60 gross.
- (3.)
  1. 100%=$42.
  2. 1%=$\frac{1}{100}$ of $42=$.42.
  3. 50%=50 times $.42=$21=first discount.
  4. $42—$21=$21=first net proceeds.
- (4.)
  1. 100%=$21.
  2. 1%=$\frac{1}{100}$ of $21=$.21.
  3. 10%=10 times $.21=$2.10=second discount.
  4. $21—$2.10=$18.90=second net proceeds.
- (5.)
  1. 100%=$18.90.
  2. 1%=$.189.
  3. 5%=$.945=third discount.
  4. $18.90—$.945=$17.955=cost.
- (6.)
  1. 100%=$42.
  2. 1%=$\frac{1}{100}$ of $42=$.42.
  3. 20%=20 times $.42=$8.40=first conditional discount.
  4. $42—$8.40=$33.60=first conditional net proceeds.
- (7.)
  1. 100%=$33.60.
  2. 1%=$\frac{1}{100}$ of $33.60=$.336.
  3. 10%=10 times $.336=$3.36=second conditional discount.
  4. $33.60—$3.36=$30.24=second conditional net proceeds.
- (8.)
  1. 100%=$30.24.
  2. 1%=$\frac{1}{100}$ of $30.24=$.3024.
  3. 5%=5 times $.3024=$1.512=third conditional discount.
  4. $30.24—$1.512=$28.728=selling price.
- (9.) ∴ $28.728—$17.955=$10.773=his profit.

III. ∴ $10.773=his profit *(R. 3d p., p. 212, prob. 9.)*

PROBLEMS.

1. Bought a case of slates containing 12 doz. for $80 list, and got 45, 10, and 10% off; getting an additional 2% off for prompt payment, what did I pay per dozen for the slates?
Ans. $2.9106.

2. Bought a case of hats containing 4 doz., on which I received a discount of 40, 20, 10, 5, and $2\frac{1}{2}$% off. If I sell them at $4 a piece making a profit of 20%, what is the wholesale merchant's price per case? *Ans.* 399\frac{17207}{20007}$.

3. If I receive a discount of 20, 10, and 5% off, and sell at a discount of 10, 5, and $2\frac{1}{2}$% off; what is my % of gain?
Ans. $21\frac{7}{8}$%—.

4. A bill of goods amounted to $2400; 20% off being allowed, what was paid for the goods? *Ans.* $1920.

5. Bought goods at 25, 20, 15, and 10% off. If the sum of my discounts amounted to $162.30, what was the list price of the goods? *Ans.* $300

III. PROFIT AND LOSS.

1. ***Profit*** and ***Loss*** are terms which denote the gain or loss in business transactions.

2. ***Profit*** is the excess of the selling price above the cost.

3. ***Loss*** is the excess of the cost above the selling price.

I. A merchant reduced the price of a certain piece of cloth 5 cents per yard, and thereby reduced his profit on the cloth from 10% to 8%; what was the cost of the cloth per yard?

II.
1. 100%=cost of cloth per yard.
2. 10%=his profit before reduction.
3. 8%=his profit after reduction.
4. 10%—8%=2%=his reduction.
5. 5¢=reduction.
6. ∴ 2%=5¢,
7. 1%=$\frac{1}{2}$ of 5¢=$2\frac{1}{2}$¢, and
8. 100%=100 times $2\frac{1}{2}$¢=$2.50=cost per yard.

III. ∴ $2.50=cost of cloth per yard.

(*R. 3d p., p. 211, prob. 13.*)

I. A dealer sold two horses for $150 each; on one he gained 25% and on the other he lost 25%; how much did he lose in the transaction?

II.
(1.) 100%=cost of the first horse;
(2.) 25%=gain.
(3.) 100%+25%=125%=selling price of first horse.
(4.) $150=selling price.
(5.) ∴ 125%=$150,
(6.) 1%=$\frac{1}{125}$ of $150=$1.20, and
(7.) 100%=100 times $1.20=$120=cost of first horse.
(8.) $150—$120=$30=gain on first horse.
(9.)
  1. 100%=cost of second horse.
  2. 25%=loss.
  3. 100%—25%=75%=selling price of second horse.
  4. $150=selling price.
  5. ∴ 75%=$150,
  6. 1%=$\frac{1}{75}$ of $150=$2, and
  7. 100%=100 times $2=$200=cost of second horse.
(10.) $200—$150=$50=loss on second horse.
(11.) $50—$30=$20=loss in the transaction.

III. ∴ He lost $20 in the transaction.

(*R. 3d p., p. 211, prob. 12.*)

I. A speculator in real estate sold a house and lot for $12000, which sale afford him a profit of $33\frac{1}{3}$% on the cost; he then invested the \$12000 in city lots, which he was obliged to sell at a loss of $33\frac{1}{3}\%$; how much did he lose by the two transactions?

II.
(1.) 100%=cost of the house and lot.
(2.) $33\frac{1}{3}\%$=gain.
(3.) $100\%+33\frac{1}{3}\%=133\frac{1}{3}\%$=selling price of house and lot.
(4.) \$12000=selling price of the house and lot.
(5.) $\therefore 133\frac{1}{3}\%$=\$12000.
(6.) $1\%=\frac{1}{133\frac{1}{3}}$ of \$12000=\$90.
(7.) 100%=100 times \$90=\$9000=cost of house and lot.
(8.) \$12000—\$9000=\$3000=gain on house and lot.
(9.)
1. 100%=\$12000.
2. $1\%=\frac{1}{100}$ of \$12000=\$120.
3. $33\frac{1}{3}\%=33\frac{1}{3}$ times \$120=\$4000=loss on city lots.

(10.) \$4000—\$3000=\$1000=loss by the two transactions.

III. ∴ \$1000=his loss by the two transactions.

(*R. 3d p., p. 211, prob. 15.*)

I. A dealer sold two horses for the same price; on one he gained 20%, and on the other he lost 20%; his whole loss was \$25; what did each horse cost?

II.
(1.) 100%=selling price of each horse.
(2.)
1. 100%=cost of first horse.
2. 20%=gain on the first horse.
3. $100\%+20\%=120\%$=selling price of first horse.
4. $\therefore 120\%=100\%$, from (1),
5. $1\%=\frac{1}{120}$ of $100\%=\frac{5}{6}\%$, and
6. 100%=100 times $\frac{5}{6}\%=83\frac{1}{3}\%$=cost of first horse in terms of the selling price.
7. $100\%-83\frac{1}{3}\%=16\frac{2}{3}\%$=gain on first horse.

(3.)
1. 100%=cost of the second horse.
2. 20%=loss on second horse.
3. $100\%-20\%=80\%$=selling price of second horse.
4. $\therefore 80\%=100\%$, from (1),
5. $1\%=\frac{1}{80}$ of $100\%=1\frac{1}{4}\%$, and
6. 100%=100 times $1\frac{1}{4}\%=125\%$=cost of second horse in terms of the selling price.
7. $125\%-100\%=25\%$=loss on the second horse.

(4.) $25\%-16\frac{2}{3}\%=8\frac{1}{3}\%$=whole loss.
(5.) \$25=whole loss.
(6.) $\therefore 8\frac{1}{3}\%$=\$25,
(7.) $1\%=\frac{1}{8\frac{1}{3}}$ of \$25=\$3, and
(8.) 100%=100 times \$3=\$300=selling price of each horse.
(9.) $83\frac{1}{3}\%=83\frac{1}{3}$ times \$3=\$250=cost of first horse.
(10.) 125%=125 times \$3=\$375=cost of second horse.

III. ∴ $250=cost of the first horse, and
$375=cost of second horse.

I. What % is lost if $\frac{2}{3}$ of cost equals $\frac{3}{4}$ of selling price?

II.
1. $\frac{3}{4}$ of selling price=$\frac{2}{3}$ of cost.
2. $\frac{1}{4}$ of selling price=$\frac{1}{3}$ of $\frac{2}{3}$ of cost=$\frac{2}{9}$ of cost.
3. $\frac{4}{4}$ of selling price=4 times $\frac{2}{9}$ of cost=$\frac{8}{9}$ of cost.
4. $\frac{9}{9}$=cost.
5. $\frac{8}{9}$=selling price.
6. $\frac{9}{9}-\frac{8}{9}=\frac{1}{9}$=loss.
7. $\frac{9}{9}$=100%.
8. $\frac{1}{9}=\frac{1}{9}$ of 100%=$11\frac{1}{9}$%, loss.

III. ∴ Loss=$11\frac{1}{9}$%.

I. Paid $125 for a horse, and traded him for another, giving 60% additional money. For the second horse I received a third and $25. I then sold the third horse for $150; what was my % of profit or loss?

II.
(1.) 100%=$125,
(2.) 1%=$\frac{1}{100}$ of $125=$1.25, and
(3.) 60%=60 times $1.25=$75=additional money paid for the second horse.
(4.) $125+$75=$200=cost of second horse.
(5.) $150=selling price of the third horse.
(6.) $150+$25=$175=selling price of second horse.
(7.) $200—$175=$25=loss in the transaction.
(8.) 1. $200=100%,
2. $1=$\frac{1}{200}$ of 100%=$\frac{1}{2}$%, and
3. $25=25 times $\frac{1}{2}$%=$12\frac{1}{2}$%=my loss.

III. ∴ My loss is $12\frac{1}{2}$%. (*R. H. A., p. 201, prob. 4.*)

I. If I buy at $4 and sell at $1, how many % do I lose?

II.
1. $4=cost.
2. $1=selling price.
3. $4—$1=$3=loss.
4. $4=100%.
5. $1=$\frac{1}{4}$ of 100%=25%.
6. $3=3 times 25%=75%=loss.

III. ∴ 75%=loss.

I. A and B each lost $5, which was $2\frac{7}{9}$% of A's and $3\frac{1}{3}$% of B's money; which had the most, and how much?

II.
(1.) 100%=A's money.
(2.) $2\frac{7}{9}$%=what he lost.
(3.) \$5=what he lost.
(4.) ∴ $2\frac{7}{9}$%=\$5,
(5.) 1%=$\frac{1}{2\frac{7}{9}}$ of \$5=\$1.80, and
(6.) 100%=100 times \$1.80=\$180=A's money.
(7.)
1. 100%=B's money.
2. $3\frac{1}{3}$%=what he lost.
3. \$5=what he lost.
4. ∴ $3\frac{1}{3}$%=\$5,
5. 1%=$\frac{1}{3\frac{1}{3}}$ of \$5=\$1.50, and
6. 100%=100 times \$1.50=\$150=B's money.
(8.) \$180—\$150=\$30=excess of A's money over B's.

III. ∴ A had \$30 more than B. (*R. H. A., p. 203, prob. 5.*)

I. Mr. A bought a horse and carriage, paying twice as much for the horse as for the carriage. He afterward sold the horse for 25% more than he gave for it, and the carriage for 20% less than he gave for it, receiving \$577.50; what was the cost of each?

II.
(1.) 100%=cost of the carriage.
(2.) 200%=cost of the horse.
(3.) 20%=loss on the carriage.
(4.) 100%—20%=80%=selling price of the carriage.
(5.)
1. 100%=200%,
2. 1%=$\frac{1}{100}$ of 200%=2%, and
3. 25%=25 times 2%=50%=gain on the horse.
4. 200%+50%=250%=selling price of the horse.
(6.) 80%+250%=330%=selling price of both.
(7.) \$577.50=selling price of both.
(8.) ∴ 330%=\$577.50,
(9.) 1%=$\frac{1}{330}$ of \$577.50=\$1.75, and
(10.) 100%=100 times \$1.75=\$175=cost of carriage.
(11.) 200%=200 times \$1.75=\$350=cost of the horse.

III. ∴ \$175=cost of the carriage, and \$350=cost of the horse.

(*Milne's prac., p. 259, prob. 19.*)

I. Mr. A. sold a horse for \$198, which was 10% less than he asked for him, and his asking price was 10% more than the horse cost him. What did the horse cost him?

II.
- (1.) 100%=cost of the horse.
- (2.) 100%+10%=110%=asking price.
- (3.)
  1. 100%=110%,
  2. 1%=$\frac{1}{100}$ of 110%=1$\frac{1}{10}$%, and
  3. 10%=10 times 1$\frac{1}{10}$%=11%=reduction from asking price.
- (4.) 110%−11%=99%=selling price.
- (5.) $198=selling price.
- (6.) ∴ 99%=$198,
- (7.) 1%=$\frac{1}{99}$ of $198=$2, and
- (8.) 100%=100 times $2=$200=cost of the horse.

III. ∴ $200=cost of horse. (*Milne's prac., p. 259, prob. 23.*)

I. What must be asked for apples which cost me $3 per bbl., that I may reduce my asking price 20% and still gain 20% on the cost?

II.
- (1.) 100%=$3.
- (2.) 1%=$\frac{1}{100}$ of $3=$.03, and
- (3.) 20%=20 times $.03=$.60=gain.
- (4.) $3.00+$.60=$3.60=selling price.
- (5.)
  1. 100%=asking price.
  2. 20%=reduction.
  3. 100%−20%=80%=selling price.
  4. $3.60=selling price.
  5. ∴ 80%=$3.60,
  6. 1%=$\frac{1}{80}$ of $3.60=$.045, and
  7. 100%=100 times $.045=$4.50=asking price.

III. ∴ $4.50=asking price. (*Milne's prac., p. 261, prob. 38.*)

I. A merchant sold a quantity of goods at a gain of 20%. If, however, he had purchased them for $60 less than he did, his gain would have been 25%. What did the goods cost him?

II.
- (1.) 100%=actual cost of goods.
- (2.) 20%=gain.
- (3.) 100%+20%=120%=actual selling price.
- (4.) 100%−$60=supposed cost.
- (5.)
  1. 100%=100%−$60,
  2. 1%=$\frac{1}{100}$ of (100%−$60)=1%−$.60, and
  3. 25%=25 times (1%−$.60)=25%−$15=supposed gain.
- (6.) (100%−$60)+(25%−$15)=125%−$75=actual selling price.
- (7.) ∴ 125%−$75=120%,
- (8.) 5%=$75,
- (9.) 1%=$\frac{1}{5}$ of $75=$15, and
- (10.) 100%=100 times $15=$1500=cost of the goods.

III. ∴ $1500=cost of goods. (*Milne's prac., p. 261, prob. 40.*)

Note.—The selling price is the same in the last condition of this problem as in the first. Hence we have the selling price in the last condition equal to the selling price in the first as shown in step (7.)

I. I sold an article at 20% gain, had it cost me $300 more, I would have lost 20%; find the cost.

II.
(1.) 100%=actual cost of the article.
(2.) 20%=actual gain.
(3.) 100%+20%=120%=actual selling price.
(4.) 100%+$300=supposed cost.
(5.) 1. 100%=100%+$300,
2. 1%=$\frac{1}{100}$ of (100%+$300)=1%+$3, and
3. 20%=20 times (1%+$3)=20%+$60=supposed loss.
(6.) (100%+$300)—(20%+$60)=80%+$240=actual selling price.
(7.) ∴ 120%=80%+$240.
(8.) 40%=$240,
(9.) 1%=$\frac{1}{40}$ of $240=$6, and
(10.) 100%=100 times $6=$600=cost of the article.

III. ∴ $600=cost of the article.

(*R. H. A., p. 409, prob. 85.*)

I. A man wishing to sell a horse and a cow, asked three times as much for the horse as for the cow, but, finding no purchaser, he reduced the price of the horse 20%, and the price of the cow 10%, and sold them for $165. What did he get for each?

II.
(1.) 100%=asking price of the cow.
(2.) 300%=asking price of the horse.
(3.) 10%=reduction on the price of the cow.
(4.) 100%—10%=90%=selling price of the cow.
(5.) 1. 100%=300%,
2. 1%=$\frac{1}{100}$ of 300%=3%, and
3. 20%=20 times 3%=60%=reduction on horse.
(6.) 300%—60%=240%=selling price of the horse.
(7.) 90%+240%=330%=selling price of both.
(8.) $165=selling price of both
(9.) ∴ 330%=$165,
(10.) 1%=$\frac{1}{330}$ of $165=$.50, and
(11.) 90%=90 times $.50=$45=selling price of cow.
(12.) 240%=240 times $.50=$120=selling price of horse.

III. ∴ $45=amount he received for the cow, and
$120=amount he received for the horse.

PROBLEMS.

1. What price must a man ask for a horse that cost him \$200, that he may fall 20% on his asking price and still gain 20%? *Ans.* \$300.

2. A man paid \$150 for a horse which he offered in trade at a price he was willing to discount at 40% for cash, as he would then gain 20%. What was his trading price? *Ans.* \$300.

3. A man gained 20% by selling his house for \$3600. What did it cost him? *Ans.* \$3000.

4. A gained 120% by selling sugar at 8¢ per pound. What did the sugar cost him per pound? *Ans.* $3\frac{7}{11}$¢.

5 How must cloth, costing \$3.50 a yard, be marked that a merchant may deduct 15% from the marked price and still gain 15%? *Ans.* \$4.73$\frac{9}{17}$.

6. Sold a piece of carpeting for \$240, and lost 20%; what selling price would have given me a gain of 20%? *Ans.* \$360.

7. Sold two carriages for \$240 apiece, and gained 20% on one and lost 20% on the other; how much did I gain or lose in the transaction? *Ans.* Lost \$20.

8. Sold goods at a gain of 25% and investing the proceeds, sold at a loss of 25%; what was my % of gain or loss. *Ans.* $6\frac{1}{4}$%.

9. A man sold a horse and carriage for \$597, gaining by the sale, 25% on the horse and 10% on the cost of the carriage. If $\frac{4}{4}$ of the cost of the horse equals $\frac{2}{3}$ of the the cost of carriage, what was the cost of each? *Ans.* Carriage \$270; horse \$240.

10. If $\frac{4}{9}$ of the selling price is gain, what is the profit? *Ans.* 80%.

11. If $\frac{1}{2}$ of an article be sold for the cost of $\frac{1}{3}$ of it, what is the rate of loss? *Ans.* $33\frac{1}{3}$%.

12. I sold two houses for the same sum; on one I gained 25% and on the other I lost 25%. My whole loss was \$240; what did each house cost? *Ans.* First \$1440, second \$2400.

13. My tailor informs me that it will take $10\frac{1}{4}$ sq. yd. of cloth to make me a full suit of clothes. The cloth I am about to buy is $1\frac{7}{8}$ yards wide and on sponging it will shrink 5% in length and width. How many yards will it take for my new suit? *Ans.* $6\frac{62}{1083}$ yd.

14. A grocer buys coffee at 15¢ per ℔. to the amount of \$90 worth, and sells it at the same price by Troy weight; find the % of gain or loss. *Ans.* Gain $21\frac{19}{36}$%.

15. I spent \$260 for apples at \$1.30 per bushel; after retaining a part for my own use, I sold the rest at a profit of 40%, clearing \$13 on the whole cost. How many bushels did I retain?
Ans. 50 bu.

16 How must cloth costing \$3.50 per yard, be marked that the merchant may deduct 15% from the marked price and still make 15% profit? *Ans.* \$4.735.

17. I sold goods at a gain of 20%. If they had cost me \$250 more than they did, I would have lost 20% by the sale. How much did the goods cost me? *Ans.* \$500.

18. A merchant bought cloth at \$3.25 per yard, and after keeping it 6 months sold it at \$3.75 per yard. What was his gain %, reckoning 6% per annum for the use of money?
Ans. 12%+.

IV. STOCKS AND BONDS.

1. ***Stocks*** is a general term applied to bonds, state and national, and to certificates of stocks belong to corporations.

3. ***A Bond*** is a written or printed obligation, under seal, securing the payment of a certain sum of money at or before a specified time.

3. ***Stock*** is the capital of the corporation invested in business; and is divided into *Shares*, usually of \$100 each.

4. ***An Assessment*** is a sum of money required of the stockholders in proportion to their amount of stock.

5. ***A Dividend*** is a sum of money to be paid to the stockholders in proportion to their amounts of stock.

6. ***The Par Value*** of money, stocks, drafts, etc., is the nominal value on their face.

7. ***The Market Value*** is the sum for which they sell.

8. ***Discount*** is the excess of the par value of money, stocks, drafts, etc., over their market value.

9. ***Premium*** is the excess of their market value over their par value.

10. ***Brokerage*** is the sum paid an agent for buying stocks, bonds, etc.

I. At $\frac{1}{4}\%$ brokerage, a broker received \$10 for making an investment in bank stock; how many shares did he buy?

II.
1. 100%=par value of stock.
2. $\frac{1}{4}\%$=brokerage.
3. \$10=brokerage.
4. $\therefore \frac{1}{4}\%$=\$10,
5. 1%=4 times \$10=\$40, and
6. 100%=100 times \$40=\$4000=par value of stock.
7. \$100=par value of one share.
8. \$4000=par value of 4000÷100, or 40 shares.

III. ∴ 40=number of shares.

I. How many shares of railroad stock at 4% premium can be bought for \$9360?

II.
1. 100%=par value of stock I can buy.
2. 4%=premium.
3. 104%=price of what I buy.
4. \$9360=price of what I buy.
5. $\therefore 104\%$=\$9360.
6. $1\%=\frac{1}{104}$ of \$9360=\$90.
7. 100%=100 times \$90=\$9000=par value.
8. \$100=par value of one share.
9. \$9000=par value of 9000÷100, or 90 shares.

III. ∴ 90=number of shares that can be bought.

I. When gold is at 105, what is the value of a gold dollar in currency?

II.
1. 105¢; or 105% in currency=100¢; or 100% in gold.
2. 1¢; or 1% in currency=$.95\frac{5}{21}$¢; or $.95\frac{5}{21}\%$ in gold.
3. 100¢; or 100% in currency=$95\frac{5}{21}$¢; or $95\frac{5}{21}\%$ in gold.

III. ∴ \$1 in currency is worth $95\frac{5}{21}$¢ in gold.

I. In 1864, the "greenback" dollar was worth only $35\frac{5}{7}$¢ in gold; what was the price of gold?

II.
1. $35\frac{5}{7}$¢; or $35\frac{5}{7}\%$ in gold=100¢; or 100% in currency.
2. 1¢; or 1% in gold=$\frac{1}{35\frac{5}{7}}$ of 100¢; or 100%=2.8¢; or 2.8% in currency.
3. 100¢; or 100% in gold=100 times 2.8¢; or 2.8%=280¢; or 280% in currency.

III. ∴ \$1 in gold was worth \$2.80 in currency.

(R. 3d p., p. 217, prob. 8.)

I. Bought stock at 10% discount, which rose to 5% premium and sold for cash. Paying a debt of \$33, I invested the balance in stock at 2% premium, which at par, left me \$11 less than at first; how much money had I at first?

II.
(1.) 100%=my money at first.
(2.) 100%=par value of stock.
(3.) 10%=discount.
(4.) $100\%-10\%=90\%$=market value.
(5.) $\therefore 90\%=100\%$, my money; because that is the amount invested.
(6.) $1\%=\frac{1}{90}$ of $100\%=1\frac{1}{9}\%$, and
(7.) $100\%=100$ times $1\frac{1}{9}\%=111\frac{1}{9}\%$=par value of the stock in terms of my money.
(8.)
1. $100\%=111\frac{1}{9}\%$
2. $1\%=1\frac{1}{9}\%$, and
3. $5\%=5$ times $1\frac{1}{9}\%=5\frac{5}{9}\%$=premium on stock in terms of my money.
4. $111\frac{1}{9}\%+5\frac{5}{9}\%=116\frac{2}{3}\%$ = what I received for the stock.
5. $116\frac{2}{3}\%-\$33$=amount invested in second stock.

(9.)
1. 100%=par value of second stock.
2. 2%=premium.
3. $100\%+2\%=102\%$=market value of second stock.
4. $116\frac{2}{3}\%-\$33$=market value of second stock
5. $\therefore 102\%=116\frac{2}{3}\%-\$33$,
6. $1\%=\frac{1}{102}$ of $(116\frac{2}{3}\%-\$33)=1\frac{22}{153}\%-\$.32\frac{6}{17}$,
7. $100\%=100$ times $(1\frac{22}{153}\%-\$.32\frac{6}{17})=114\frac{58}{153}\%-\$32\frac{6}{17}$=par value of second stock.

(10.) $114\frac{58}{153}\%-\$32\frac{6}{17}$=what I received for the second stock, since I sold them at par.
(11.) $\therefore 114\frac{58}{153}\%-\$32\frac{6}{17}=100\%-\$11$, by the last condition of the problem.
(12.) $14\frac{58}{153}\%=\$21\frac{6}{17}$,
(13.) $1\%=\dfrac{1}{14\frac{58}{153}}$ of $\$21\frac{6}{17}=\$1.485$, and
(14.) $100\%=100$ times $\$1.485=\$148.50$.

III. $\therefore$ I had $148.50 at first. (*R. H. A., p. 212, prob. 8.*)

I. Bought $8000 in gold at 110%, brokerage $\frac{1}{8}\%$; what did I pay for the gold in currency?

II.
1. 100%=par value of gold.
2. 110%=market value.
3. $\frac{1}{8}\%$=brokerage.
4. $110\%+\frac{1}{8}\%=110\frac{1}{8}\%$=entire cost.
5. $100\%=\$8000$,
6. $1\%=\frac{1}{100}$ of $\$8000=\$80$, and
7. $110\frac{1}{8}\%=110\frac{1}{8}$ times $\$80=\$8810$=cost of gold in currency.

III. $\therefore$ $8000 in gold costs $8810 in currency.

I. What income in currency would a man receive by investing $5220 in U. S. 5–20, 6% bonds at 116%, when gold is worth 105?

II.
- (1.) 100%=par value of the bonds.
- (2.) 116%=market value.
- (3.) $5220=market value.
- (4.) ∴ 116%=$5220.
- (5.) 1%=$\frac{1}{116}$ of $5220=$45.
- (6.) 100%=100 times $45=$4500=par value of bonds.
- (7.)
  1. 100%=$4500.
  2. 1%=$\frac{1}{100}$ of $4500=$45.
  3. 6%=6 times $45=$270=income in gold.
- (8.) $1.00 in gold=$1.05 in currency.
- (9.) $270 in gold=270 times $1.05=$283.50 in currency.

III. ∴ $283.50=income in currency.

(R. 3d p., p. 217, prob. 5.)

I. What % of income do U. S. $4\frac{1}{2}$% bonds, at 108, yield when gold is 105%?

II.
- (1.) 100%=amount invested in the bonds.
- (2.) 100%=par value of bonds.
- (3.) 108%=market value.
- (4.) ∴ 108%=100%, from (1).
- (5.) 1%=$\frac{1}{108}$ of 100%=$\frac{25}{27}$%, [of amount invested.
- (6.) 100%=100 times $\frac{25}{27}$%=$92\frac{7}{9}$%=par value in terms
- (7.)
  1. 100%=$92\frac{7}{9}$%.
  2. 1%=$\frac{1}{100}$ of $92\frac{7}{9}$%=$\frac{25}{27}$%,
  3. $4\frac{1}{2}$%=$4\frac{1}{2}$ times $\frac{25}{27}$%=$4\frac{1}{6}$%=income in gold.
- (8.) 100% in gold=105% in currency.
- (9.) 1% in gold=$\frac{1}{100}$ of 105%=$1\frac{1}{20}$% in currency.
- (10.) $4\frac{1}{6}$% in gold=$4\frac{1}{6}$ times $1\frac{1}{20}$%=$4\frac{3}{8}$% in currency.

III. ∴ Income in currency=$4\frac{3}{8}$%.

Note.—This is a general solution of the preceding problem. Since there is no special amount given, we represent the amount invested by 100%. The market value and the amount invested being the same, we have 108%=100% as shown in (4).

I. A man bought Michigan Central at 120, and sold at 124%; what % of the investment did he gain?

II.
1. 124%=selling price.
2. 120%=cost.
3. 124%−120%=4%=gain.
4. 120%=100% of itself.
5. 1%=$\frac{1}{120}$ of 100%=$\frac{5}{6}$%,
6. 4%=4 times $\frac{5}{6}$%=$3\frac{1}{3}$%=gain on the investment.

III. ∴ He gained $3\frac{1}{3}$% on the investment.

I. What sum invested in U. S. 5's of 1881, at 118, yielded an annual income of $1921 in currency, when gold was at 113?

II.
(1.) $1.13 in currency=$1 in gold.
(2.) $1 in currency=$\frac{1}{1.13}$ of $1=$$\frac{100}{113}$ in gold, and
(3.) $1921 in currency=1921 times $$\frac{100}{113}$=$1700=income in gold.
(4.) 100%=par value of the bonds.
(5.) 5%=income in gold.
(6.) $1700=income in gold.
(7.) ∴ 5%=$1700,
(8.) 1%=$\frac{1}{5}$ of $1700=$340, and
(9.) 100%=100 times $340=$34000=par value of the bonds.
(10.) 1. 100%=$34000,
2. 1%=$\frac{1}{100}$ of $34000=$340, and
3. 118%=118 times $340=$40120=market value, or amount invested.

III. ∴ $40120=amount invested.

SECOND SOLUTION.

II.
(1.) 100%=amount invested in currency.
(2.) 100%=par value.
(3.) 118%=market value.
(4.) ∴ 118%=100%, from (1.)
(5.) 1%=$\frac{1}{118}$ of 100%=$\frac{50}{59}$%, and
(6.) 100%=100 times $\frac{50}{59}$%=84$\frac{44}{59}$%=par value in terms of the investment.
(7.) 1. 100%=84$\frac{44}{59}$%.
2. 1%=$\frac{1}{100}$ of 84$\frac{44}{59}$%=$\frac{50}{59}$%, and
3. 5%=5 times $\frac{50}{59}$%=4$\frac{14}{59}$%=income in gold.
(8.) 1. 100% in gold=113% in currency,
2. 1% in gold=1$\frac{13}{100}$% in currency, and
3. 4$\frac{14}{59}$% in gold=4$\frac{14}{59}$ times 1$\frac{13}{100}$%=4$\frac{93}{118}$%=income in currency.
(9.) $1921=income in currency.
(10.) ∴ 4$\frac{93}{118}$%=$1921,
(11.) 1%=$\frac{1}{4\frac{93}{118}}$ of $1921=$401.20, and
(12.) 100%=100 times $401.20=$40120=amount invested in currency.

III. ∴ $40120=amount invested. (*R. 3d p., p. 218, prob. 8.*)

I. How many shares of stock bought at 95$\frac{1}{4}$%, and sold at 105, brokerage $\frac{1}{4}$% on each transaction, will yield an income of $925?

II.
1. 100%=par value of stock.
2. $95\frac{1}{4}\%$=market value of stock.
3. $\frac{1}{4}\%$=brokerage.
4. $95\frac{1}{4}+\frac{1}{4}\%=95\frac{1}{2}\%$=entire cost.
5. 105%=selling price+brokerage.
6. $\frac{1}{4}\%$=brokerage.
7. $105\%-\frac{1}{4}\%=104\frac{3}{4}\%$=selling price.
8. $104\frac{3}{4}\%-95\frac{1}{2}\%=9\frac{1}{4}\%$=gain.
9. \$925=gain.
10. $\therefore 9\frac{1}{4}\%=\$925$,
11. $1\%=\frac{1}{9\frac{1}{4}}$ of \$925=\$100, and
12. 100%=100 times \$100=\$10000=par value of stock.
13. \$100=par value one share.
14. \$10000=par value $10000\div100$, or 100 shares.

III. ∴ 100=number of shares. *(R. 3d p., p. 218, prob. 9.)*

I. If I invest all my money in 5% furnace stock salable at 75%, my income will be \$180; how much must I borrow to make an investment in 5% state stock selling at 102%, to have that income?

II.
(1.)
1. 100%=par value of furnace stock.
2. 5%=income.
3. \$180=income.
4. $\therefore 5\%=\$180$,
5. $1\%=\frac{1}{5}$ of \$180=\$36, and
6. 100%=100 times \$36=\$3600=par value of furnace stock.

(2.)
1. $100\%=\$3600$,
2. $1\%=\frac{1}{100}$ of \$3600=\$36, and
\$. 75%=75 times \$36=\$2700=market value of furnace stock.

(3.)
1. 100%=par value of state stock.
2. 6%=income.
3. \$180=income.
4. $\therefore 6\%=\$180$,
5. $1\%=\frac{1}{6}$ of \$180=\$30, and
6. 100%=100 times \$30=\$3000=par value of state stock.

(4.)
1. $100\%=\$3000$,
2. $1\%=\frac{1}{100}$ of \$3000=\$30, and
3. 102%=102 times \$30=\$3060=market value of state stock.

(5.) \$3060—\$2700=\$360=what I must borrow.

III. ∴ I must borrow \$360. *(R. H. A., p. 225, prob. 2.)*

I. When U. S. 4% bonds are quoted at 106, what yearly income will be received in gold from bonds that can be bought for \$4982?

II.
(1.) 100%=par value of the bonds.
(2.) 106%=market value.
(3.) \$4982=market value, or amount invested.
(4.) ∴ 106%=\$4982,
(5.) 1%=$\frac{1}{106}$ of \$4982=\$47, and
(6.) 100%=100 times \$47=\$4700.
(7.) 1. 100%=\$4700,
2. 1%=$\frac{1}{100}$ of \$4700=\$47, and
3. 4%=4 times \$47=\$188=income in gold.

III. ∴ \$188=income in gold. (*R. 3p., p. 218, prob. 11.*)

I. The sale of my farm cost me \$500, but I gave the proceeds to a broker, allowing him $\frac{1}{2}$%, to purchase railroad stock then in the market at 102%; the farm paid 5% income, equal to \$2075, but the stock will pay \$2025 more; what is the rate of dividend?

II.
(1.) 100%=value of the farm.
(2.) 5%=income on the farm.
(3.) \$2075=income on the farm.
(4.) ∴ 5%=\$2075,
(5.) 1%=$\frac{1}{5}$ of \$2075=\$415, and
(6.) 100%=100 times \$415=\$41500=value of farm.
(7.) \$41500—\$500=\$41000=amount invested in stock.
(8.) 1. 100%=par value of the stock.
2. 102%=market value, or amount invested.
3. $\frac{1}{2}$%=brokerage
4. 102%+$\frac{1}{2}$%=102$\frac{1}{2}$%=entire cost of stock.
5. ∴ 102$\frac{1}{2}$%=\$41000,
6. 1%=$\frac{1}{102\frac{1}{2}}$ of \$41000=\$400, and
7. 100%=100 times \$400=\$40000=par value of the railroad stock.
(9.) 1. \$2075+\$2025=\$4100=income on railroad stock.
2. \$40000=100%,
3. \$1%=$\frac{1}{40000}$ of 100%=$\frac{1}{400}$%, and
4. \$4100=4100 times $\frac{1}{400}$%=10$\frac{1}{4}$%=rate of dividend.

III. ∴ 10$\frac{1}{4}$%=rate of dividend. (*R. H. A., p. 224, prob. 4.*)

I. What must be paid for 6% bonds to realize an income of 8% on the investment?

II.
1. 100%=amount invested.
2. 6%=income on the par value of the bonds.
3. 8%=income on the investment.
4. ∴ 8% of investment=6% of the par value,
5. 1% of investment=$\frac{1}{8}$ of 6%=$\frac{3}{4}$% of the par value, and
6. 100% of investment=100 times $\frac{3}{4}$%=75% of par value.

III. ∴ Must pay 75% to make 8% on the investment.

Note.—It must be borne in mind that 100% of any quantity is the quantity itself. ∴ 100% of the amount invested equals the amount invested. It must also be remembered that the income on the par value is equal to the income on the investment. Suppose I buy a 500-dollar 6% bond for \$400. The income on the par value, or face of the bond is 6% of \$500, or \$30. But \$30 is $7\frac{1}{2}\%$ of \$400, the amount invested. Hence, the truth of step 4 in the above solution.

I. Which is the better investment, buying 9% stock at 25% advance, or 6% stock at 25% discount.

II. A.
(1.) 100%=amount invested in the 9% stock,
(2.) 100%=par value.
(3.) 25%=premium.
(4.) 100%+25%=125%=market value.
(5.) ∴ 125%=100%,
(6.) $1\%=\frac{1}{125}$ of $100\%=\frac{4}{5}\%$, and
(7.) 100%=100 times $\frac{4}{5}\%=80\%$=par value in terms of the investment.
(8.) 1. 100%=80%,
2. $1\%=\frac{1}{100}$ of $80\%=\frac{4}{5}\%$, and
3. 9%=9 times $\frac{4}{5}\%=7\frac{1}{5}\%$=income of 9% stock.

B.
(1.) 100%=amount invested in 6% stock.
(2.) 100%=par value of 6% stock.
(3.) 25%=discount.
(4.) 100%−25%=75%=market value.
(5.) ∴ 75%=100%.
(6.) $1\%=\frac{1}{75}$ of $100\%=1\frac{1}{3}\%$, and
(7.) 100%=100 times $1\frac{1}{3}\%=133\frac{1}{3}\%$=par value of the 6% stock in terms of the investment.
(8.) 1. $100\%=133\frac{1}{3}\%$,
2. $1\%=\frac{1}{100}$ of $133\frac{1}{3}\%=1\frac{1}{3}\%$, and
3. 6%=6 times $1\frac{1}{3}\%=8\%$=income of 6% stock.

III. ∴ The latter is the better investment, since it pays 8%−$7\frac{1}{5}\%$, or $\frac{4}{5}\%$ more income on the investment.

(*Greenleaf's N. A., p. 298, prob. 5.*)

I. If I pay $87\frac{1}{2}\%$ for railroad bonds that yield an annual income of 7%, what % do I get on my investment?

II.
(1.) 100%=investment.
(2.) 100%=par value.
(3.) $87\frac{1}{2}\%$=market value, or amount invested.
(4.) ∴ $87\frac{1}{2}\%=100\%$, from (1.)
(5.) $1\%=\frac{1}{87\frac{1}{2}}$ of $100\%=1\frac{1}{7}\%$, and
(6.) 100%=100 times $1\frac{1}{7}\%=114\frac{2}{7}\%$=par value in terms of the investment.
(7.) 1. $100\%=114\frac{2}{7}\%$,
2. $1\%=\frac{1}{100}$ of $114\frac{2}{7}\%=1\frac{1}{7}\%$, and
3. 7%=7 times $1\frac{1}{7}\%=8\%$=income on the investment.

III. ∴ 8%=income on the investment.

I. A banker owns $2\frac{1}{2}\%$ stocks at 10% below par, and 3% stocks at 15% below par. The income from the former is $66\frac{2}{3}\%$ more than from the latter, and the investment in the latter is \$11400 less than in the former; required the whole investment and income.

II.

(1.) 100%=investment in the former.

(2.) $100\%-\$11400$=investment in the latter.

(3.)
1. 100%=par value of the former.
2. 10%=discount of the former.
3. $100\%-10\%=90\%$=market value, or amount invested in former.
4. $\therefore 90\%=100\%$, from (1),
5. $1\%=\frac{1}{90}$ of $100\%=1\frac{1}{9}\%$, and
6. $100\%=100$ times $1\frac{1}{9}\%=111\frac{1}{9}\%$=par value of former in terms of the investment.

(4.)
1. $100\%=111\frac{1}{9}\%$,
2. $1\%=\frac{1}{100}$ of $111\frac{1}{9}\%=1\frac{1}{9}\%$, and
3. $2\frac{1}{2}\%=2\frac{1}{2}$ times $1\frac{1}{9}\%=2\frac{7}{9}\%$=income of former in terms of the investment.

(5.)
1. 100%=par value of the latter.
2. 15%=discount.
3. $100\%-15\%=85\%$=market value, or amount invested in the latter.
4. $\therefore 85\%=100\%-\$11400$, from (2),
5. $1\%=\frac{1}{85}$ of $(100\%-\$11400)=1\frac{3}{17}\%-\$134\frac{2}{17}$,
6. $100\%=100$ times $(1\frac{3}{17}\%-\$134\frac{2}{17})=117\frac{11}{17}\%-\$13411\frac{13}{17}$=par value of latter in terms of former.

(6.)
1. $100\%=117\frac{11}{17}\%-\$13411\frac{13}{17}$,
2. $1\%=\frac{1}{100}$ of $(117\frac{11}{17}\%-\$13411\frac{13}{17})=1\frac{3}{17}\%-\$134\frac{2}{17}$, and
3. $3\%=3$ times $(1\frac{3}{17}\%-\$134\frac{2}{17})=3\frac{9}{17}\%-\$402\frac{6}{17}$=income of latter in terms of the investment.

(7.)
1. 100%=income of the latter.
2. $100\%+66\frac{2}{3}\%=166\frac{2}{3}\%$=income of the former.
3. $2\frac{7}{9}\%$=income of the former.
4. $\therefore 166\frac{2}{3}\%=2\frac{7}{9}\%$,
5. $1\%=\frac{1}{166\frac{2}{3}}$ of $2\frac{7}{9}\%=\frac{1}{60}\%$, and
6. $100\%=100$ times $\frac{1}{60}\%=1\frac{2}{3}\%$=income of latter in terms of income of former.

(8.) $3\frac{9}{17}\%-\$402\frac{6}{17}$=income of the latter.

(9.) $\therefore 3\frac{9}{17}\%-\$402\frac{6}{17}=1\frac{2}{3}\%$.

(10.) $1\frac{44}{51}\%=\$402\frac{6}{17}\%$,

(11.) $1\%=\frac{1}{1\frac{44}{51}}$ of $\$402\frac{6}{17}=\$216$,

(12.) $100\%=100$ times $\$216=\$21600$=investment in former.

(13.) $100\%-\$11400=\$21600-\$11400=\$10200$=investment in latter.

(14.) $2\frac{7}{9}\%=2\frac{7}{9}$ times $\$216=\$600$=income of former.

(15.) $3\frac{9}{17}\%-\$402\frac{6}{17}=3\frac{9}{17}$ times $216—$402 $\frac{6}{17}$=$360= income of latter.

(16.) $21600+$10200=$31800=whole investment.

(17.) $600+$360=$960=whole income.

III. ∴ $31800=whole investment, and $960=whole income. (*R. H. A., p. 225, prob. 4.*)

I. W. F. Baird, through his broker, invested a certain sum of money in Philadelphia 6's at $115\frac{1}{2}\%$, and three times as much in Union Pacific 7's at $89\frac{1}{2}\%$, brokerage $\frac{1}{2}\%$ in both cases; how much was invested in each kind of stock if his annual income is $9920?

II.

(1.) 100%=amount invested in Philadelphia 6's.

(2.) 300%=amount invested in Union Pacific 7's.

(3.)
1. 100%=par value of Philadelphia 6's.
2. $115\frac{1}{2}\%$=market value.
3. $\frac{1}{2}\%$=brokerage.
4. $115\frac{1}{2}\%+\frac{1}{2}\%=116\%$=entire cost of Phila. 6's.
5. ∴ 116%=100%.
6. $1\%=\frac{1}{116}$ of $100\%=\frac{25}{29}\%$, and
7. 100%=100 times $\frac{25}{29}\%=86\frac{6}{29}\%$=par value of Philadelphia 6's in terms of investment.

(4.)
1. $100\%=86\frac{6}{29}\%$,
2. $1\%=\frac{1}{100}$ of $86\frac{6}{29}\%$, and
3. 6%=6 times $\frac{25}{29}\%=5\frac{5}{29}\%$=income of Philadelphia 6's in terms of investment.

(5.)
1. 100%=par value of Union Pacific 7's.
2. $89\frac{1}{2}\%$=market value.
3. $\frac{1}{2}\%$=brokerage.
4. $89\frac{1}{2}\%+\frac{1}{2}\%=90\%$=entire cost of Union Pacific 7's.
5. ∴ 90%=300%,
6. $1\%=\frac{1}{90}$ of $300\%=3\frac{1}{3}\%$, and
7. 100%=100 times $3\frac{1}{3}\%=333\frac{1}{3}\%$=par value of Union Pacific 7's.

(6.)
1. $100\%=333\frac{1}{3}\%$,
2. $1\%=\frac{1}{100}$ of $333\frac{1}{3}\%=3\frac{1}{3}\%$, and
3. 7%=7 times $3\frac{1}{3}\%=23\frac{1}{3}\%$=income of Union Pacific 7's in terms of investment.

(7.) $5\frac{5}{29}\%+23\frac{1}{3}\%=28\frac{44}{87}\%$=whole income.

(8.) $9920=whole income.

(9.) ∴ $28\frac{44}{87}\%$=$9920,

(10.) $1\%=\frac{1}{28\frac{44}{87}}$ of $9920=$348,

(11.) 100%=100 times $348=$34800=amount invested [in Philadelphia 7's.

(12.) 300%=300 times $348=$104500= amount invested in Union Pacific 7's.

III. ∴ $34800=amount invested in Philadelphia 6's, and
$104400=amount invested in Union Pacific 7's.
(*R. H. A., p. 225, prob. 6.*)

I. Thomas Reed bought 6% mining stock at $114\frac{1}{2}\%$, and 4% furnace stock at 112%, brokerage $\frac{1}{2}\%$; the latter cost him $430 more than the former, but yielded the same income; what did each cost him?

II.
(1.) 100%=amount invested in mining stock.
(2.) 100%+$430=amount invested in furnace stock.
(3.)
1. 100%=par value of mining stock.
2. $114\frac{1}{2}\%$=market value.
3. $\frac{1}{2}\%$=brokerage.
4. $114\frac{1}{2}\%+\frac{1}{2}\%=115\%$=entire cost.
5. ∴ 115%=100%, from (1),
6. $1\%=\frac{1}{115}$ of $100\%=\frac{20}{23}\%$, and
7. 100%=100 times $\frac{20}{23}\%=96\frac{22}{23}\%$=par value of mining stock in terms of investment.

(4.)
1. $100\%=96\frac{22}{23}\%$,
2. $1\%=\frac{1}{100}$ of $96\frac{22}{23}\%=\frac{20}{23}\%$, and
3. 6%=6 times $\frac{20}{23}\%=5\frac{5}{23}\%$=income of mining stock in terms of investment.

(5.)
1. 100%=par value of furnace stock.
2. 112%=market value.
3. $\frac{1}{2}\%$=brokerage.
4. $112\%+\frac{1}{2}\%=112\frac{1}{2}\%$=entire cost.
5. ∴ $112\frac{1}{2}\%=100\%+\$430$,
6. $1\%=\dfrac{1}{112\frac{1}{2}}$ of $(100\%+\$430)=\frac{8}{9}\%+\$3\frac{37}{45}$, and
7. 100%=100 times $(\frac{8}{9}\%+\$3\frac{37}{45})=88\frac{8}{9}\%+\$382\frac{2}{9}$= par value of furnace stock in terms of investm't.

(6.)
1. $100\%=88\frac{8}{9}\%+\$382\frac{2}{9}$,
2. $1\%=\frac{1}{100}$ of $(88\frac{8}{9}\%+\$382\frac{2}{9})=\frac{8}{9}\%+\$3\frac{37}{45}$, and
3. 4%=4 times $(\frac{8}{9}\%+\$3\frac{37}{45})=3\frac{5}{9}\%+\$15\frac{13}{45}$=income of furnace stock in terms of the investment.

(7.) ∴ $5\frac{5}{23}\%=3\frac{5}{9}\%+\$15\frac{13}{45}$, by the conditions of the problem,
(8.) $1\frac{137}{207}\%=\$15\frac{13}{45}$,
(9.) $1\%=\dfrac{1}{1\frac{137}{207}}$ of $\$15\frac{13}{45}=\$9.20$, and
(10.) 100%=100 times $9.20=$920=amount invested in mining stock.
(11.) 100%+$430=$1350=amount invested in furnace stock. (*R. H. A., p. 225, prob. 7.*)

III. ∴ $920=amount invested in mining stock, and
$1350=amount invested in furnace stock.

I. Suppose 10% state stock is 20% better in market than 4% railroad stock; if A.'s income be $500 from each, how much money has he paid for each, the whole investment bringing $6\frac{2}{333}$%?

II.

(1.)
1. 100%=par value of state stock.
2. 10%=income.
3. $500=income.
4. ∴ 10%=$500,
5. 1%=$\frac{1}{10}$ of $500=$50, and
6. 100%=100 times $50=$5000=par value of state stock.

(2.)
1. 100%=par value of railroad stock.
2. 4%=income.
3. $500=income.
4. ∴ 4%=$500,
5. 1%=$\frac{1}{4}$ of $500=$125, and
6. 100%=100 times $125=$12500=par value of railroad stock.

(3.) $5000=$\frac{2}{5}$ of $12500, *i. e.*, the face of state stock is $\frac{2}{5}$ of face of railroad stock.

(4.)
1. 100%=whole investment.
2. $6\frac{2}{333}$%=income of whole investment.
3. $500+$500=$1000=income of whole investment.
4. ∴ $6\frac{2}{333}$%=$1000,
5. 1%=$\frac{1}{6\frac{2}{333}}$ of $1000=$166.50, and
6. 100%=100 times $166.50=$16650=whole investment.

(5.)
1. 100%=investment in railroad stock.
2.
   - 1′. 40%=$\frac{2}{5}$ of 100%=investment in state stock, excluding the 20% excess.
   - 2′. 100%=40%,
   - 3′. 1%=$\frac{1}{100}$ of 40%=$\frac{2}{5}$%, and
   - 4′. 20%=20 times $\frac{2}{5}$% = 8%=excess of state stock over same amount of railroad stock.
3. 40%+8%=48%=investment in state stock.
4. 100%+48%=148%=whole investment.
5. $16650=whole investment.
6. ∴ 148%=$16650.
7. 1%=$\frac{1}{148}$ of $16650=$112.50,
8. 100%=100 times $112.50=$11250=investment in railroad stock.
9. 48%=48 times $112.50=$5400=investment in state stock

III. ∴ $11250=amount invested in railroad stock, and
$5400=amount invested in state stock.

(*R. H. A., p. 227, prob. 5.*)

PROBLEMS.

1. What could I afford to pay for bonds yielding an annual income of 9% to invest my money so as to realize 6% on the investment? *Ans.* 150%.

2. What must I pay for Chicago, Burlington & Quincy Railroad stock that bears 6% that my annual income on the investment may yield 5%? *Ans.* 120%.

3. Bought 75 shares N. Y., P. & O. Railroad stock at 105%, and sold them at $108\frac{1}{2}$%; how much did I gain in the transaction? *Ans.* $262.50.

4. How many shares of bank stock at 5% premium, can be bought for $7665? *Ans.* 73.

5. A broker bought stock at 4% discount, and, selling them at 3% premium, gained $1400; how many shares did he buy? *Ans.* 200.

6. At what price must I buy 15% stock that it may yield the same income as 4% stock purchased at 90%? *Ans.* $337\frac{1}{2}$%.

7. How much must I pay for New York 6's so that I may realize an income of 9%? *Ans.* $66\frac{2}{3}$%.

8. At what price must I buy 7% stock so that they may yield an income equivalent to 10% stocks at par? *Ans.* 70%.

9. What sum must I invest in U. S. 6's at 118% to secure an annual income of $1800? *Ans.* $35400.

10. Which is the more profitable, and how much, to invest $5000 in 6% stock purchased at 75%, or 5% stock purchased at 60%? *Ans*, The latter; $$16\frac{2}{3}$.

11. If a man who had $5000 U. S. 6's of 1881 should sell them at 115%, and invest in U. S. 10-40's purchased at 105%, would he gain or lose and how much? *Ans.* Loss $26.19.

12. When gold is at 120, what is a "greenback" dollar worth? *Ans.* $83\frac{1}{3}$¢.

13. Suppose the market value of 5% bank stock to be $11\frac{1}{9}$% higher than 8% corporation bonds; I realize 8% on my investment, and my income from each is $180; what did I invest in each? *Ans.* $$2923.07\frac{9}{13}$ in former, and $$1576.92\frac{4}{13}$ in latter.

14. A bought 5% railroad stock at $109\frac{1}{2}$%, and $4\frac{1}{2}$% pike stock at $107\frac{1}{2}$%, brokerage $\frac{1}{2}$%; the former cost $100 less than the latter but yielded the same income; what did each cost him?

Ans. $1100 cost of former, and $1200 cost of latter.

15. What rate % of income shall I receive if I buy U. S. 5's at a premium of 10%, and receive payment at par in 15 years?

Ans. $3\frac{31}{33}\%$.

16. Suppose the market value of 6% corporation stock is 20% less than 5% state stock; if my income be $1200 from each, what did I pay for each if the whole investment brings 6%?

Ans. $16000, and $24000.

17. I bought $2\frac{1}{2}\%$ stock at 80%, and $4\frac{1}{2}\%$ stock at 86%. The income on the former was $44\frac{4}{9}\%$ more than on the latter, but my investment is $22140 less in the latter than in the former; what do I realize on my investment? *Ans.* $3\frac{221}{293}\%$.

Hint.—Find the whole investment, and whole income as in the problem on page 75. Then find what % the whole income is of the whole investment.

18. Invested in U. S. $4\frac{1}{2}$'s at 105, brokerage $\frac{1}{2}\%$; $\frac{4}{5}$ as much in U. P. 6's at $119\frac{2}{3}$, brokerage $\frac{1}{3}\%$; and 3 times as much in N. Y. 7's, at $87\frac{1}{4}$, brokerage $\frac{1}{4}\%$. If my entire income is $1702, find my investment. (*School Visitor, vol. 12, p. 97.*) *Ans.* $25320.

19. A. paid $1075 for U. S. 5-20 6% bonds at $7\frac{1}{2}\%$ premium, interest payable semi-annually in gold. When the average premium on gold was 112%, did he make more or less than B. who invested an equal sum in railroad stock at 14% below par, which paid a semi-annual dividend of 4%?

Ans. A. makes $16.40 less than B. every six months.

20. I invested $4200 in railroad stock at 105, and sold it at 80%; how much must I borrow at 4% so that by investing all I have in 6% bonds at 8% interest, payable annually, I may retrieve my loss in one year? *Ans.* $18600.

V. INSURANCE.

1. ***Insurance*** is indemnity against loss or damage.

2. Insurance.

- 1. Property Insurance.
  - 1. Fire Insurance.
  - 2. Marine Insurance.
- 2. Personal Insurance.
  - 1. Life Insurance.
  - 2. Accident Insurance.
  - 3. Health Insurance.

3. ***Property Insurance*** is the indemnity against loss or damage of property.

4. ***Personal Insurance*** is indemnity against loss of life or health.

5. ***Fire Insurance*** is indemnity against loss by fire.

6. ***Marine Insurance*** is indemnity against the dangers of navigation

7. ***Life Insurance*** is a contract in which a company agrees, in consideration of certain premiums received, to pay a certain sum to the heirs or assigns of the insured at his death, or to himself if he attains a certain age.

8. ***Accident Insurance*** is indemnity against loss by accident.

9. ***Health Insurance*** is a weekly indemnity in case of sickness.

10. ***The Insurer***, or ***Underwriter***, is the party, or company, that undertakes the risk.

11. ***The Risk*** is the particular danger against which the insurer undertakes.

12. ***The Insured*** is the party protected against loss.

13. ***The Premium*** is the sum paid for insurance; and is a certain per cent. of the amount insured.

14. ***The Amount***, or ***Valuation***, is the sum for which the premium is paid.

I. My house is permanently insured for \$1800, by a deposit of ten annual premiums, the rate per year being $\frac{3}{4}\%$; how much did I deposit, and if, on terminating the insurance, I receive my deposit less 5% ; how much do I get?

II.
- (1.) 100%=\$1800,
- (2.) $1\%=\frac{1}{100}$ of \$1800=\$18, and
- (3.) $\frac{3}{4}\%=\frac{3}{4}$ times \$18=\$13.50=one annual deposit.
- (4.) \$135=10 times \$13.50=ten annual deposits.
- (5.)
  1. 100%=\$135,
  2. $1\%=\frac{1}{100}$ of \$135=\$1.35, and
  3. 5%=5 times \$1.35=\$6.75=deduction.
- (6.) \$135—\$6.75=\$128.25=what I received.

III. ∴ \$135=amount deposited, and \$128.25=amount received.

(R. H. A., p. 230, prob. 5.)

I. An insurance company having a risk of \$25000, at $\frac{9}{10}\%$, reinsured \$10000, at $\frac{4}{5}\%$, with another office. and \$5000, at 1%, with another; how much did it clear above what it paid?

II.
- (1.) $100\%=\$25000$,
- (2.) $1\%=\frac{1}{100}$ of $\$25000=\$250$, and
- (3.) $\frac{9}{10}\%=\frac{9}{10}$ times $\$250=\$225$=what the company received for taking the risk.
- (4.)
  1. $10000=amount the company reinsured at $\frac{4}{5}\%$.
  2. $100\%=\$10000$,
  3. $1\%=\frac{1}{100}$ of $\$10000=\$100$, and
  4. $\frac{4}{5}\%=\frac{4}{5}$ times $\$100=\$80$=what the company paid for reinsuring $10000.
- (5.)
  1. $5000=amount reinsured in another office at 1%.
  2. $100\%=\$5000$,
  3. $1\%=\frac{1}{100}$ of $\$5000=\$50$=what the company paid [for reinsuring $5000.
  4. $\$80+\$50=\$130$=what the company paid out.
  5. $\$225-\$130=\$95$=what it cleared.

III. ∴ $95=what the company cleared.

(R. H. A., p. 230, prob. 7.)

I. I took a risk at $4\frac{1}{2}\%$; reinsured $\frac{2}{5}$ of it at 2%, and $\frac{1}{4}$ of it at $2\frac{1}{2}\%$; what rate of insurance do I get on what is left?

II.
- (1.) 100%=whole risk.
- (2.) $1\frac{1}{2}\%$=premium.
- (3.)
  1. $40\%=\frac{2}{5}$ of 100%=amount reinsured at 2%.
  2. $100\%=40\%$,
  3. $1\%=\frac{1}{100}$ of $40\%=\frac{2}{5}\%$, and
  4. $2\%=2$ times $\frac{2}{5}\%=\frac{4}{5}\%$=amount I pay out for reinsuring $\frac{2}{5}$ of the risk.
- (4.)
  1. $25\%=\frac{1}{4}$ of 100%=second part reinsured.
  2. $100\%=25\%$.
  3. $1\%=\frac{1}{100}$ of $25\%=\frac{1}{4}\%$, and
  4. $2\frac{1}{2}\%=2\frac{1}{2}$ times $\frac{1}{4}\%=\frac{5}{8}\%$=amount I paid out for reinsuring $\frac{1}{4}$ of the risk.
- (5.) $\frac{4}{5}\%+\frac{5}{8}\%=1\frac{17}{40}\%$=amount of premiums paid out.
- (6.) $1\frac{1}{2}\%-1\frac{17}{40}\%=\frac{3}{40}\%$=amount of premium I had left.
- (7.) $40\%+25\%=65\%$=whole amount reinsured.
- (8.) $100\%-65\%=35\%$=risk left on which I received $\frac{3}{40}\%$ premium.
- (9.)
  1. $35\%=100\%$ of itself.
  2. $1\%=\frac{1}{35}$ of $100\%=2\frac{6}{7}\%$, and
  3. $\frac{3}{40}\%=\frac{3}{40}$ times $2\frac{6}{7}\%=\frac{3}{14}\%$=rate of premium

III. ∴ $\frac{3}{14}\%$=rate of insurance I receive.

(R. H. A., p. 231, prob. 6.)

Remark.—35% is the base and $\frac{3}{40}\%$ is the percentage, and we wish to know what per cent. $\frac{3}{40}\%$ is of 35%.

I. Took a risk at 2%; reinsured $10000 of it at $2\frac{1}{8}\%$ and $8000 at $1\frac{3}{4}\%$; my share of the premium was $207.50; what sum was insured?

II.
- (1.)
  1. $100\%=\$10000$,
  2. $1\%=\frac{1}{100}$ of $\$10000=\$100$, and
  3. $2\frac{1}{8}\%=2\frac{1}{8}$ times $\$100=\$212.50$=amount paid out on $10000 reinsured.
- (2.)
  1. $100\%=\$8000$,
  2. $1\%=\frac{1}{100}$ of $\$8000=\$80$, and
  3. $1\frac{3}{4}\%=1\frac{3}{4}$ times $\$80=\$140$=amount paid out on $8000 reinsured.
- (3.) $\$212.50+\$140=\$352.50$=whole amount paid out.
- (4.) $207.50=what I realize.
- (5.) $\therefore$ $\$352.50+207.50=\$560$=premium on whole risk.
- (6.) 100%=risk.
- (7.) 2%=premium.
- (8.) $560=premium.
- (9.) $\therefore$ $2\%=\$560$,
- (10.) $1\%=\frac{1}{2}$ of $\$560=\$280$, and
- (11.) $100\%=100$ times $\$280=\$28000$=risk.

III. $\therefore$ $28000=risk. (*R. H. A., p. 232, prob. 6.*)

I. I can insure my house for $2500 at $\frac{8}{10}\%$ premium annually, or permanently by paying down 12 annual premiums; which should I prefer, and how much will I gain by it if money is worth 6% per annum to me?

II.
- (1.) $100\%=\$2500$.
- (2.) $1\%=\frac{1}{100}$ of $\$2500=\$25$, and
- (3.) $\frac{8}{10}\%=\frac{8}{10}$ times $\$25=\$20$=one annual premium.
- (4.) $\$240=12$ times $20=twelve annual premiums.
- (5.)
  1. 100%=the amount that will produce $20 annually at 6%.
  2. 6%=interest.
  3. $20=interest.
  4. $\therefore$ $6\%=\$20$,
  5. $1\%=\frac{1}{6}$ of $\$20=\$3\frac{1}{3}$, and
  6. $100\%=100$ times $\$3\frac{1}{3}=\$333\frac{1}{3}$=the amount that will produce $20 annually at 6%.
- (6.) $\$333\frac{1}{3}+\$20=\$353\frac{1}{3}$=amount I would have to pay down by the former condition.
- (7.) $\therefore$ $\$353\frac{1}{3}-\$240=\$113\frac{1}{3}$=gain by the latter condition.

III. $\therefore$
- The latter is the better.
- $\$113\frac{1}{3}$=gain.

Remark.—In (6) we add $20, since a payment must be made immediately. $\$333\frac{1}{3}$ will not produce that sum until the end of the year.

I. The Mutual Fire Insurance Company insured a building and its stock for $\frac{2}{3}$ of its value, charging $1\frac{3}{4}\%$. The Union Insurance Company relieved them of $\frac{1}{4}$ of the risk, at $1\frac{1}{2}\%$. The building and stock being destroyed by fire, the Union lost \$49000 less than the Mutual; what amount of money did the owners of the building and stock lose?

II.

(1.) 100%=value of the building and stock.

(2.) $66\frac{2}{3}\%=\frac{2}{3}$ of 100%=amount insured.

(3.) $1\frac{3}{4}\%$=rate of insurance.

(4.)
1. $100\%=66\frac{2}{3}\%$,
2. $1\%=\frac{1}{100}$ of $66\frac{2}{3}\%=\frac{2}{3}\%$, and
3. $1\frac{3}{4}\%=1\frac{3}{4}$ times $\frac{2}{3}\%=1\frac{1}{6}\%$=what Mutual received from the owners of the building and stock.

(5.) $16\frac{2}{3}\%=\frac{1}{4}$ of $66\frac{2}{3}\%$=amount of which the Union relieved the Mutual.

(6.)
1. $100\%=16\frac{2}{3}\%$,
2. $1\%=\frac{1}{100}$ of $16\frac{2}{3}\%=\frac{1}{6}\%$, and
3. $1\frac{1}{2}\%=1\frac{1}{2}$ times $\frac{1}{6}\%=\frac{1}{4}\%$=what the Mutual paid the Union for taking the risk of $16\frac{2}{3}\%$.

(7.) $16\frac{2}{3}\%+1\frac{1}{6}\%=17\frac{5}{6}\%$=whole amount the Mutual received.

(8.) $66\frac{2}{3}\%+\frac{1}{4}\%=66\frac{11}{12}\%$=whole amount the Mutual paid out.

(9.) $\therefore 66\frac{11}{12}\%-17\frac{5}{6}\%=49\frac{1}{12}\%$ = amount the Mutual lost.

(10.) $16\frac{2}{3}\%$=amount the Union paid the Mutual.

(11.) $\frac{1}{4}\%$=amount the Union received from the Mutual.

(12.) $\therefore 16\frac{2}{3}\%-\frac{1}{4}\%=16\frac{5}{12}\%$=amount the Union lost.

(13.) $49\frac{1}{12}\%-16\frac{5}{12}\%=32\frac{2}{3}\%$=what the Mutual lost more than the Union.

(14.) \$49000=what the Mutual lost more than the Union.

(15.) $\therefore 32\frac{2}{3}\%$=\$49000,

(16.) $1\%=\frac{1}{32\frac{2}{3}}$ of \$49000=\$1500, and

(17.) 100%=100 times \$1500=\$150000=value of building and stock.

(18.) $66\frac{2}{3}\%=66\frac{2}{3}$ times \$1500=\$100000=amount insured.

(19.) $33\frac{1}{3}\%=33\frac{1}{3}$ times \$1500=\$50000=what the owners lost, it not being insured.

(20.)
1. 100%=\$100000,
2. $1\%=\frac{1}{100}$ of \$100000=\$1000, and
3. $1\frac{3}{4}\%=1\frac{3}{4}$ times \$1000=\$1750=what the owners paid the Mutual for insurance.

(21.) $\therefore$ \$50000 + \$1750 = \$51750=whole amount the owners lost.

III. $\therefore$ The owners of the building and stock lost \$51750.

PROBLEMS.

1. At $1\frac{3}{8}\%$, the premium for insuring my store was \$89.10; what was the amount of the insurance? *Ans.* \$6480.

2. The premium for insuring a tannery for $\frac{3}{4}$ of its value, at $1\frac{2}{3}\%$, was \$145.60; what was the value of the tannery? *Ans.* \$11648.

3. A store and its goods are worth \$6370. What sum must be insured, at 2%, to cover both property and premium? *Ans.* ——.

4. The premium for insuring \$9870 was \$690.90; what was the rate? *Ans.* 7%.

5. A merchant whose stock of goods was valued at \$30000, insured it for $\frac{3}{4}$ of its value, at $\frac{3}{4}\%$. In a fire he saved \$5000 of the goods. What was his loss? What was the loss of the insurance companies? *Ans.* ——.

6. A man paid \$180 for insuring his saw mill for $\frac{2}{3}$ of its value at 3%; what was the value of the mill? *Ans.* ——.

7. A house which has been insured for \$3500 for 10 years, at $\frac{2}{5}\%$ a year, was destroyed by fire; how much did the money received from the company exceed the cost of premiums? *Ans.* ——.

8. Took a risk on a house worth \$40000, at 2%; reinsured $\frac{1}{2}$ of it for $2\frac{1}{4}\%$, and $\frac{1}{4}$ of it at $2\frac{1}{2}\%$; in each case the amount covers premium; how much do I gain? *Ans.* \$99.558.

9. Took a risk at $1\frac{3}{4}\%$; reinsured $\frac{2}{3}$ of it at $2\frac{1}{4}\%$; my share of the premium was \$43; what was the amount of the risk? *Ans.* \$17200.

10. Took a risk at $2\frac{1}{4}\%$; reinsured $\frac{1}{2}$ of it at a rate equal to 3% of the whole, by which I lost \$37.50. What was the value of the risk? *Ans.* \$5000.

CHAPTER XII.

INTEREST.

I. SIMPLE INTEREST.

1. ***Interest*** is money paid by the borrower to the lender for the use of money.

2. ***The Principal*** is the sum of money for which interest is paid.

3. ***The Rate*** of interest is the rate per cent. on $1 for a certain time.

4. ***The Time*** is the period during which the money is on interest.

5. ***The Amount*** is the sum of the principal and interest.

6. ***Simple Interest*** is interest on the principal only.

7. ***Legal Interest*** is at the rate fixed by law.

8. ***Usury*** is interest at a rate greater than that allowed by law.

Let P=the principal,
r=the interest on $1 for one year,
$R=1+r$=amount of $1 for one year,
n=the number of years,
A=amount of P for n years,
Pr=simple interest on P for a year,
Pnr=simple interest on P for n years.
$P+Pnr=P(1+nr)$=amount of P for n years.
A=amount of P for n years.

$$\therefore A=P+Pnr=P(1+nr) \ldots . (\text{I.});$$

$$\therefore P=\frac{A}{1+nr} \ldots . (\text{II.});$$

$$\therefore Pnr=A-P.$$

$$\therefore P=\frac{A-P}{nr}=\frac{\textit{Interest}}{nr} \ldots . (\text{III.});$$

$$\therefore r=\frac{A-P}{Pn} \ldots . (\text{IV.}); \text{ and}$$

$$\therefore n=\frac{A-P}{Pr} \ldots . (\text{V.}).$$

When any three of the quantities A, P, n, r are given, the fourth may be found.

CASE I.

Given {Principal, Rate, and Time,} to find the interest. Formula, $I=Prn$.

I. Find the interest of \$300 for two years at 6%.

By formula,

Interest Prn=\$300×.06×2=\$36.

By 100% method.

II.
1. 100%=\$300,
2. 1%=$\frac{1}{100}$ of \$300=\$3, and
3. 6%=6 times \$3=\$18=interest for one year.
4. \$36=2 times \$18=interest for 2 years.

III. ∴ \$36=interest on \$300 at 6% for 2 years.

CASE II.

Given {Principal, Rate, and Interest,} to find the time. Formula, $n=\frac{A-P}{Pr}$.

I. In what time, at 5%, will \$60 amount to \$72?

By formula,

$$n=\frac{A-P}{Pr}=\frac{\$72-\$60}{\$60\times.05}=4 \text{ years.}$$

By 100% method.

II.
1. \$72=amount.
2. \$60=principal.
3. \$72−\$60=\$12=interest for a certain time.
4. 100%=\$60,
5. 1%=$\frac{1}{100}$ of \$60=\$$\frac{3}{5}$, and
6. 5%=5 times \$$\frac{3}{5}$=\$3=interest for one year.
7. \$12=interest for 12÷3, or 4 years.

III. ∴ \$60 at 5% will amount to \$72 in 4 years.

CASE III.

Given {Principal, Time, and Interest,} to find the rate. Formula, $r=\frac{A-P}{Pn}$.

I. I borrowed \$600 for two years and paid \$48 interest; what rate did I pay?

By formula,

$$r=\frac{A-P}{Pn}=\frac{I}{Pn}=\frac{\$48}{\$600\times2}=.04=4\%.$$

By 100% method.

II.
1. \$48=interest for 2 years.
2. \$24=$\frac{1}{2}$ of \$48=interest for 1 year.
3. \$600=100%,
4. \$1=$\frac{1}{600}$ of 100%=$\frac{1}{6}$%, and
5. \$24=24 times $\frac{1}{6}$%=4%.

III. ∴ I paid 4% interest.

CASE IV.

Given {Time, Rate, and Interest.} to find the principal. Formula, $P=\frac{A-P}{nr}=\frac{I}{nr}$.

I. The interest for 3 years, at 9%, is $21.60; what is the principal?

By formula,

$$P=\frac{A-P}{nr}=\frac{I}{nr}=\frac{\$21.60}{3\times.09}=\$80.$$

By 100% method.

II.
1. $21.60=interest for 3 years.
2. $7.20=$\frac{1}{3}$ of $21.60=interest for 1 year.
3. 100%=principal.
4. 9%=interest for 1 year.
5. $7.20=interest for 1 year.
6. ∴ 9%=$7.20,
7. 1%=$\frac{1}{9}$ of $7.20=$.80, and
8. 100%=100 times $.80=$80=principal.

III. ∴ $80=the principal.

CASE V.

Given {Time, Rate, and Amount} to find the principal. Formula, $P=\frac{A}{1+nr}$.

I. What principal will amount to $936 in 5 years, at 6%?

By formula,

$$P=\frac{A}{1+nr}=\frac{\$936}{1+5\times.06}=\$720.$$

By 100% method.

II.
1. 100%=principal.
2. 6%=interest for 1 year.
3. 30%=5 times 6%=interest for 5 years.
4. 100%+30%=130%=amount.
5. $936=amount.
6. ∴ 130%=$936,
7. 1%=$\frac{1}{130}$ of $936=$7.20, and
8. 100%=100 times $7.20=$720=principal.

III. ∴ $720=the principal that will amount to $936 in 5 years at 6%.

I. In what time will any sum quadruple itself at 8%?

II.
1. 100%=principal. Then
2. 400%=the amount.
3. ∴ 400%—100%=300%=interest.
4. 8%=interest for 1 year.
5. 300%=interest for 300÷8, or $37\frac{1}{2}$ years.

III. ∴ Any principal will quadruple itself in $37\frac{1}{2}$ years at 8%.

II. TRUE DISCOUNT.

1. ***Discount*** on a debt payable by agreement at some future time, is a deduction made for "cash," or present payment; and arises from the consideration of the *present worth* of the debt.

2. ***Present Worth*** is that sum of money which, put on interest for the given time and rate, will amount to the debt at its maturity.

3. ***True Discount*** is the difference between the present worth and the whole debt.

Since P will amount to A in n years, P may be considered equivalent to A due at the end of n years.

∴ P may be regarded as the present worth of a given future sum A.

$$\therefore P=\frac{A}{1+nr}$$

I. Find the present worth of $590, due in 3 years, the rate of interest being 6%.

By formula,

$$P=\frac{A}{1+nr}=\frac{\$590}{1+3\times.06}=\$500.$$

By 100% method.

II.
1. 100%=present worth.
2. 6%=interest on present worth for 1 year.
3. 18%=3×6%=interest for 3 years.
4. 100%+18%=118%=amount, or debt.
5. $590=debt.
6. ∴ 118%=$590,
7. 1%=$\frac{1}{118}$ of $590=$5, and
8. 100%=100 times $5=$500=present worth.

III. ∴ $500=present worth of $590 due in 3 years at 6%.

I. A merchant buys a bill of goods amounting to \$2480; he can have 4 months credit, or 5% off for cash: if money is worth only 10% to him, what will he gain by paying cash?

II.
- (1.) 100%=present worth of the debt.
- (2.) 10%=interest on present worth for 1 year.
- (3.) $3\frac{1}{3}$%=interest for 4 months.
- (4.) 100%+$3\frac{1}{3}$%=$103\frac{1}{3}$%=amount of present worth, which equals the debt, by definition.
- (5.) \$2480=the debt.
- (6.) ∴ $103\frac{1}{3}$%=\$2480,
- (7.) 1 %=$\frac{1}{103\frac{1}{3}}$ of \$2480=\$24, and
- (8.) 100%=100 times \$24=\$2400=present worth.
- (9.) \$2480—\$2400=\$80=true discount.
- (10.)
  1. 100%=\$2480.
  2. 1%=$\frac{1}{100}$ of \$2480=\$24.80,
  3. 5%=5 times \$24.80=\$124=trade discount, or discount for cash.
- (11.) ∴ \$124—\$80=\$44=his gain by paying cash.

III. ∴ He would gain \$44 by paying cash.

(R. 3d p., p. 258, prob. 10.)

Remark.—It is clear that \$2480—\$124,=\$2356 would pay for the goods cash. If the merchant had this sum of money on hand, it would, in 4 months, at 10%, produce \$78.53$\frac{1}{3}$ interest. But if he pays his debt he will make \$124. Hence he will gain \$124—\$78.53$\frac{1}{3}$=\$45.46$\frac{2}{3}$.

III. BANK DISCOUNT.

1. ***Bank Discount*** is simple interest on the face of a note, calculated from the day of discount to the day of maturity, and paid in advance.

2. ***The Proceeds*** of a note is the amount which remains after deducting the discount from the face.

CASE I.

Given {Face of note, Rate, and Time,} to find the discount and proceeds.

Formulæ, $\begin{cases} D=F\times r\times n \\ P=F-D. \end{cases}$

I. What is the bank discount of \$770 for 90 days, at 6%?

By formula,

$$D=F\times r\times n=\$770\times .06\times\frac{(90+3)}{360}=\$11.935.$$

By 100% method.

II.
1. 100% = \$770;
2. 1% = $\frac{1}{100}$ of \$770 = \$7.70, and
3. 6% = 6 times \$7.70 = \$46.20 = discount for 1 year.
4. \$11.935 = $\frac{93}{360}$ of \$46.20 = discount for 93 days.

III. ∴ \$11.935 = bank discount on \$770 for 90 days at 6%.

CASE II.

Given { Proceeds, Time, and Rate, } to find the face of the note.

Formula, $F=\frac{P}{1-rn}$.

I. For what sum must a note be made, so that when discounted at a bank, for 90 days, at 6% the proceeds will be \$393.80?

By formula,

$$F=\frac{P}{1-rn}=\frac{\$393.80}{1-.06\times\frac{93}{360}}=\$400.$$

By 100% method.

II.
1. 100% = face of the note.
2. 6% = discount for one year.
3. $1\frac{11}{20}$% = $\frac{93}{360}$ of 6% = discount for 93 days.
4. 100% − $1\frac{11}{20}$% = $98\frac{9}{20}$% = proceeds.
5. \$393.80 = proceeds.
6. ∴ $98\frac{9}{20}$% = \$393.80,
7. 1% = $\frac{1}{98\frac{9}{20}}$ of \$393.80 = \$4, and
8. 100% = 100 times \$4 = \$400 = face of the note.

III. ∴ \$400 = face of the note.

CASE III.

Given rate of bank discount, to find the corresponding rate of interest.

Formula, rate of $I=\frac{r}{1-rn}$.

I. What is the rate of interest when a 60 day note is discounted at 8% per annum?

By formula,

$$\text{rate of } I=\frac{r}{1-rn}=\frac{.08}{(1-\frac{63}{360}\times .08)}=.08\tfrac{56}{493}=8\tfrac{56}{493}\%.$$

By 100% method.

II.
1. $100\% =$ face of note.
2. $8\% =$ discount for 1 year.
3. $1\frac{2}{5}\% = \frac{63}{360}$ of $8\% =$ discount for 63 days.
4. $100\% - 1\frac{2}{5}\% = 98\frac{3}{5}\% =$ proceeds.
5. $98\frac{3}{5}\% = 100\%$ of itself.
6. $1\% = \frac{1}{98\frac{3}{5}}$ of $100\% = \frac{500}{493}\%$, and.
7. $8\% = 8$ times $\frac{500}{493}\% = 8\frac{56}{493}\% =$ rate of interest.

III. ∴ The rate of interest on a 60 day note discounted at 8% per annum $= 8\frac{56}{493}\%$.

CASE IV.

Given the rate of interest, to find the corresponding rate of discount. Formula, $r = \frac{\text{rate of } I.}{1 + n \times \text{rate of } I.}$

I. What is the rate of discount on a 60 day note which yields 10% interest?

By formula,

$$r = \frac{\text{rate of } I.}{1 + n \times \text{rate of } I.} = \frac{.10}{1 + \frac{63}{360} \times .10} = .09\frac{337}{407} = 9\frac{337}{407}\%.$$

By 100% method.

II.
1. $100\% =$ proceeds.
2. $10\% =$ interest on proceeds for 1 year.
3. $1\frac{3}{4}\% = \frac{63}{360}$ of $10\% =$ interest on proceeds for 63 days.
4. $100\% + 1\frac{3}{4}\% = 101\frac{3}{4}\% =$ face of note.
5. $101\frac{3}{4}\% = 100\%$ of itself.
6. $1\% = \frac{1}{101\frac{3}{4}}$ of $100\% = \frac{400}{407}\%$, and
7. $10\% = 10$ times $\frac{400}{407}\% = 9\frac{337}{407}\%$.

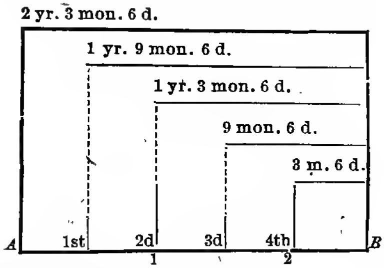
III. ∴ The rate of discount $= 9\frac{337}{407}\%$.

Note.—It must be borne in mind that the interest on the proceeds is equal to the discount on the face of the note.

IV. ANNUAL INTEREST.

1. *Annual Interest* is the simple interest of the principal and each year's interest from the time of its accruing until settlement.

I. No interest having been paid, find the amount due Sept. 7, 1877, on a note of \$500, dated June 1, 1875, with interest at 6%, payable semi-annually.



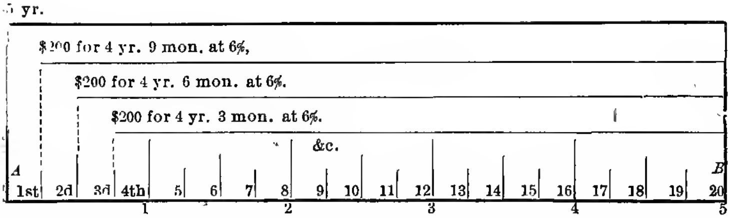
II. (1.) 100%=\$500, 1877—9—7
(2.) 1%=$\frac{1}{100}$ of \$500=\$5, and 1875—6—1
(3.) 6%=6 times \$5=\$30=simple interest for 1 year. 2—3—6
(4.) \$68=$2\frac{4}{15}$ times \$30=simple interest for 2 years, 3 months, 6 days.
(5.) \$15=$\frac{1}{2}$ of \$30=semi-annual interest.
(6.)
1. 100%=\$15,
2. 1%=$\frac{1}{100}$ of \$15=\$.15, and
3. 6%=6 times \$.15=\$.90=interest on one semi-annual interest for 1 year.
4. \$3.885=$4\frac{19}{60}$ times \$.90=interest on one semi-annual interest for the sum of the periods each draws int.

(7.) ∴ \$500+\$68+\$3.885=\$571.885=amount of the note.

III. ∴ \$571.885=amount of the note.

Explanation.—At the end of six months there is \$15 interest due; and, since it was not paid at that time, it drew interest from that time to the time of settlement, which is 1 yr. 9 mon. 6 da. At the end of the next six months, or at the end of the first year, there is another \$15 due; and, since it was not paid at that time, it drew interest from that time to the time of settlement, which is 1 yr. 3 mon. 6 da. In like manner, the third semi-annual interest drew interest for 9 mon. 6 da., and the fourth for 3 mon. 6 da. This is the same as one semi-annual interest drawing interest for the sum of 1 yr. 9 mon. 6 da., 1 yr. 3 mon. 6 da., 9 mon. 6 da., 3 mon. 6 da. In the diagram, the line *A B* represents 2 yr. 3 mon. 6 day., *A* 1 represents the first year the note run, and 1-2 represents the second year the note run. Between *A* and 1 is a small mark that denotes the semi-annual period; also one between 1 and 2. By such diagrams, the time for which to compute interest on the simple interest may be easily found.

I. The interest of U. S. 4% bonds is payable quarterly in gold; granting that the income from them might be immediately invested, at 6%, what would the income on 20 1000-dollar bonds amount to in 5 years, with gold at 105?



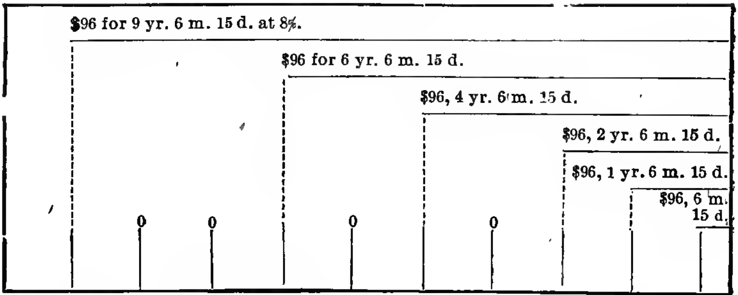
II.
(1.) $1000=par value of one bond.
(2.) $20000=par value of 20 bonds.
(3.) 100%=$20000,
(4.) 1%=$\frac{1}{100}$ of $20000=$200, and
(5.) 4%=4 times $200=$800=income for one year.
(6.) $4000=5 times $800=income for five years.
(7.) $200=$\frac{1}{4}$ of $800=interest due at the end of first quarter, and which draws interest to time of settlement.
(8.)
1. 100%=$200,
2. 1%=$\frac{1}{100}$ of $200=$2, and
3. 6%=6 times $2=$12=interest on quarterly interest for one year.
4. $570=47$\frac{1}{2}$ times $12=interest on quarterly interest for the sum of 4$\frac{3}{4}$ yr.+4$\frac{1}{2}$ yr.+4$\frac{1}{4}$ yr.+. . . . . +$\frac{1}{4}$ yr., or 47$\frac{1}{2}$ years.

(9.) ∴ $4000+$570=$4570=income of bonds in gold.
(10.) $1.00 in gold=$1.05 in currency.
(11.) $4570 in gold=4570 times $1.05=$4798.50 in currency.

III. ∴ The bonds yield $4798.50 in currency.

Explanation.—It must be borne in mind that the quarterly interest, $200, is put on interest at 6% as soon as it is due. At the end of the first quarter there is $200 due which draws interest at 6% for the remaining time, 4 years, 9 months. The second quarterly interest is due at the end of six months and draws interest for the remaining time, 4 years 3 months, and so on with the remaining quarterly payments. This is the same as one quarterly payment drawing interest for the sum of 4$\frac{3}{4}$ yr.+4$\frac{1}{2}$ yr. +4$\frac{1}{4}$ yr.+etc., or 47$\frac{1}{2}$ years.

I. What was due on a note of $1200, dated January 16, 1883, and due Aug. 1, 1892, and bearing interest at 8%, payable annually, if the 2, 3, 5, and 7th years' interest were paid?



II.
(1.) 100%=$1200.
(2.) 1%=$\frac{1}{100}$ of $1200=$12, and
(3.) 8%=8 times $12=$96=simple int. for one year.
(4.) $480=5×$96=five simple interests.
(5.) $48=$\frac{1}{2}$ of $96=interest for 6 months.
(6.) $4=$\frac{1}{12}$ of $48=interest for 15 days.
(7.) ∴ $532=simple interest unpaid.
(8.)
1. 100%=$96.
2. 1%=$\frac{1}{100}$ of $96=$.96, and
3. 8%=8 times $.96=$7.68=interest on one year's simple interest.
4. $193.92=25$\frac{1}{4}$ times $7.68=interest on year's simple interest for 9 yr. 6 mon. 15 da.,+6 yr. 6 mon. 15 da.,+4 yr. 6 mon. 15 da.,+2 yr. 6 mon. 15 da., +1 yr. 6 mon. 15 da.,+6 mon. 15 da., or 25 yr. 3 mon.

(9.) ∴ $532+$193.92=$725.92=amount of interest due.
(10) $1200+725.92 = $1925.92 = amount due July 1, 1892.

III. ∴ $1925.92=whole amount due Aug. 1, 1892.

V. COMPOUND INTEREST.

1. ***Compound Interest*** is interest on a principal formed by adding interest to a former principal.

Let P=principal on compound interest.
r=rate,
$R=(1+r)$=amount of one dollar for 1 year.
$P(1+r)=PR$=amount of P dollars for 1 year.
$P(1+r)^2=PR^2$=amount of P dollars for 2 years.
$P(1+r)^3=PR^3$=amount of P dollars for 3 years.
$P(1+r)^n=PR^n$=amount of P dollars for n years.

Let A=amount of P dollars in n years, and
I=the compound interest of P dollars for n years.

Then $I=PR^n-P$. . . . . I.
$A=PR^n$. . . . . . . II.
∴ $P=\frac{A}{R^n}$. . . . . III.

$\therefore R=\sqrt[n]{\frac{A}{P}}$ . . . . IV. Applying logarithms to $R^n=\frac{A}{P}$,

n log. R=log. A—log. P, whence

$n=\frac{\text{log. } A - \text{log. } P}{\text{log. } R}$ . . . V.

When compound interest is payable semi-annually.

$P\ (1+\frac{r}{2})$=amount of P dollars for $\frac{1}{2}$ year.

$P\ (1+\frac{r}{2})^2$=amount of P dollars for 1 year.

$P\ (1+\frac{r}{2})^{2n}$=amount of P dollars for n years.

$\therefore A=P\ (1+\frac{r}{2})^{2n}$, when payable semi-annually.

When compound interest is payable quarterly,

$P\ (1+\frac{r}{4})$=amount of P dollars for $\frac{1}{4}$ year.

$P\ (1+\frac{r}{4})^2$=amount of P dollars for $\frac{1}{2}$ year.

$P\ (1+\frac{r}{4})^3$=amount of P dollars for $\frac{3}{4}$ year.

$P\ (1+\frac{r}{4})^4$=amount of P dollars for 1 year.

$P\ (1+\frac{r}{4})^{4n}$=amount of P dollars for n years.

$\therefore A=P\ (1+\frac{r}{4})^{4n}$.

When the interest is payable monthly,

$A=P\ (1+\frac{r}{12})^{12n}$.

When the interest is payable q times a year,

$A=P\ (1+\frac{r}{q})^{qn}$.

CASE I.

Given {Principal, Rate, and Time,} to find the compound interest and amount.

Formulæ, $\begin{cases} I=PR^n-P, \\ A=PR^n. \end{cases}$

I. Find the compound interest and amount of \$500 for 3 years at 6%.

By formulæ,

$A=PR^n=\$500\times(1+.06)^3=\$595.508$, and

$I=PR^n-P=\$500\times(1+.06)^3-\$500=\$95.508$.

Remark.—In compound interest, the 100% method becomes very tedious.

By 100% method.

II.
(1.) 100%=\$500,
(2.) 1%=$\frac{1}{100}$ of \$500=\$5,
(3.) 6%=6 times \$5=\$30=interest for 1 year.
(4.) \$500+\$30=\$530=amount, or principal for the second year.
(5.)
1. 100%=\$530,
2. 1%=$\frac{1}{100}$ of \$530=\$5.30,
3. 6%=6 times \$5.30=\$31.80=interest for second year.
4. \$530+\$31.80=\$561.80=amount, or principal for the third year.

(6.) 1. $100\%=\$561.80$,
2. $1\%=\frac{1}{100}$ of $\$561.80=\$5.618$, and [year.
3. $6\%=6$ times $\$5.618=\$33.708$=interest for third
4. $561.80+\$33.708=\$595.508$=amount at end of the third year.

(7.) $\$595.508-\$500=\$95.508$=compound interest.

III. ∴ $95.508=compound interest, and
$595.508=compound amount.

CASE II.

Given Principal, Rate, and Compound Interest, to find the time.

Formula, $n=\frac{\log. A-\log. P}{\log. R}$.

I. In what time will $8000 amount to $12000, at 6% compound interest?

By formula,

$$n=\frac{\log. A-\log. P}{\log. R}=\frac{\log. 12000-\log. 8000}{\log. 1.06}=$$

$\frac{4.079181-3.903090}{.025306}=6$ yr. 11 mon. 15 da. We may solve the problem thus: $\$8000(1.06)^n=\$12000$, whence $(1.06)^n=12000\div 8000=1.50$. Referring to a table of compound amounts and passing down the column of 6%, we find this amount between 6 years and 7 years.

The amount for 6 years is 1.4185191; the amount for required time is 1.50. ∴ There is a difference of 1.50—1.4185191, or .0814809. The difference for the year between 6 and 7 is .0851112. .0851112=amount for the whole period between 6 and 7, .0814809=amount for $\frac{814809}{851112}$ of the period or, 11 mon. 15 da. ∴ The whole time=6 yr. 11 mon. 15 da.

CASE III.

Given Principal, Compound Interest or Amount, and Time, to find the rate.

Formula, $r=\sqrt[n]{\frac{P}{A}}-1$.

I. At what rate, by compound interest, will $1000 amount to $1593.85 in 8 years?

By formula,

$$r=\sqrt[n]{\frac{A}{P}}-1=\sqrt[8]{\frac{\$1593.85}{\$1000}}-1=.06=6\%.$$

CASE IV.

Given $\left\{\begin{array}{l}\text{Compound Interest or Amount}\\ \text{Time, and}\\ \text{Rate,}\end{array}\right\}$ to find the principal.

Formulæ, $P=\begin{cases}\dfrac{A}{R^n}, \text{or}\\[2ex] \dfrac{I}{R^n-1}\end{cases}$

I. What principal, at compound interest will amount to 27062.85 in 7 years at 4%?

By formula,

$$P=\frac{A}{R^n}=\frac{27062.85}{(1.04)^7}=\$20565.54$$

CHAPTER XIII.

ANNUITIES.

1. ***An Annuity*** is a sum of money payable at yearly, or other regular intervals.

2. ***Annuities*** $\begin{cases}\text{1. Perpetual, or}\\ \text{2. Limited;}\\ \text{3. Certain, or}\\ \text{4. Contingent.}\end{cases}$

3. ***A Perpetual Annuity*** is one that continues forever.

4. ***A Limited Annuity*** ceases at a certain time.

5. ***A Certain Annuity*** begins and ends at fixed times.

6. ***A Contingent Annuity*** begins or ends with the happening of a contingent event.

7. ***An Immediate Annuity*** is one that begins at once.

8. ***A Deferred Annuity*** is one that does not begin immediately.

9. ***The Final or Forborne*** value of an annuity is the amount of the whole accumulated debt and interest, at the time the annuity ceases.

10. ***The Present Value*** of an annuity is that sum, which, put at interest for the given time and given rate, will amount to the initial value.

11. ***The Initial Value*** of an annuity is the value of a deferred annuity at the time it commences.

CASE I.

Given { Annuity, Time, and Rate, } to find the initial value of a perpetuity.

I. What is the initial value of a perpetual annuity of $300 a year, allowing interest at 6%?

II.
1. 100%=initial value.
2. 6%=interest for 1 year.
3. $300=interest for 1 year.
4. ∴ 6%=$300.
5. 1%=$\frac{1}{6}$ of $300=$50, and
6. 100%=100 times $50=$5000=initial value.

III. ∴ Initial value=$5000. (*R. H. A., p. 310, prob. 1.*)

I. What is the initial value of a perpetual leasehold of $2500 a year payable quarterly, interest payable semi-annually at 6%; 6% payable annually; 6% payable quarterly?

II.

A.
1. Let S=the annuity. Then S=the amount due in 3 months.
2. $S+S(1+\frac{r}{4})$=amount due in 6 months.
3. ∴ $A=S+S(1+.01\frac{1}{2})=\$625+\$625(1.01\frac{1}{2})=$ 1259.37\frac{1}{2}$=amount due at the end of 6 months.
4. 100%=initial value.
5. 3%=semi-annual annuity.
6. 1259.37\frac{1}{2}$=semi-annual annuity.
7. ∴ 3%=1259.37\frac{1}{2}$.
8. 1%=$\frac{1}{3}$ of 1259.37\frac{1}{2}$=$419.7916$\frac{2}{3}$, and
9. 100%=100 times 419.7916\frac{2}{3}$=initial value.

B.
1. Let S=amount due in 3 months. Then
2. $S+S(1+\frac{r}{4})$=amount due in 6 months, [and
3. $S+S(1+\frac{r}{4})+S(1+\frac{2r}{4})$=amount due in 9 months,
4. $S+S(1+\frac{r}{4})+S(1+\frac{2r}{4})+S(1+\frac{3r}{4})$= amount due in 1 year.
5. ∴ $A=\$625+\$625(1+\frac{.06}{4})+\$625(1+\frac{.12}{4})+\$625$-[$(1+\frac{.18}{4})$=$2556.25.
6. 100%=initial value.
7. 6%=annuity.
8. $2556.25=annuity.
9. ∴ 6%=$2556.25.
10. 1%=$\frac{1}{6}$ of $2556.25=$426.0416$\frac{2}{3}$, and
11. 100%=100 times 426.0416\frac{2}{3}$=$42604.16$\frac{2}{3}$=initial [value.

C.
1. 100%=initial value.
2. 1$\frac{1}{2}$%=quarterly annuity.
3. $625=quarterly annuity.
4. ∴ 1$\frac{1}{2}$%=$625.
5. 1%=$\frac{1}{1\frac{1}{2}}$ of $625=$416.6666$\frac{2}{3}$, and
6. 100%=100 times 416.6666\frac{2}{3}$=$41666.66$\frac{2}{3}$.

III. ∴ $\begin{cases} \text{Initial value of A=\$41979.16}\frac{2}{3}, \\ \text{Initial value of B=\$42604.16}\frac{2}{3}, \text{ and} \\ \text{Initial value of C=\$41666.66}\frac{2}{3}. \end{cases}$

(*R. H. A., p. 310, prob. 5.*)

CASE II.

Given $\begin{cases} \text{Annuity,} \\ \text{Interval,} \\ \text{Rate, and} \\ \text{Time the perpetuity is deferred,} \end{cases}$ to find the present value of a deferred perpetuity.

Let S=the annuity, r=the rate, and $R=1+r$. Then by Case I., the initial value of S is $S\div r$. To find the present value of the initial value, we use formula III., compound interest. ∴ $P=\frac{S}{r(1+r)^t}=\frac{S}{R^t(R-1)}$ in which t is the time the perpetuity is deferred.

I. Find the present value of a perpetuity of $250 a year, deferred 8 years, allowing 6% interest.

By formula,

$$P=\frac{S}{R^t(R-1)}=\frac{\$250}{(1+.06)^8(1+.06-1)}=\frac{\$250}{.06(1.06)^8}=\$2614.22.$$

By 100% method.

II. (1.) 100%=initial value.
(2.) 6%=annuity.
(3.) $250=annuity.
(4.) ∴ 6%=$250.
(5.) 1%=$\frac{1}{6}$ of $250=$41$\frac{2}{3}$, and
(6.) 100%=100 times 41\frac{2}{3}$=$4166.66$\frac{2}{3}$=initial value.
(7.)
1. 100%=present value of 4166.66\frac{2}{3}$ due in 8 years at 6%.
2. 159.38481%=$(1.06)^8\times100\%$=compound amount of the present value for 8 yr. at 6%.
3. ∴ 159.38481%=4166.66\frac{2}{3}$,
4. 1%=$\frac{1}{159.38481}$ of 4166.66\frac{2}{3}$=$26.1422, and
5. 100%=100 times $26.1422 = $2614.22 = present value.

III. ∴ The present value of a perpetuity of $250 a year deferred 8 years at 6% interest=$2614.22.

I. Find the present value of an estate which, in 5 years, is to pay $325 a year forever; interest 8%, payable semi-annually.

By formula,

$$P=\frac{S}{[(1+\frac{r}{2})^2-1]R^t}=\frac{\$325}{[(1.04)^2-1](1.04)^{10}}=\frac{\$325}{.0816(1.04)^{10}}=$$
$2690.67.

By 100% method.

II.
- (1.) 100%=initial value.
- (2.) 4%=amount due in 6 months.
- (3.) 4%+(1.04)×4%=8.16%=amount due in 1 year.
- (4.) $325=amount due in 1 year.
- (5.) ∴ 8.16%=$325,
- (6.) 1%=$\frac{1}{8.16}$ of $325=$39.828431, and
- (7.) 100%=100 times $39.828431= $3982.8431 = initial value.
- (8.)
  1. 100%=present value of $3982.8431.
  2. 148.024428%=$(1.04)^{10}$×100%=compound amount of 100% for 5 yr. at 8%.
  3. ∴148.024428%=$3982.8431,
  4. 1%=$\frac{1}{148.024428}$ of $3982.8431=$26.9067, and
  5. 100% = 100 times $26.9067 = $2690.67 = present value.

III. ∴ $2690.67=present value of the estate.

(*R. H. A., p. 311, prob. 4.*)

Explanation.—The initial value is a sum of money which placed on interest at 8% payable semi-annually will produce $325 per year. But 8% payable semi-annually is the same as 8.16% payable annually. Hence 8.16% is the annual payment. But $325 is the annual payment. Hence 8.16%=$325, from which we find that $3982.8431 is the initial value, or the amount that will produce $325 per year. Then the present value of a sum of money that will pay $325 is $3982.8431 if the payments are to begin at once, but $3982.8431÷$(1.04)^{10}$ if the payments are not to begin until the end of 5 years.

CASE III.

Given {Rate, Annuity, Time to run, and Interval} to find the present valve of an annuity certain.

(*a*) Let P denote the present value. The amount of P for n years$=PR^n=A$.

Let S=the payment, or amount due the first year.

$S+SR$=the amount due the second year.

$S+SR+SR^2$=the amount due the third year.

$S+SR+SR^2+SR^3$ = the amount due the fourth year.

$S+SR+SR^2+SR^3+\ldots\ldots+SR^{n-1}$=amount due the nth year.

$\therefore A=S+SR+SR^2+SR^3+\ldots\ldots+SR^{n-1}$ .... (1)

$AR=SR+SR^2+SR^3+SR^4+\ldots\ldots SR^n$ ..... (2), by multiplying (1) by R.

$AR-A=SR^n-S$...(3), by subtracting (1) from (2).

$\therefore A=\frac{S(R^n-1)}{R-1}$ .... (4.) But $PR^n=A$.

$$\therefore\ PR^n=\frac{S(R^n-1)}{(R-1)}\ldots\ldots(5.),\text{ whence}$$

$$P=\frac{S(R^n-1)}{R^n(R-1)}=\frac{S}{R-1}\times\frac{R^n-1}{R^n}\ldots\ldots(6).$$

(*b.*) When the annuity is to begin at a certain time, and then to continue a certain time.

Let p=the number of years the annuity is deferred, and q= the number of years the annuity continues. Then

$P'=\frac{S}{R-1}\times\frac{R^{p+q}-1}{R^{p+q}}$=the present value of an annuity S, for the time $(p+q)$ years, and

$P''=\frac{S}{R-1}\times\frac{R^p-1}{R^p}$=the present value of an annuity S, for p years.

$$\therefore\ P=P'-P''=\frac{S}{R-1}\times\frac{R^{p+q}-1}{R^{p+q}}-\frac{S}{R-1}\times\frac{R^p-1}{R^p}=\frac{S}{R-1}\left(\frac{R^{p+q}-1}{R^{p+q}}-\frac{R^p-1}{R^p}\right)=\frac{S}{R-1}\left[1-\frac{1}{R^{p+q}}-\left(1-\frac{1}{R^p}\right)\right]=\frac{S}{R-1}\left(\frac{1}{R^p}-\frac{1}{R^{p+q}}\right)=\frac{S}{R-1}\times\frac{R^q-1}{R^{p+q}}\ldots(7.)$$

I. Find the present value of an annuity of \$250, payable annually for 30 years at 5%.

Given S, n, and r.

By formula,

$$P=\frac{S}{R-1}\times\frac{R^n-1}{R^n}=\frac{\$250}{.05}\times\frac{(1.05)^{30}-1}{(1.05)^{30}}=\$3843.1135.$$

By 100% method.

II.
- (1.) 100%=initial value.
- (2.) 5%=annuity.
- (3.) \$250=annuity.
- (4.) ∴ 5%=\$250,
- (5.) 1%=$\frac{1}{5}$ of \$250=\$50, and
- (6.) 100%=100 times \$50%=\$5000=initial value of an *immediate* perpetuity of \$250 per year.
- (7.)
  1. 100%=present value of an annuity *deferred* 30 years.
  2. 432.19424%=$(1.05)^{30}\times100\%$=amount of present value for 30 years.
  3. ∴ 432.19424%=\$5000,
  4. 1%=$\frac{1}{432.19424}$ of \$5000=\$11.568865, and
  5. 100%=100 times \$11.568865=\$1156.8865=present value of annuity of \$250 deferred 30 years.
- (8.) ∴ \$5000—\$1156.8865=\$3843.1135=present value of an annuity continuing 30 years.

III. ∴ \$3843.1135=present value of an annuity of \$250, payable annually for 30 years, *Remark.*—Since \$5000 is the initial value which, in this case, is also the present value of an immediate perpetual annuity, or perpetuity of \$250, and \$1156.8865 the present value of an annuity of \$250 deferred 30 years, \$5000—\$1156.8865=\$3843.1135= the present value of an annuity of \$250 continuing for 30 years at 5%.

I. Find the present value of an annuity of \$826.50, to commence in 3 years and run 13 years, 9 months, interest 6%, payable semi-annually.

Given S=\$826.50, r=.06, p=3 years, and $q=13\frac{3}{4}$ years.

When interest is payable semi-annually, $R=(1+\frac{r}{2})^2$.

By formula (7),

$$P=\frac{S}{R-1}\times\frac{R^q-1}{R^{(p+q)}}=\frac{\$826.50}{.0609}\times\frac{(1.0609)^{13\frac{3}{4}}-1}{(1.0609)^{16\frac{3}{4}}}=\$6324.69.$$

By 100% method.

II.
- (1.) 100%=initial value.
- (2.) 3%=amount due in 6 months.
- (3.) 3%+3% (1.03)=6.09%=amount due in 1 year.
- (4.) \$826.50=amount due in 1 year.
- (5.) ∴ 6.09%=\$826.50,
- (6.) 1%=$\frac{1}{6.09}$ of \$826.50=\$135.712643, and
- (7.) 100% = 100 times \$135.712643 = \$13571.2643= initial value
- (8.)
  1. 100%=present value of a perpetuity of \$826.50 deferred 3 years.
  2. 119.40523%=$(1.0609)^2$ times 100%=amount of present value for 3 years.
  3. ∴ 119 40523%=\$13571.2643,
  4. 1%=$\frac{1}{119.40523}$ of \$13571.2643=\$113.6586,
  5. 100%=100 times \$113.6586=\$11365.86=present value of such a perpetuity deferred 3 years
- (9.)
  1. 100%=present value of such a perpetuity deferred $16\frac{3}{4}$ years.
  2. 269.212027%=$(1.0609)^{16\frac{3}{4}}$ times 100% =amount of present value for $16\frac{3}{4}$ years
  3. ∴ 269.212027%=\$13571.2643,
  4. 1%=$\frac{1}{269.212027}$ of \$13571.2643=\$50.4117,
  5. 100%=100 times \$50.4117 = \$5041.17 = present value of such a perpetuity deferred $16\frac{3}{4}$ years.
- (10.) ∴ \$11365.86—\$5041.17=\$6324.69=present value of an annuity of \$826.50 deferred 3 years and continuing $13\frac{3}{4}$ years.

III. ∴ \$6324.69=present value of \$826.50, etc.

If the annuity is to begin in p years and continue forever, the formula,

$$P=\frac{S}{R-1}\times\frac{R^q-1}{R^{p+q}} \text{ becomes } P=\frac{S}{R^p(R-1)}.$$

For, since $P=\frac{S}{R-1}\left[\left(1-\frac{1}{R^{p+q}}\right)-\left(1-\frac{1}{R^p}\right)\right]$, if $q=\infty$, the quantity

$1-\frac{1}{R^{p+q}}=1-\frac{1}{R^{p+\infty}}=1-\frac{1}{\infty}=1-0$, approaches 1 as its limit,

and we have $P=\frac{S}{R-1}\left[(1-0)-\left(1-\frac{1}{R^p}\right)\right]=\frac{S}{(R-1)R^p}$.

I. Find the present value of a perpetual annuity of \$1000 to begin in 3 years, at 4% interest.

By formula,

$$P=\frac{S}{(R-1)R^p}=\frac{\$1000}{.04\times(1.04)^3}=\$22224.92=\text{present value of the annuity.}$$

By 100% method.

II.
- (1.) 100%=initial value.
- (2.) 4%=annuity.
- (3.) \$1000=annuity.
- (4.) ∴ 4%=\$1000,
- (5.) 1%=$\frac{1}{4}$ of \$1000=\$250, and
- (6.) 100%=100 times \$250=\$25000=initial value of [\$1000.
- (7.)
  1. 100%=present value.
  2. 112.4864% = $(1.04)^3$ times 100% = amount of present value for 3 years at 4%.
  3. ∴ 112.4864%=\$25000,
  4. 1%=$\frac{1}{112.4864}$ of \$25000=\$222.2492, and
  5. 100%=100 times \$222.2492=\$22224.92=present value.

III. ∴ \$22224.92=present value of an annuity of \$1000 to begin in 3 years at 4%.

CASE IV.

Given {Annuity, Rate, Interval, and Time to run,} to find the final or forborne value.

Let S=amount due first year.

$S+SR$=amount due second year.

$S+SR+SR^2$=amount due third year.

$S+SR+SR^2+SR^3$=amount due the fourth year.

$S+SR+SR^2+SR^3+\ldots\ldots+SR^{n-1}$= amount due the nth year.

Let A=amount due the nth year.

$\therefore A=S+SR+SR^2+SR^3+\ldots\ldots+SR^{n-1}\ldots$ (1).

$AR=SR+SR^2+SR^3+SR^4+\ldots\ldots+SR^n\ldots$ ..(2), by multiplying (1) by R. [from (2).

$\therefore AR-A=SR^n-S\ldots\ldots$ (3), by subtracting (1).

$\therefore A=\dfrac{S(R^n-1)}{R-1}\ldots\ldots$ (4.)

I. A pays $25 a year for tobacco; how much better off would he have been in 40 years if he had invested it at 10% per annum?

By formula,

$A=\dfrac{S}{R-1}\times(R^n-1)=\dfrac{\$25}{.10}\times[(1.10)^{40}-1]=\$11064.8139.$

By 100% method.

II.
1. 100%=initial value.
2. 10%=annuity.
3. $25=annuity.
4. ∴ 10%=$25,
5. 1%=$\frac{1}{10}$ of $25=$2.50, and
6. 100%=100 times $2.50=$250=initial value.
7. $44.2592556=[(1.10)^{40}-1]×$1=compound interest of $1 for 40 yr. at 10%.
8. ∴ $11064.8139=44.2592556×$250=compound int. of [$250 for 40 yr. at 10%.

III. ∴ He would be $11064.8139 better off.

Remark.—$250 placed on interest at 10% will produce $25 per year. If this interest be put on interest at 10%, instead of spending it for tobacco, it will amount to $11064.8139 in 40 years. This would be a very sensible and profitable investment for every young man to make, who is a slave to the pernicious habit.

I. An annuity, at simple interest 6%, in 14 years, amounted to $116.76; what would have been the difference, had it been at compound interest 6%?

II.
(1.) 100%=initial value, or the principal that would produce the annuity.
(2.) 6%=annuity for 1 year.
(3.) 84%=14×6%=annuity for 14 years.
(4.)
 1. 100%=6%,
 2. 1%=$\frac{1}{100}$ of 6%=$\frac{3}{50}$%, and
 3. 6%=6 times $\frac{3}{50}$%=$\frac{9}{25}$%=interest on annuity for [1 year.
 4. 32.76%=91 times $\frac{9}{25}$%=interest on annuity for (1+2+3+ ..... +14), or 91 years.
(5.) 84%+32.76%=116.76%=whole amount of the annuity.
(6.) $116.76=whole amount of the annuity.
(7.) ∴ 116.76%=$116.76,
(8.) 1%=$\frac{1}{116.76}$ of $116,76=$1, and
(9.) 100%=100 times $1=$100=initial value.
(10.) 6%=6 times $1=$6=annuity.

(11.) $1.260904=[(1.06)^{14}-1]\times\$1$= compound interest on $1 for 14 yrs. at 6%.
(12.) $126.0904=1.260904×$100= compound interest on $100 for 14 yrs. at 6%.
(13.) ∴ $126.0904—$116.76=$9.3304=difference.

III. ∴ The difference=$9.3304.

CASE V.

Given { Final Value or Present Value, Rate, and Time to run, } to find the **annuity.**

Solving $P=\frac{S}{R-1}\times\frac{R^n-1}{R^n}$ with respect to S and we have $S=\frac{P(R-1)R^n}{R^n-1}=rP\times\frac{R^n}{R^n-1}$ . . . . . . (1). If A=the final or forborne value, by the formula in the last case, we have $A=\frac{S}{R-1}\times R^n-1$. Solving this with respect to S, we have.

$$S=\frac{(R-1)A}{R^n-1}=\frac{rA}{R^n-1} \quad \ldots\ldots (2).$$

I. How much a year should I pay, to secure **$15000 at the** end of 17 years, interest 7%?

By formula (2),

$$S=\frac{rA}{R^n-1}=\frac{.07\times\$15000}{(1.07)^{17}-1}=\$486.38.$$

By 100% method.

II.
(1.) 100%=annuity.
(2.) 7%=annuity.
(3.) ∴ 7%=100%,
(4.) 1%=$\frac{1}{7}$ of 100%=$14\frac{2}{7}$%, and
(5.) 100%=100 times $14\frac{2}{7}$%=$1428\frac{4}{7}$%=initial value.
(6.)
  1. 100%=present value of $1428\frac{4}{7}$% due in 17 years.
  2. 315.8815%=amount of present value for 17 years.
  3. ∴ 315.8815%=$1428\frac{4}{7}$%,
  4. 1%=$\frac{1}{315.8815}$ of $1428\frac{4}{7}$%=4.522591%, and
  5. 100%=100 times 4.522591%=452.2591%=present value.
(7.) ∴ $1428\frac{4}{7}$% — 452.2591% = 976.3223% = present value of an annuity running 17 years.
(8.) 3.1588152%=$(1.07)^{17}$ times 1%=amount of 1% for 17 years.
(9.) 3084.0217%=$(1.07)^{17}$ times 976.3223%=amount of 976.3223% for 17 years at 7%.
(10.) $15000=amount, or final value.
(11.) ∴ 3084.0217%=$15000.
(12.) 1%=$\frac{1}{3084.0217}$ of $15000=$4.8638, and
(13.) 100%=100×$4.8638=$486.38=annuity.

III. ∴ I must pay $486.38.

CASE VI.

Given $\left\{\begin{array}{l}\text{Annuity,}\\ \text{Present Value of the Annuity, and}\\ \text{Rate,}\end{array}\right\}$ to find time it runs.

In formula (6), Case III., we have $P=\frac{S}{R-1}\times\frac{R^n-1}{R^n}$, whence $\frac{R^n-1}{R^n}=\frac{P(R-1)}{S}$, or $1-\frac{1}{R^n}=\frac{Pr}{S}$, $\frac{1}{R^n}=1-\frac{Pr}{S}=\frac{S-Pr}{S}$.

$\therefore R^n=\frac{S}{S-Pr}$ . . . . . (1). Applying logarithms,

$n\log. R=\log.\left(\frac{S}{S-Pr}\right)$.

$\therefore n=\log.\left(\frac{S}{S-Pr}\right)\div\log. R=\frac{\log. S-\log.(S-Pr)}{\log. R}$ . . . . . (2).

I. In how many years can a debt of $1,000,000, drawing interest at 6%, be discharged by a sinking fund of $80,000 per year?

By formula (2),

$$n=\frac{\log. S-\log.(S-Pr)}{\log. R}=\frac{\log. 80000-\log.(80000-1000000\times.06)}{\log. 1.06}$$
$$=\frac{\log. 80000-\log. 20000}{\log. 1.06}=\frac{4.903090-4.301030}{.025306}=\frac{.602060}{.025306}=23.857$$
years.

By another method.

Assume $1,000,000 to be the present value of an annuity of $80000 a year. Then $12.50 may be considered as the present value of $1 for the same time and rate. By reference to a table of present worths $12.50, which is 1000000÷80000, will be found to be between 23 and 24 years.

Note.—A table of present worths may be computed by formula (6.), Case III., in which put $S=\$1$.

I. In what time will a debt of $10000, drawing interest at 6%, be paid by installments of $1000 a year.

By formula,

$$n=\frac{\log. S-\log.(S-Pr)}{\log. R}=\frac{\log. 1000-\log.(1000-10000\times.06)}{\log. 1.06}$$
$$=\frac{3-2.602060}{.025306}=15.725 \text{ years}=15 \text{ yr. } 8 \text{ mo. } 21 \text{ da.}$$

By another method.

Assume $10000 to be the present value of an annuity of $1000 a year. Then $10000÷1000=$10=the present value of $1 for the same time and rate. By referring to a table of present worth we find this amount between 15 and 16 years. ∴ The time is 15 years +

The compound amount of $10000 for 15 yr. at 6%= $23965.58
The final value of $1000 for 15 years at 6%= $23275.97
Balance= $ 689.61

This balance, $689.61, will require a fraction of a year to discharge it. The part of a year required, will be such a fraction of a year as the amount of $689.61 for the *fraction* of a year is of $1000.

6% of $689.61 for the *fraction* of a year=$41.3766× *fraction* of a year.

∴ $689.61+$41.3766× *fraction* of a year=the amount of $689.61 for the *fraction* of a year. This amount divided by $1000, a yearly payment, will give the *fraction*.

$$\therefore \frac{\$689.61+\$41.3766\times fraction}{\$1000}=fraction, \text{ whence}$$

$\$689.61+\$41.3766\times fraction=\$1000\times fraction$

$\therefore \$1000\times fraction-\$41.3766\times fraction=\$689.61$, or

$\therefore \$958.628\times fraction=\$689.61$.

$$\therefore fraction=\frac{689.61}{958.628}=8 \text{ months, 19 days.}$$

∴ The whole time=15 yr. 8 mon. 19 da.

CASE VII.

Given { Annuity, Time to Run, and Present Value of an Annuity, } to find the rate of interest.

From the formula (6), Case III, $P=\frac{S}{R-1}\times\frac{R^n-1}{R^n}$, we obtain $\frac{R^n-1}{rR^n}=\frac{P}{S}$ .... (1). This is the simplest expression we can obtain for the rate as the equation is of the nth degree and can not be solved in a general manner.

I. If an immediate annuity of $80, running 14 yr., sells for $650, what is the rate?

By formula,

$$\frac{R^n-1}{rR^n}=\frac{P}{S}=\frac{\$650}{\$80}=8.125, \text{ or}$$

$\frac{1}{r}\frac{1}{(1+r)^{14}}=8.125$. Solving this equation by the method of Double Position, we find $r=8\%+$.

By another method.

$650÷$80=8.125. By referring to a table of present worths of $1, corresponding to 14 years, we find it to be between 8 and 9%.

PROBLEMS.

1. What is the amount of an annuity of $1000, forborne 15 years, at 3½% compound interest? *Ans.* $19295.125

2. What will an annuity of $30 payable semi-annually, amount to, in arrears 3 years at 7% compound interest?
Ans.——

3. What is the present worth of an annuity of $500 to continue 40 years at 7%? *Ans.*——

4. What is the present worth of an annuity of $200, for 7 years, at 5%? *Ans.* $1152.27.

5. A father presents to his daughter, for 8 years, a rental of $70 per annum, payable yearly, and the reversion for 12 years succeeding to his son. What is the present value of the gift to his son, allowing 4% compound interest? *Ans.*——

6. A yearly pension which has been forborne for 6 years, at 6%, amounts to $279; what was the pension? *Ans.* $480.03.

7. A perpetual annuity of $100 a year is sold for $2000; at what rate is the interest reckoned? *Ans.*——

8. A perpetual annuity of $1000 beginning at the end of 10 years, is to be purchased. If interest is reckoned at 3½%, what should be paid for it? *Ans.*——

9. If a clergyman's salary of $700 per annum is 6 years in arrears, how much is due, allowing compound interest at 6%?
Ans. $4882.72.

10. A soldier's pension of $350 per annum is 5 years in arrears; allowing 5% compound interest, what is due him?
Ans. $1933.97.

11. What annual payment will meet principal and interest of a debt of $2000 due in 4 year a 8% compound interest? *Ans.*—

12. What is the present worth of a perpetual annuity of $600 at 6% per annum? *Ans.* $10000.

13. What is the present value of an annuity of $1000, to commence at the end of 15 years, and continue forever, at 6% per annum? *Ans.* $6954.40.

14. To what sum will an annuity of $120 for 20 years amount at 6% per annum? *Ans.* $4414.27.

15. A debt of $8000 at 6% compound interest, is discharged by eight equal annual installments; what was the annual installment? *Ans.* $1288.286.

CHAPTER XIV.

MISCELLANEOUS PROBLEMS,

INVOLVING THE VARIOUS APPLICATIONS OF PERCENTAGE.

I. Sold a cow for \$25, losing $16\frac{2}{3}\%$; bought another and sold it at a gain of 16%; I neither gained nor lost on the two; what was the cost of each?

II. A.
1. 100%=cost of the first cow.
2. $16\frac{2}{3}\%$=loss.
3. $100\%-16\frac{2}{3}\%=83\frac{1}{3}\%$=selling price.
4. \$25=selling price.
5. ∴ $83\frac{1}{3}\%$=\$25,
6. $1\%=\frac{1}{83\frac{1}{3}}$ of \$25=\$.30, and
7. 100%=100 times \$.30=\$30=cost of first cow.
8. \$30—\$25=\$5, loss on the first cow, and gain on second cow.

B.
1. 100%=cost of second cow.
2. 16%=gain.
3. \$5=gain.
4. ∴ 16%=\$5.
5. $1\%=\frac{1}{16}$ of \$5=\$.3125, and
6. 100%=100 times \$.3125=\$31.25=cost of second cow.

III. . . \$30=cost of first cow, and \$31.25=cost of second cow.

Remark.—Since I lost \$5 on the first cow, and neither gained nor lost on the two, I must have gained \$5 on the second cow. ∴ 16%=\$5.

I. There have been two equal annual payments on a 6% note of \$175, given 2 years ago this day. The balance is \$154.40; what was each payment?

II.
(1.) 100%=a payment.
(2.) 100%=\$175,
(3.) $1\%=\frac{1}{100}$ of \$175=\$1.75, and
(4.) 6%=6 times \$1.75=\$10.50=interest for 1 year.
(5.) \$175+\$10.50=\$185.50=amount before paying the payment.
(6.) \$185.50—100% =amount left after paying the payment.
(7.)
1. 100%=\$185.50—100%,
2. $1\%=\frac{1}{100}$ of (\$185.50—100%)=\$1.855—1%, and
3. 6%=6 times (\$1.855—6%)=\$11.13—6%=interest for second year.
4. \$185.50—100%+\$11.13—6%=\$196.63—106% = amount before paying the last payment.
5. \$196.63 — 106% — 100% = \$196.63 — 206% = amount left after paying the last payment.

(8.) $154.40=amount after paying the last payment.
(9.) ∴ $154.40=$196.63—206%.
(10.) 206%=$196.63—$154.40=$42.23,
(11.) 1%=$\frac{1}{206}$ of $42.23=$.205, and
(12.) 100%=100 times $.205=$20.50=the payment.

III. ∴ $20.50=the payment.

Remark.—In this solution we are obliged to use the minus sign, —, which is no obstacle to the student of algebra, but to the student of arithmetic it may seem insurmountable. To avoid this sign, we give another solution.

II.
(1.) 100%=the payment. Then
(2.) $154.40+100%=amount of the debt at the end of of the second year.
(3.) 100%=principal that produced this amount.
(4.) 6%=interest.
(5.) 106%=amount.
(6.) ∴ 106%=$154.40+100%, [and
(7.) 1%=$\frac{1}{106}$ of ($154.40+100%)=$1.4566$\frac{2}{53}$+$\frac{50}{53}$%,
(8.) 100%=100 times (1.4566\frac{2}{53}$+$\frac{50}{53}$%) = 145.66\frac{2}{53}$+94$\frac{18}{53}$%=amount at end of the first year after paying off the payment.
(9.) 145.66\frac{2}{53}$+94$\frac{18}{53}$%+100%= 145.66\frac{2}{53}$ + 194$\frac{18}{53}$% =amount before paying off the payment= amount at end of first year.
(10.)
1. 100%=the principal that produced it.
2. 6%=interest.
3. 106%=amount.
4. ∴ 106%=145.66\frac{2}{53}$+194$\frac{18}{53}$%,
5. 1%=$\frac{1}{106}$ of (145.66\frac{2}{53}$+194$\frac{18}{53}$%)=$1.37$\frac{1167}{2809}$ + 1.83$\frac{953}{2809}$%, and
6. 100% = 100 times (1.37\frac{1167}{2809}$ + 1.83$\frac{953}{2809}$%) = 137\frac{1167}{2809}$+183$\frac{953}{2809}$%=the amount at first.

(11.) $175=the amount at first.
(12.) ∴ 137\frac{1167}{2809}$+183$\frac{953}{2809}$%=$175.
(13.) 183$\frac{953}{2809}$%=37\frac{1742}{2809}$,
(14.) 1%=37\frac{1742}{2809}$÷183$\frac{953}{2809}$=$.205, and
(15.) 100%=100 times $.205=$20.50=the payment.

III. ∴ $20.50=the payment. (*R. H. A., p. 264, prob. 5.*)

Explanation.—$154.40=the amount after paying off the last payment. ∴ $154.40+100%=amount before paying of the last payment, or it equals the debt at the end of the first year plus the interest on this debt for the second year. ∴ We let 100%= the debt at the end of the first year, 106%=amount of 100% for 1 year. ∴ 106%=$154.40+100%. Then proceed as in the solution.

I. If a merchant sells $\frac{3}{4}$ of an article for what $\frac{7}{8}$ of it cost, what is his gain %?

II.
1. 100%=cost of whole article.
2. $87\frac{1}{2}$%=$\frac{7}{8}$ of 100%=cost of $\frac{7}{8}$ of the article.
3. $87\frac{1}{2}$%=selling price of $\frac{3}{4}$ of the article.
4. $29\frac{1}{6}$%=$\frac{1}{3}$ of $87\frac{1}{2}$%=selling price of $\frac{1}{4}$ of the article.
5. $116\frac{2}{3}$%=4 times $29\frac{1}{6}$% = selling price of the whole article.
6. ∴ $116\frac{2}{3}$%—100%=$16\frac{2}{3}$%=gain.

III. ∴ $16\frac{2}{3}$%=his gain. (*Milne's Prac., p. 360, prob. 51.*)

I. A merchant sold goods to a certain amount, on a commission of 4%, and having remitted the net proceeds to the owner, received $\frac{1}{4}$% for prompt payment, which amounted to $15.60. What was his commission?

II.
(1.) 100%=cost of goods.
(2.) 4%=commission.
(3.) 100%—4%=96%=net proceeds.
(4.)
1. $\frac{1}{4}$%=amount received for prompt payment.
2. $15.60=amount received for prompt payment.
3. ∴ $\frac{1}{4}$%=$15.60.
4. 1%=4 times $15.60=$62.40.
5. 100%=100 times $62.40=$6240=net proceeds.
(5.) ∴ 96%=$6240.
(6.) 1%=$\frac{1}{96}$ of $6240=$65, and
(7.) 100%=100 times $65=$6500=cost of goods.
(8.)
1. 100%=$6500.
2. 1%=$\frac{1}{100}$ of $6500=$65, and
3. 4%=4 times $65=$260=his commission.

III. ∴ His commission=$260.
(*Greenleaf's N. A., p. 441, prob. 11.*)

I. If I sell 30 yards of cloth for $132, and gain 10%, how ought I to sell it a yard to lose 25%?

II.
(1.) $132=selling price of 30 yards.
(2.) $4.40=$132÷30=selling price of one yard.
(3.) 100%=cost of one yard.
(4.) 10%=gain.
(5.) 100%+10%=110%=selling price per yard.
(6.) $4.40=selling price per yard.
(7.) ∴ 110%=$4.40.
(8.) 1%=$\frac{1}{110}$ of $4.40=$.04,
(9.) 100%=100 times $.04=$4=cost per yard.
(10.)
1. 100%=$4.
2. 1%=$\frac{1}{100}$ of $4=$.04,
3. 25%=25 times $.04=$1=loss.
4. ∴ $4—$1=$3=selling price per yard to lose 25%.

III. ∴ I must sell it at $3 per yard to lose 25%.
(*Stoddard's Complete, p. 206, prob. 9.*)

I. A merchant receives on commission three kinds of flour; from A he receives 20 barrels, from B 25 barrels, and from C 40 barrels. He finds that A's flour is 10% better than B's, and that B's is 20% better than C's. He sells the whole at \$6 per barrel. What in justice should each man receive?

II.
(1.) \$6=selling price of 1 barrel.
(2.) \$510=selling price of (20+25+40), or 85 barrels.
(3.) 100%=value of C's flour per barrel.
(4.) 120%=value of B's flour per barrel.
(5.) 1. 100%=120%.
2. 1%=$\frac{1}{100}$ of 120%=$1\frac{1}{5}$%,
3. 10%=10 times $1\frac{1}{5}$%=12%.
(6.) 120%+12%=132%=value of A's flour per barrel.
(7.) 4000%=40 times 100%=what C received.
(8.) 3000%=25 times 120%=what B received.
(9.) 2640%=20 times 132%=what A received.
(10.) 9640%=4000%+3000%+2640%=what all rec'd.
(11.) \$510=what all received.
(12.) ∴ 9640%=\$510.
(13.) 1%=$\frac{1}{9640}$ of \$510=\$.52$\frac{211}{241}$, and
(14.) 4000%=4000 times \$.52$\frac{211}{241}$=\$211$\frac{149}{241}$=what C received.
(15.) 3000%=3000 times \$.52$\frac{211}{241}$=\$158$\frac{172}{241}$=what B received.
(16.) 2640%=2640 times \$.52$\frac{211}{241}$=\$139$\frac{161}{241}$=what A received.

III. ∴ \$139$\frac{161}{241}$=A's share,
\$158$\frac{172}{241}$=B's share, and
\$211$\frac{149}{241}$=C's share.

(*Greenleaf's National Arith. p. 442.*)

I. $\frac{3}{4}$ of B's money equals A's money. What % is A's money less than B's, and what % is B's money more than A's?

II.
1. 100%=B's money.
2. 75%=$\frac{3}{4}$ of 100%=A's money.
3. 100%−75%=25%=excess of B's money over A's.
4. 75%=100% of itself,
5. 1%=$\frac{1}{75}$ of 100%=$1\frac{1}{3}$%, and
6. 25%=25 times $1\frac{1}{3}$%=$33\frac{1}{3}$%=the % B's money is more than A's.

III. ∴ A's money is 25% less than B's, and
B's money is $33\frac{1}{3}$% more than A's money.

(*Stod. Comp., p. 203, prob. 19.*)

I. At what price must an article which cost 30 cents be marked, to allow a discount of $12\frac{1}{2}$% and yield a net profit of $16\frac{2}{3}$%?

II.
(1.) $100\%=30¢$,
(2.) $1\%=\frac{1}{100}$ of $30¢=\frac{3}{10}¢$, and
(3.) $16\frac{2}{3}\%=16\frac{2}{3}$ times $\frac{3}{10}¢=5¢$=profit.
(4.) $30¢+5¢=35¢$=the price at which it must sell to gain $16\frac{2}{3}\%$.
(5.)
1. 100%=marked price.
2. $12\frac{1}{2}\%$=discount from marked price.
3. $100\%-12\frac{1}{2}\%=87\frac{1}{2}\%$=selling price.
4. $35¢$=selling price.
5. $\therefore 87\frac{1}{2}\%=35¢$.
6. $1\%=\dfrac{1}{87\frac{1}{2}}$ of $35¢=.40¢$, and
7. $100\%=100$ times $.40¢=40¢$=marked price.

III. $\therefore 40¢$=marked price.

(*Seymour's Prac., p. 203, prob. 4.*)

I. Had an article cost 10% less, the number of % gain would have been 15% more; what was the gain?

II.
1. *100%*=selling price.
2. 100%=actual cost price.
3. *100%*—100%=gain.
4. 100%—10%=90%=supposed cost.
5. *100%*—90%=conditional gain.
6. 90%=100% of itself.
7. $1\%=\frac{1}{90}$ of $100\%=1\frac{1}{9}\%$.
8. *100%*—90%=(*100*—90) times $1\frac{1}{9}\%=\frac{10}{9}\times$*100%*—100% =conditional gain %. [difference.
9. $\therefore \frac{10}{9}\times$*100%*—100%—(*100%* — 100%) = $\frac{1}{9}\times$*100%* =
10. 15%=difference.
11. $\therefore \frac{1}{9}\times$*100%*=15%. [the actual cost.
12. *100%*=9 times 15%=135%=selling price in terms of
13. $\therefore$ 135%—100%=35%=gain.

III. $\therefore$ 35%=gain. (*R. H. A., p. 406, prob. 87.*)

A literal solution.

Let S=selling price and C=the cost. Then $S-C$=gain and $\dfrac{S-C}{C}$=rate of gain. $S-\frac{9}{10}C$=conditional gain and $\dfrac{S-\frac{9}{10}C}{\frac{9}{10}C}=\dfrac{\frac{10}{9}S-C}{C}$=conditional rate of gain. $\therefore \dfrac{\frac{10}{9}S-C}{C}-\dfrac{S-C}{C}=\frac{3}{20}$, or $\frac{1}{9}S=\frac{3}{20}C$, whence $S=\frac{27}{20}C=1.35C$. $\therefore 1.35C-C=.35C$=gain. $\therefore$ Rate of gain$=.35C\div C=.35=35\%$.

I. In the erection of my house I paid three times as much for material as for labor. Had I paid 6% more for labor, and 10% more for material, my house would have cost \$3052. What did it cost me?

II.
(1.) 100%=cost of labor.
(2.) 300%=3 times 100%=cost of material.
(3.) 1. 100%=100%,
2. 1%=1%, and
3. 6%=6%.
4. 100%+6%=106%=supposed cost of labor.
(4.) 1. 100%=300%,
2. 1%=$\frac{1}{100}$ of 300%=3%, and
3. 10%=10 times 3%=30%.
4. 300%+30%=330%=supposed cost of material.
(5.) 330%+106%=436%=supposed cost of house.
(6.) \$3052=supposed cost of house.
(7.) ∴ 436%=\$3052,
(8.) 1%=$\frac{1}{436}$ of \$3052=\$7, and
(9.) 100%=100 times \$7=\$700=cost of labor.
(10.) 300%=300 times \$7=\$2100=cost of material.
(11.) \$2100+\$700=\$2800=cost of house.

III. ∴ \$2800=cost of the house.

I. I invest $\frac{2}{3}$ as much in 8% canal stock at 104%, as in 6% gas stock at 117%; if my income from both is \$1200, how much did I pay for each, and what was the income from each?

II.
(1.) 100%=investment in gas stock. Then
(2.) 66$\frac{2}{3}$%=investment in canal stock.
(3.) 1. 100%=par value of the gas stock.
2. 117%=market value of the gas stock.
3. ∴ 117%=100%, from (1),
4. 1%=$\frac{1}{117}$ of 100%=$\frac{100}{117}$%, and
5. 100%=100 times $\frac{100}{117}$%=85$\frac{25}{39}$%=par value in terms of the investment.
(5.) 1. 100%=85$\frac{25}{39}$%,
2. 1%=$\frac{100}{117}$%, and
3. 6%=6 times $\frac{100}{117}$%=5$\frac{5}{39}$%=income of gas stock.
(6.) 1. 100%=par value of canal stock.
2. 104%=market value.
3. ∴ 104%=66$\frac{2}{3}$%,
4. 1%=$\frac{1}{104}$ of 66$\frac{2}{3}$%=$\frac{25}{39}$%, and
5. 100%=100 times $\frac{25}{39}$%=64$\frac{4}{39}$%.
(6.) 1. 100%=64$\frac{4}{39}$%,
2. 1%=$\frac{1}{100}$ of 64$\frac{4}{39}$%=$\frac{25}{39}$%, and
3. 8%=8 times $\frac{25}{39}$%=5$\frac{5}{39}$%=income of canal stock.
(7.) 5$\frac{5}{39}$%+5$\frac{5}{39}$=10$\frac{10}{39}$%=income from both.
(8.) \$1200=income from both.
(9.) ∴ 10$\frac{10}{39}$%=\$1200,
(10.) 1%=$\frac{1}{10\frac{10}{39}}$ of \$1200=\$117, and

(11.) $100\%=100$ times $\$117=\$11700$=investment in gas stock.

(12.) $66\frac{2}{3}\%=66\frac{2}{3}$ times $\$117=\$7800$=investment in canal stock.

(13.) $5\frac{5}{39}\%=5\frac{5}{39}$ times $\$117=\$600$=income from each.

III. ∴ $600=income from each.
$11700=investment in gas stock, and
$7800=investment in canal stock.

I. A man bought two horses for $300; he sold them for $250 apiece. The gain on one was 5% more than on the other; what was the gain on each?

II.
1. $300=cost of both.
2. $500=$250+$250=selling price of both.
3. $500—$300=$200=gain on both.
4. 100%=gain on first horse. Then
5. 105%=gain on second horse.
6. 205%=100%+105%=gain on both.
7. $200=gain on both.
8. ∴ 205%=$200.
9. $1\%=\frac{1}{205}$ of $\$200=\$\frac{40}{41}$, and
10. $100\%=100$ times $\$\frac{40}{41}=\$97.56\frac{4}{41}$=gain on the first.
11. $105\%=105$ times $\$\frac{40}{41}=\$102.43\frac{37}{41}$=gain on the second.

III. ∴ $\$97.56\frac{4}{41}$=gain on the first, and
$\$102.43\frac{37}{41}$=gain on the second.

Note.—In this solution, it is assumed that the gain on one was 5% of the gain on the other more than the other, and this is the usual assumption. But the problem really means that the per cent. of gain on one, computed on its cost, was 5% more than the per cent. of gain on the other, computed on its cost. By this assumption, the problem is algebraic. The following is the solution: Let x=the cost of the first horse, and $\$300-x$, the cost of the second. Then $\$250-x$=gain on first, and $\$250-(\$300-x)=x-\$50$, the gain on the second. $(\$250-x)\div x$= rate of gain on the first, and $(x-\$50)\div(\$300-x)$, the rate of gain on the second. Then $(250-x)\div x-(x-50)\div(300-x)=\frac{1}{20}$. Whence, by clearing of fractions, transposing and, combining, $x^2-10300x=-1500000$, $x=5150\pm 50\sqrt{10009}=\$147.7755$, the cost of the first horse. $\$300-x=\$152.2245$, the cost of the second horse. $\$250-x=\$102.2245$, gain on the first horse, and $x-\$50=\$97.7755$, the gain on the second horse.

I. An agent sells produce at 2% commission, invests the proceeds in flour at 3% commission; his whole commission was $75. How many barrels of flour did he buy at $5 per barrel?

II.
(1.) 100%=value of the produce.
(2.) 2%=the commission.
(3.) $100\%-2\%=98\%$= net proceeds, or amount invested in the flour.
(4.)
1. 100%=cost of the flour.
2. 3%=commission on flour.
3. $100\%+3\%=103\%$=whole cost of the flour.
4. $\therefore 103\%=98\%$,
5. $1\%=\frac{1}{103}$ of $98\%=\frac{98}{103}\%$, and
6. $100\%=100\times\frac{98}{103}\%=95\frac{15}{103}\%$ = cost of flour in terms of the value of the produce.
7. $98\%-95\frac{15}{103}\%=2\frac{88}{103}\%$=commission on flour.

(5.) $2\%+2\frac{88}{103}\%=4\frac{88}{103}\%$=whole commission.
(6.) \$75=whole commission.
(7.) $\therefore 4\frac{88}{103}\%=\$75$,
(8.) $1\%=\$75\div4\frac{88}{103}=\$15.45$, and
(9.) $100\%=100$ times \$15.45=\$1545 = value of the produce.
(10.) $95\frac{15}{103}\%=95\frac{15}{103}$ times \$15.45=\$1470 = value of the the flour.
(11.) \$5=cost of 1 barrel.
(12.) \$1470=cost of $1470\div5$, or 294 barrels.

III. ∴ The agent bought 294 barrels of flour.

I. A distiller sold his whisky, losing 4%; keeping \$18 of the proceeds, he gave the remainder to an agent to buy rye at 8% commission; he lost in all \$32; what was the whisky worth?

II
(1.) 100%=value of the whisky.
(2.) 4%=loss.
(3.) $100\%-4\%=96\%$=amount he had left.
(4.) $96\%-\$18$=amount he invested in rye.
(5.)
1. 100%=cost of the rye.
2. 8%=commission on the rye.
3. $100\%+8\%=108\%$=whole cost of rye.
4. $\therefore 108\%=96\%-\$18$,
5. $1\%=\frac{1}{108}$ of $(96\%-\$18)=\frac{8}{9}\%-\$.16\frac{2}{3}$, and
6. $100\%=100$ times $(\frac{8}{9}\%-\$.16\frac{2}{3})=88\frac{8}{9}\%-\$16.66\frac{2}{3}$ =cost of rye.
7. $8\%=8$ times $(\frac{8}{9}\%-\$.16\frac{2}{3})=7\frac{1}{9}\%-\$1.33\frac{1}{3}$=commission on rye.

(6.) $4\%+(7\frac{1}{9}\%-\$1.33\frac{1}{3})=11\frac{1}{9}\%-\$1.33\frac{1}{3}$=whole loss.
(7.) \$32=whole loss.
(8.) $\therefore 11\frac{1}{9}\%-\$1.33\frac{1}{3}=\$32$
(9.) $11\frac{1}{9}\%=\$33.33\frac{1}{3}$,
(10.) $1\%=\frac{1}{11\frac{1}{9}}$ of $\$33.33\frac{1}{3}=\$3$, and
(11.) $100\%=100$ times \$3=\$300=value of the whisky.

III. ∴ \$300=value of the whisky.

(R. H. A., p. 406, prob 91.)

I. What will be the cost in New Orleans of a draft on New York, payable 60 days after sight, for \$5000, exchange being at $1\frac{1}{2}\%$ premium?

II.
1. 100%=face of the draft.
2. $1\frac{1}{2}\%$=premium.
3. $100\%+1\frac{1}{2}\%=101\frac{1}{2}\%$=rate of exchange.
4. 5%=discount for one year.
5. $\frac{7}{8}\%=\frac{63}{360}$ of 5%=discount for 63 days.
6. ∴ $101\frac{1}{2}\%-\frac{7}{8}\%=100\frac{5}{8}\%$=cost of the draft
7. 100%=\$5000.
8. $1\%=\frac{1}{100}$ of \$5000=\$50, and
9. $100\frac{5}{8}\%=100\frac{5}{8}$ times \$50=\$5031.25=cost of the draft.

III. ∴ \$5031.25=cost of the draft.

Explanation.—Observe that since the draft is not to be paid in New York for 63 days, the banker in New Orleans, who has the use of the money for that time allows the drawer discount on the face of the draft for that time, which goes (1) towards reducing the premium if there be any, and (2) towards reducing the face of the draft.

Note.—The rate of exchange between two places or countries depends upon the course of trade. Suppose the trade between New York and New Orleans is such that New York owes New Orleans \$10,250,000 and New Orleans owes New York \$13,000,000. There is a "balance of trade" of \$2,750,000 against New Orleans and in favor of New York. Hence, the demand in New Orleans for drafts on New York is greater than the demand in New York for drafts on New Orleans and, therefore, the drafts are at a premium in New Orleans. But if New York owes New Orleans \$13,000,000 and New Orleans owes New York \$10,250,000, the "balance of trade," \$2,750,000, is *against* New York and in *favor* of New Orleans. Hence, the demand in New Orleans for drafts on New York is less than the demand in New York for drafts on New Orleans and, therefore, the drafts are at a discount in New Orleans.

If the trade between the two places is the same, the rate of exchange is at par.

The reason why the banks in New York should charge a premium, when the balance of trade is against them, is that they must be at the expense of actually sending money to the New Orleans banks or be charged interest on their unpaid balance; the reason why the New Orleans banks will sell at a discount is that they are willing to sell for less than the face of a draft in order to get the money owed them in New York immediately.

Exchange is charged from $\frac{1}{8}$ to $\frac{1}{2}\%$, and is designed to cover the cost of transporting the funds from one place to another.

I. What will a 30 days' draft on New Orleans for \$7216.85 cost, at $\frac{3}{8}\%$ discount, interest 6%?

II.
1. 100%=face of draft.
2. $\frac{3}{8}\%$=discount.
3. $100\%-\frac{3}{8}\%=99\frac{5}{8}\%$=face less the discount.
4. 6%=bank discount for 1 year.
5. $\frac{11}{20}\%=\frac{33}{360}$ of 6%=bank discount for 33 days.
6. $99\frac{5}{8}\%-\frac{11}{20}\%=99\frac{3}{40}\%$=cost of the draft.
7. 100%=\$7216.85,
8. $1\%=\frac{1}{100}$ of \$7216.85=\$72.1685, and
9. $99\frac{3}{40}\%=99\frac{3}{40}$ times \$72.1685=\$7150.094=cost of the draft.

III. ∴ \$7150.094=cost of the draft.

I. The aggregate face value of two notes is \$761.70 and each has 1 year 3 months to run; one of the notes I had discounted at 10% true discount and the other at 10% bank discount, and realized from both notes \$671.50. Find the face value of both notes.

II.
(1.) 100%=face of note discounted at bank discount.
(2.) \$761.70—100%=face of note discounted at true discount.
(3.) 10%=bank discount for 1 year.
(4.) $12\frac{1}{2}\%$=bank discount for 1 year 3 months.
(5.)
1. 100%=present worth of second note.
2. 10%=interest on present worth for 1 year.
3. $12\frac{1}{2}\%$=interest for 1 year 3 months.
4. $100\%+12\frac{1}{2}\%=112\frac{1}{2}\%$=amount of present worth.
5. \$761.70—100%=amount of the present worth.
6. ∴ $112\frac{1}{2}\%$=\$761.70—100%,
7. $1\%=\frac{1}{112\frac{1}{2}}$ of (\$761.70—100%)=\$6.7706$\frac{2}{3}$—$\frac{8}{9}\%$,
8. 100%=100 times (\$6.7706$\frac{2}{3}$—$\frac{8}{9}\%$)=\$677.06$\frac{2}{3}$—$88\frac{8}{9}\%$=present worth.

(6.) \$761.70—100%—(\$677.06$\frac{2}{3}$—$88\frac{8}{9}\%$)=\$84.63$\frac{1}{3}$—$11\frac{1}{9}\%$=true discount.
(7.) \$84.63$\frac{1}{3}$—$11\frac{1}{9}\%+12\frac{1}{2}\%$=\$84.63$\frac{1}{3}$+$1\frac{7}{18}\%$=whole discount.
(8.) \$761.70—\$671.50=\$90.20=whole discount.
(9.) ∴\$84.63$\frac{1}{3}$+$1\frac{7}{18}\%$=\$90.20,
(10.) $1\frac{7}{18}\%$=\$5.56$\frac{2}{3}$,
(11.) $1\%=\frac{1}{1\frac{7}{18}}$ of \$5.56$\frac{2}{3}$=\$4.008, and
(12.) 100%=100 times \$4.008=\$400.80=face of note discounted at bank discount.
(13.) \$761.70—100%=\$761.70—\$400.80=\$360.90=face of note discounted at true discount.

III. ∴ { \$400.80=face of note discounted at bank discount, and \$360.90=face of note discounted at true discount.

I. A merchant sold part of his goods at a profit of 20%, and the remainder at a loss of 11%. His goods cost $1000 and his gain was $100; how mnch was sold at a profit?

II.
(1.) 100%=cost of goods sold at a profit. Then
(2.) $1000—100%=cost of goods sold at a loss.
(3.) 20%=profit on 100%, the part sold at a profit.
(4.) 1. 100%=$1000—100%.
2. 1%=$\frac{1}{100}$ of ($1000—100%)=$10—1%,
3. 11%=11 times ($10—1%)=$110—11%=loss on the remainder.
(5.) ∴ 20%—($110—11%)=31%—$110=gain.
(6.) $100=gain.
(7.) ∴ 31%—$110=$100.
(8.) 31%=$210,
(9.) 1%=$\frac{1}{31}$ of $210=$6$\frac{24}{31}$,
(10.) 100%=100 times 6\frac{24}{31}$=677.41$\frac{29}{31}$=part sold at a profit.

III. ∴ 677.41\frac{29}{31}$=value of the part sold at a profit.

I. By discounting a note at 20% per annum, I get 22½% per annum interest; how long does the note run?

II.
1. 22½% of the proceeds=20% of the face of the note.
2. 1% of the proceeds=$\frac{1}{22\frac{1}{2}}$ of 20%=$\frac{8}{9}$% of the face of the note.
3. 100% of the proceeds=100 timss $\frac{8}{9}$%=88$\frac{8}{9}$% of the face of the note.
4. 100%=face of the note.
5. 88$\frac{8}{9}$%=proceeds.
6. 100%—88$\frac{8}{9}$%=11$\frac{1}{9}$%=discount for a certain time.
7. 20%=discount for 360 days.
8. 1%=discount for $\frac{1}{20}$ of 360 days, or 18 days.
9. 11$\frac{1}{9}$%=discount for 11$\frac{1}{9}$ times 18 days, or 200 days.

III. The note was discounted for 200 days.

I. A man bought a farm for $5000, agreeing to pay principal and interest in 5 equal annual instalments. What will be the annual payment including interest at 6%?

II.
1. 100%=one annual payment.
2. . . 100%=amount paid at end of the fifth year since the debt was then discharged.
3. 100%=principal that drew interest the fifth year.
4. 6%=interest on this principal.
5. ∴ 100%+6%=106%=amount of this principal.
6. ∴ 106%=100%=the annual payment.
7. 1%=$\frac{1}{106}$ of 100%=$\frac{50}{53}$%, and
8. 100%=100 times $\frac{50}{53}$%=94$\frac{18}{53}$%=principal at the beginning of the fifth year.
9. 94$\frac{18}{53}$%+100%=194$\frac{18}{53}$%=amount at the end of the fourth year.

II. (2.)
1. 100%=principal at the beginning of the fourth year.
2. 6%=interest on this principal.
3. $100\%+6\%=106\%$=amount.
4. $\therefore 106\%=194\frac{18}{53}\%$,
5. $1\%=\frac{1}{106}$ of $194\frac{18}{53}\%=1.83\frac{958}{2809}\%$, and
6. $100\%=100$ times $1.83\frac{958}{2809}\%=183\frac{958}{2809}\%$=principal at the beginning of the fourth year.
7. $183\frac{958}{2809}\%+100\%=283\frac{958}{2809}\%$=amount at the end of the third year.

(3.)
1. 100%=principal at the beginning of the third year.
2. 6%=interest.
3. $100\%+6\%=106\%$ = amount at the end of the third year.
4. $\therefore 106\%=283\frac{958}{2809}\%$,
5. $1\%=\frac{1}{106}$ of $283\frac{958}{2809}\%=2.67\frac{44841}{148877}\%$, and
6. $100\%=100$ times $2.67\frac{44841}{148877}\% = 267\frac{44841}{148877}\%$ = principal at the beginning of third year.
7. $267\frac{44841}{148877}\%+100\% = 367\frac{44841}{148877}\%$ = amount at the end of second year.

(4.)
1. 100%=principal at the beginning of second year.
2. 6%=interest.
3. $100\%+6\%=106\%$=amount at the end of second year.
4. $\therefore 106\%=367\frac{44841}{148877}\%$,
5. $1\%=\frac{1}{106}$ of $367\frac{44841}{148877}\%=3.46\frac{4028574}{7890481}\%$, and
6. $100\%=100$ times $3.46\frac{4028574}{7890481}\%=346\frac{4028574}{7890481}\%$= principal at the beginning of the second year.
7. $346\frac{4028574}{7890481}\%+100\%=446\frac{4028574}{7890481}\%$=amount at the end of first year.

(5.)
1. 100%=principal at the beginning of the first year, or the cost of farm.
2. 6%=interest.
3. $100\%+6\%=106\%$=amount at end of first year.
4. $\therefore 106\%=446\frac{4028574}{7890481}\%$,
5. $1\%=\frac{1}{106}$ of $446\frac{4028574}{7890481}\%=4.21\frac{98852447}{418195493}\%$, and
6. $100\%=100$ times $4.21\frac{98852447}{418195493}\%=421\frac{98852447}{418195493}\%$=cost of the farm.

(6.) \$5000=cost of the farm.
(7.) $\therefore 421\frac{98852447}{418195493}\%=\$5000$,
(8.) $1\%=\$5000\div421\frac{98852447}{418195493}=\$11.8698+$, and
(9.) $100\%=100$ times $\$11.8698=\$1186.98+$=the annual payment.

III. $\therefore$ \$1186.98+=the annual payment.

(*Milne's Prac., p. 361, prob. 63.*)

I. A and B have \$4700; $\frac{\frac{3\%}{4}\%}{\frac{1\%}{2}\%}\%$ of A's share equals $\frac{\frac{2}{3\%}\%}{60\%}\%$ of B's share; how much has each?

II.
1. $\frac{3\%}{4}\%=\frac{\frac{3}{100}}{4}\%=\frac{3}{400}\%=\frac{3}{40000}$.
2. $\frac{1\%}{2}\%=\frac{\frac{1}{100}}{2}\%=\frac{1}{200}\%=\frac{1}{20000}$.
3. $\therefore \frac{\frac{3\%}{4}\%}{\frac{1\%}{2}\%}=\frac{\frac{3}{40000}}{\frac{1}{20000}}\%=1\frac{1}{2}\%$.
4. $\frac{2}{3\%}\%=\frac{2}{\frac{3}{100}}\%=\frac{200}{3}\%=\frac{200}{300}=\frac{2}{3}$.
5. $60\%=\frac{60}{100}=\frac{3}{5}$.
6. $\therefore \frac{\frac{2}{3\%}\%}{60\%}\%=\frac{\frac{2}{3}}{\frac{3}{5}}\%=1\frac{1}{9}\%$.
7. $\therefore 1\frac{1}{2}\%$ of A's$=1\frac{1}{9}\%$ of B's,
8. 1% of A's$=\frac{1}{1\frac{1}{2}}$ of $1\frac{1}{9}\%=\frac{20}{27}\%$ of B's, and
9. 100% of A's$=100$ times $\frac{20}{27}\%=74\frac{2}{27}\%$ of B's.
10. $100\%=$B's share.
11. $74\frac{2}{27}\%=$A's share.
12. $100\%+74\frac{2}{27}\%=174\frac{2}{27}\%=$sum of their shares.
13. $\$4700=$sum of their shares.
14. $\therefore 174\frac{2}{27}\%=\$4700$,
15. $1\%=\frac{1}{174\frac{2}{27}}$ of $\$4700=\$27$, and
16. $100\%=100$ times $\$27=\$2700=$B's share.
17. $74\frac{2}{27}\%=74\frac{2}{27}$ times $\$27=\$2000=$A's share.

III. $\therefore$ $\$2700=$B's share and
$\$2000=$A's share.

CHAPTER XV.

RATIO AND PROPORTION.

1. ***Ratio*** is the relative magnitude of one quantity as compared with another of the same kind; thus, the ratio of 12 apples to 4 apples is 3.

The first quanity, 12 apples, is called the *Antecedent*, and the second quantity, 4 apples, the consequent. Taken together they are called *Terms* of the ratio, or a *Couplet*.

2. ***The Sign*** of ratio is the colon, :, the common sign of division with the horizontal line omitted.

Note.—Olney says, "There is a common notion among us, that the French express a ratio by dividing the consequent by the antecedent, while the English express it by dividing the antecedent by the consequent. Such is not the fact. French, German, and English writers agree in the above definition. In fact, the Germans very generally use the sign : instead of ÷; and by all, the two signs are used as exact equivalents." Some writers, however, divide the consequent by the antecedent, as $a : b = \frac{b}{a}$. This is according to Webster's definition and illustration. To my mind, to divide the antecedent by the consequent is more simple and philosophical and should be universally adopted by all writers on mathematics.

3. ***A Direct Ratio*** is the quotient of the antecedent divided by the consequent.

4. ***An Indirect Ratio*** is the quotient of the consequent by the antecedent.

5. A ratio of ***Greater Inequality*** is a ratio greater than unity; as, 7:3.

6. A ratio of ***Less Inequality*** is a ratio less than unity; as, 4:5.

7. ***A Compound Ratio*** is the product of the corresponding terms of several simple ratios. Thus, the compound ratio of 1:3, 5:4, and 7:2 is $1\times5\times7:3\times4\times2$.

8. ***A Duplicate Ratio*** is the ratio of the squares of two numbers.

9. ***A Triplicate Ratio*** is the ratio of the cubes of two numbers; as, $a^3 : b^3$.

10. ***A Subduplicate Ratio*** is the ratio of the square roots of two numbers; as, $\sqrt{a} : \sqrt{b}$.

11. ***A Subtriplicate Ratio*** is the ratio of the cube roots of two numbers; as, $\sqrt[3]{a} : \sqrt[3]{b}$.

PROPORTION.

12. ***Proportion*** is an equality of ratios. The equality is indicated by the ordinary sign of equality or by the double colon, ::. Thus, $a : b = c : d$, or $a : b :: c : d$.

13. ***The Extremes*** of a proportion are the first and fourth terms.

14. ***The Means*** are the second and third terms.

15. ***A Mean Proportional*** between two quantities is a quantity to which either of the two quantities bears the same ratio that the mean does to the other of the two.

16. ***A Continued Proportion*** is a succession of equal ratios, in which each consequent is the antecedent of the next ratio.

17. ***A Compound Proportion*** is an expression of equality between a compound and a simple ratio.

18. ***A Conjoined Proportion*** is a proportion which has each antecedent of a compound ratio equal in value to its consequent. The first of each pair of equivalent terms is an antecedent, and the term following, a consequent. This is also called the "Chain rule."

I. What is the ratio of $\frac{1}{2}$ to $\frac{2}{3}$?
$\frac{1}{2} \div \frac{2}{3} = \frac{1}{2} \times \frac{3}{2} = \frac{3}{4}$, the ratio.

I. What is the ratio of 10 bu. to $1\frac{3}{7}$ bu.?
10 bu. $\div\ 1\frac{3}{7}$ bu. $= 10 \times \frac{7}{10} = 7$, the ratio.

I. What is the ratio of 25 apples to 75 boxes?
Ans. No ratio; for no number of times one will produce the other

In a true proportion, we must always have greater : less :: greater : less or less : greater :: less : greater. The test for the truth of a proportion is that the product of the means equals the product of the extremes.

I. If a 5-cent loaf weighs 7oz. when flour is $8 per barrel, how much should it weigh when flour is $7.50 per barrel?

It should evidently weigh more.
∴ less : greater :: less : greater.
$7.50 : $8.00 :: 7oz : (?= $7\frac{7}{15}$oz.)

I. If a staff 3 feet long, casts a shadow 2 feet, how high is the steeple whose shadow at the same time is 75 feet?

Since the steeple casts a longer shadow than the staff, it is evidently higher than the staff.
∴ less : greater :: less : greater.
2 feet : 75 feet :: 3 feet : (?= $112\frac{1}{2}$ feet.)

I. What number is that which being divided by one more than itself, gives $\frac{1}{7}$ for a quotient?

II.
1. Let $\frac{2}{2}$=number. Then
2. $\dfrac{\frac{2}{2}}{\frac{2}{2}+1} = \frac{1}{7}$ or $\frac{2}{2} : \frac{2}{2}+1 :: 1 : 7$, whence
3. $7(\frac{2}{2}) = 1(\frac{2}{2}+1)$ or
4. $\frac{14}{2} = \frac{2}{2}+1$; whence
5. $\frac{12}{2} = 1$,
6. $\frac{1}{2} = \frac{1}{12}$, and
7. $\frac{2}{2} = 2$ times $\frac{1}{12} = \frac{1}{6}$=number.

III. ∴ $\frac{1}{6}$=the number.

I. What number divided by 3 more than itself gives $\frac{7}{9}$ for a quotient?

II.
1. Let $\frac{2}{2}$=the number. Then
2. $\frac{\frac{2}{2}}{\frac{2}{2}+3}=\frac{7}{9}$ or, putting this in the form of a proportion,
3. $\frac{2}{2}:\frac{2}{2}+3::7:9$. [the product of the extremes.
4. $\therefore \frac{18}{2}=\frac{14}{2}+21$, the product of the means being equal to
5. $\frac{18}{2}-\frac{14}{2}=\frac{4}{2}=21$,
6. $\frac{1}{2}=\frac{1}{4}$ of $21=5\frac{1}{4}$, and
7. $\frac{2}{2}=2$ times $5\frac{1}{4}=10\frac{1}{2}$=the number.

III. $\therefore 10\frac{1}{2}$=the number.

I. If 7 lb. of coffee is equal in value to 5 lb. of tea, and 3 lb. of tea to 13 lb. of sugar, 39 lb. of sugar to 24 lb. of rice, 12 lb. of rice to 7 lb. of butter, 8 lb. of butter to 12 lb. of cheese; how many lb. of coffee are equal in value to 65 lb. of cheese?

II.
1. 7 lb. of coffee=5 lb. of tea,
2. 3 lb. of tea=13 lb. of sugar,
3. 39 lb. of sugar=24 lb. of rice,
4. 12 lb. of rice=7 lb. of butter,
5. 8 lb. of butter=12 lb. of cheese, and
6. 65 lb. of cheese= ?=39 lb. of coffee,
7. $\frac{7\times3\times39\times12\times8\times65}{5\times13\times24\times7\times12}=39$ lb.

III. $\therefore$ 65 lb. of cheese=39 lb. of coffee.

I. I can keep 10 horses or 15 cows on my farm; how many horses can I keep if I have 9 cows?

15 cows : 9 cows :: 10 horses : ?=6 horses.

10 horses—6 horses=4 horses.

$\therefore$ I can keep 4 horses with the 9 cows.

I. If 2 oxen or 3 cows eat one ton of hay in 60 days, how long will it last 4 oxen and 5 cows?

2 oxen : 4 oxen :: 3 cows : ?=6 cows.

$\therefore$ 4 oxen eat as mnch as 6 cows. If a ton of hay last 3 cows 60 days, it will last 6 cows, which are equal to 4 oxen, and 5 cows, or 11 cows, not so long.

$\therefore$ 11 cows : 3 cows :: 60 days : ?=$17\frac{3}{11}$ days.

I. If 24 men, by working 8 hours a day, can, in 18 days, dig a ditch 95 rods long, 12 feet wide at the top, 10 feet wide at the bottom, and 9 feet deep; how many men, in 24 days of 12 hours a day, will be required to dig a ditch 380 rods long, 9 feet wide at the top, 5 feet wide at the bottom, and 6 feet deep?

95 rods : 380 rods
24 days : 18 days
12 hours : 8 hours
12 feet : 9 feet
10 feet : 5 feet
9 feet : 6 feet
} :: 24 men : ?=12 men.

$$\frac{380\times18\times8\times9\times5\times6\times24}{95\times24\times12\times12\times10\times9}=12 \text{ men.}$$

I. A Louisville merchant wishes to pay $10000, which he owes in Berlin. He can buy a bill of exchange in Louisville on Berlin at the rate of $.96 for 4 reichmarks; or he is offered a circular bill through London and Paris, brokerage $\frac{1}{8}\%$ at each place, at the following rates: £1=$4.90=25.38 francs, and 5 francs=4 reichmarks. What does he gain by direct exchange?

II.
1. $.238=1 mark.
2. $10000=10000÷.238=42016.807 marks.
3. $.24=1 mark, since this is the rate of exchange.
4. ∴ $10084.033=42016.807 times $.24=42016.807 marks =direct exchange.
5. 42016.807 marks=(?=$10165.38.)
6. $4.90=£1—$\frac{1}{8}\%$ of £1=£.99$\frac{7}{8}$.
7. £1=.99$\frac{7}{8}$ times 25.38 fr.
8. 5 fr.=4 marks.
9. $\frac{42016.807\times4.90\times5}{.99\frac{7}{8}\times.99\frac{7}{8}\times25.38\times4}$=$10165.38=cost by circular exchange.
10. $10165.38—$10084.033=$81.35=gain by direct exchange.

III. ∴ $81.35=gain by direct exchange.

I. A wheel has 35 cogs; a smaller wheel working in it, 26 cogs; in how many revolutions of the larger wheel will the smaller one gain 10 revolutions?

II.
1. 35 cogs—26 cogs=9 cogs=what the smaller wheel gains on larger in 1 revolution of larger wheel.
2. 26 cogs passed through the point of contact=1 revolution of smaller wheel.
3. 1 cog passed through the point of contact=$\frac{1}{26}$ revolution of smaller wheel.
4. 9 cogs passed through the point of contact=$\frac{9}{26}$ revolution of smaller wheel.
5. ∴ In 1 revolution of larger wheel the smaller gains $\frac{9}{26}$ revolution of smaller wheel.
6. ∴ $\frac{9}{26}$ revolution gained : 10 revolutions gained :: 1 revolution of larger wheel : ?=28$\frac{8}{9}$ revolutions of larger wheel.

III. ∴ The smaller wheel will gain 10 revolutions in 28$\frac{8}{9}$ revolutions of larger wheel.

By analysis and proportion.

26 cogs passed through the point of contact=1 revolution of the smaller wheel.

35 cogs passed through the point of contact=1 revolution of the larger wheel. But when the larger wheel has made 1 revolution, 35 cogs of the smaller wheel have passed through the point of contact. If 26 cogs having passed through the point of contact make 1 revolution of the smaller wheel, how many revolutions will 35 cogs make?

By proportion, 26 cogs : 35 cogs :: 1 rev. : ?=$1\frac{9}{26}$ rev.

∴ The smaller wheel makes $1\frac{9}{26}$ revolutions while the larger wheel makes 1 revolution. ∴ The smaller gains $1\frac{9}{26}$ revolutions —1 revolution=$\frac{9}{26}$ revolution. If the smaller wheel gains $\frac{9}{26}$ revoluion in 1 revolution of the larger wheel to gain 10 revolutions on the larger wheel, the larger wheel must make more revolutions. ∴ less : greater :: less : greater.

$\frac{9}{26}$ rev. : 10 rev. :: 1 rev. of larger : ?=$28\frac{8}{9}$ rev. of larger.

I. If the velocity of sound be 1142 feet per second, and the number of pulsations in a person 70 per minute, what is the distance of a cloud, if 20 pulsations are counted between the time of seeing the flash and hearing the thunder?

II.
1. 1142 ft.=distance sound travels in 1 second.
2. 68520 ft.=60×1142 ft.=distance sound travels in 1 min., or the time of 70 pulsations.
3. ∴ If it travels 68520 feet while 70 pulsations are counted, it will travel not so far while 20 pulsations are counted.
4. ∴ greater : less :: greater : less. [145 yd. $2\frac{1}{7}$ ft.
5. 70 pul. : 20 pul. :: 68520 ft. : ?=$19577\frac{1}{7}$ ft.=3 mi. 5 fur.

III. ∴ The cloud is 3 mi. 5 fur. 145 yd. $2\frac{1}{7}$ ft. distant.

(R., 3d p., p. 289, prob. 45.)

PROBLEMS.

1. If 3 horses, in $\frac{1}{4}$ of a month eat $\frac{3}{4}$ of a ton of hay, how long will $\frac{5}{6}$ of a ton last 5 horses?

2. If a 4-cent loaf weighs 9 oz. when flour is $6 a barrel, how much ought a 5-cent loaf weigh when flour is $8 per barrel?

3. A dog is chasing a hare, which is 46 rods ahead of the dog. The dog runs 19 rods while the hare runs 17; how far must the dog run before he catches the hare?

4. If 52 men can dig a trench 355 feet long, 60 feet wide, and 8 feet deep in 15 days, how long will a trench be that is 45 feet wide and 10 feet deep, which 45 men can dig in 25 days?

5. If $\frac{1}{6}$ of 12 be 3 what will $\frac{1}{4}$ of 40 be? *Ans.* 15.

6. If 3 be $\frac{1}{6}$ of 12, what will $\frac{1}{4}$ of 40 be? *Ans.* $6\frac{2}{3}$.

7. If 18 men or 20 women do a work in 9 days, in what time can 4 men and 9 women do the same work? *Ans.* $13\frac{47}{121}$ days.

8. If 5 oxen or 7 cows eat $3\frac{4}{11}$ tons of hay in 87 days, in what time will 2 oxen and 3 cows eat the same quantity of hay? *Ans.* 105 days.

9. Divide $600 between three men, so that the second man shall receive one-third more than the first, and the third $\frac{3}{8}$ more than the second.

10. Two men in Boston hire a carriage for $25, to go to Concord, N. H., and back, the distance being 72 miles, with the privilege of taking in three more persons. Having gone 20 miles, they took in A; at Concord they took in B; and when within 30 miles of Boston, they took in C. How much shall each pay? *Ans.* First man, $\$7.609\frac{103}{108}$; second, $\$7.609\frac{103}{108}$; A, $\$5.873\frac{91}{108}$; B, $\$2,864\frac{7}{12}$; and C, $\$1.041\frac{8}{12}$.

11. Three men purchased 6750 sheep. The number of A's sheep is to the number of B's sheep as $\frac{2}{3}$ is to $3\frac{1}{3}$, and 4 times the number of C's sheep is to the number of A's sheep as $\frac{1}{3}$ is to $\frac{1}{9}$. Find the number of sheep each had.

Ans. { A's= B's= C's=

12. If $500 gain $10 in 4 months, what is the rate per cent? *Ans.* 6%.

13. If 12 men can do as much work as 25 women, and 5 women do as much as 6 boys; how many men would it take to do the work of 75 boys? *Ans.* 30 men.

14. If 5 experienced compositors in 16 days, 11 hours each, can compose 25 sheets of 24 pages in each sheet, 44 lines on a page, 8 words in a line, and 5 letters to a word; how many inexperienced compositors in 12 days, 10 hours each, will it take to compose a volume (to be printed with the same kind of type), consisting of 36 sheets, 16 pages to a sheet, 112 lines to the page, 5 words to a line, and 8 letters to a word, provided that while composing an inexperienced compositor can do only $\frac{4}{5}$ as much as an experienced compositor, and that the latter work is only $\frac{5}{8}$ as hard as the former? *Ans.* 16.

15. If A can do $\frac{2}{3}$ as much in a day as B, B can do $\frac{3}{4}$ as much as C, and C can do $\frac{4}{5}$ as much as D, and D can do $\frac{5}{6}$ as much as E, and E can do $\frac{6}{7}$ as much as F; in what time can F do as much work as A can do in 28 days? *Ans.* 8.

16. A starts on a journey, and travels 27 miles a day; 7 days after, B starts, and travels the same road, 36 miles a day; in how many days will B overtake A? *Ans.* 21 days.

17. A wheel has 45 cogs; a smaller wheel working in it, 36 cogs; in how many revolutions of the larger wheel will the smaller gain 10 revolutions? *Ans.* 40.

18. If the velocity of sound be 1142 feet per second, and the number of pulsations in a person 70 per minute, what is the distance of a cloud, if 30 pulsations are counted between the time of seeing a flash of lightning and hearing the thunder?
Ans. $5\frac{1}{2}$ mi. 108 yd. $1\frac{3}{7}$ ft.

19. If William's services are worth $\$15\frac{3}{8}$ a month, when he labors 9 hours a day, what ought he to receive for $4\frac{2}{5}$ months, when he labors 12 hours a day? *Ans.* $\$91.91\frac{1}{9}$.

20. If 300 cats kill 300 rats in 300 minutes, how many cats will kill 100 rats in 100 minutes? *Ans.* 300 cats.

CHAPTER XVI.

ANALYSIS.

1. ***Analysis,*** in mathematics, is the process of solving problems by tracing the relation of the parts.

I. What will 7 lb. of sugar cost at 5 cents a pound?

Analysis for primary classes.

If one ponnd of sugar costs 5 cents, 7 pounds will cost 7 times 5 cents, which are 35 cents.

I. If 6 lead pencils cost 30 cents, what will one lead pencil cost?
Analysis: If 6 lead pencils cost 30 cents, one lead pencil will cost as many cents as 6 is contained into 30 cents which are 5 cents.

I. If 8 oranges cost 48 cents, what will 5 oranges cost?
Analysis: If 8 oranges cost 48 cents, one orange will cost as many cents as 8 is contained into 48 cents which are 6 cents; if one orange costs 6 cents 5 oranges will cost 5 times 6 cents, which are 30 cents.

I. If a boy had 7 apples and ate 2 of them, how many had he left?
Analysis: If a boy had 7 apples and ate 2 of them, he had left the difference between 7 apples and 2 apples which are 5 apples.

I. If John had 12 cents and found 5 cents, how many cents did he then have?
Analysis: If John had 12 cents and found 5 cents, he then had the sum of 12 cents and 5 cents which are 17 cents.

Note.—If teachers in the Primary Departments would see that their pupils gave the correct analysis to such problems, their pupils would often be better prepared for the higher grades. After they are thoroughly acquainted with the analysis of such questions they may be taught to write out neat, accurate solutions with far less trouble than if allowed to give careless analysis to problems in the lower grades.

I. If 4 balls cost 36 cents, how many balls can be bought for 81 cents?

Analysis: If 4 balls cost 36 cents, one ball will cost as many cents as 4 is contained into 36 cents which are 9 cents; if one ball costs 9 cents for 81 cents there can be bought as many balls as 9 is contained into 81 which are 9 balls.

Written solution.

II.
1. 36 cents=cost of 4 balls.
2. 9 cents=36 cents÷4=cost of 1 ball.
3. 81 cents=cost of 81÷9, or 9 balls.

III. ∴ If 4 balls cost 36 cents, for 81 cents there can be bought 9 balls.

I. What number divided by $\frac{3}{5}$ will give 10 for a quotient?

II.
1. $\frac{5}{5}$=the number.
2. $\frac{5}{5}\div\frac{3}{5}=\frac{5}{5}\times\frac{5}{3}=\frac{5}{3}$=quotient
3. 10=quotient.
4. ∴ $\frac{5}{3}=10$,
5. $\frac{1}{3}=\frac{1}{5}$ of 10=2, and
6. $\frac{3}{3}$=3 times 2=6=the number.

III. ∴ 6=the number required.

I. $24 is $\frac{3}{5}$ of the cost of a barrel of wine; what did it cost?

II.
1. $\frac{5}{5}$=cost of the wine per barrel.
2. $\frac{3}{5}$ of cost=$24,
3. $\frac{1}{5}$ of cost=$\frac{1}{3}$ of $24=$8,
4. $\frac{5}{5}$ of cost=5 times $8=$40,

III. ∴ $40=cost of wine.

I. What number is that from which, if you take $\frac{3}{7}$ of itself, the remainder will be 16?

II.
1. $\frac{7}{7}$=the number.
2. $\frac{7}{7}-\frac{3}{7}=\frac{4}{7}$=remainder after taking away $\frac{3}{7}$.
3. 16=remainder.
4. ∴ $\frac{4}{7}=16$,
5. $\frac{1}{7}=\frac{1}{4}$ of 16=4, and
6. $\frac{7}{7}$=7 times 4=28=the number

III. ∴ 28=the required number.

1. AGE PROBLEMS.

I. A is 30 years old, and B is 6 years old; in how many years will A be only 4 times as old as B?

II.
1. $\frac{2}{2}$=B's age at the required time. Then
2. $\frac{8}{2}$=A's age at the required time.
3. $\frac{8}{2}-\frac{2}{2}=\frac{6}{2}$=difference of their ages.
4. 30 years—6 years=24 years=difference of their ages.
5. $\therefore \frac{6}{2}$=24 years.
6. $\frac{1}{2}=\frac{1}{6}$ of 24 years=4 years.
7. $\frac{2}{2}$=2 times 4 years=8 years, B's age at the required time.
8. $\therefore$ 8 years—6 years=2 years=the number of years hence when A will be only 4 times as old as B.

III. $\therefore$ In 2 years A will be only 4 times as old as B.

I. Jacob is twice as old as his son who is 20 years of age; how long since Jacob was 5 times as old as his son?

II.
1. 20 years=son's age at present. Then
2. 40 years=Jacob's age at present.
3. $\frac{2}{2}$=son's age at required time. Then
4. $\frac{10}{2}$=Jacob's age at required time.
5. $\therefore \frac{10}{2}-\frac{2}{2}=\frac{8}{2}$=difference of their ages.
6. 40 years—20 years=20 years=difference of their ages.
7. $\therefore \frac{8}{2}$=20 years,
8. $\frac{1}{2}=\frac{1}{8}$ of 20 years=$2\frac{1}{2}$ years, and
9. $\frac{2}{2}$=2 times $2\frac{1}{2}$ years=5 years, son's age at the required time.
10. $\therefore$ 20 years—5 years=15 years=time since Jacob was 5 times as old as his son.

III. $\therefore$ 15 years ago Jacob was 5 times as old as his son.

Remarks.—Observe that the difference between any two persons' ages is constant, that is, if the difference between A's and B's ages is 7 years now, it will be the same in any number of years from now; for, as a year is added to one's age, it is likewise added to the other's age. But the ratio of their ages is constantly changing as time goes on. If A is 3 years old and B 5 years old, A is now $\frac{4}{5}$ as old as B; but in 1 year, A's age will be 4 years and B's 6 years; A is then $\frac{2}{3}$ as old as B. In 7 years, A will be 10 years old and B 12; A will then be $\frac{10}{12}$, or $\frac{5}{6}$, as old as B, and so on. The ratio of any two persons' ages approaches unity as its limit.

I. At the time of marriage a wife's age was $\frac{3}{5}$ of the age of her husband, and 10 years after marriage her age was $\frac{7}{10}$ of the age of her husband; how old was each at the time of marriage?

1. $\frac{5}{5}$=husband's age at the time of marriage. Then
2. $\frac{3}{5}$=wife's age at the time of marriage.
3. $\frac{5}{5}$+10 years=husband's age 10 years after marriage.
4. $\frac{3}{5}$+10 years=wife's age 10 years after marriage. But

II.
5. $\frac{7}{10}+7$ years$=\frac{7}{10}$ of $(\frac{5}{5}+10$ years$)=$wife's age 10 years after marriage, by second condition of the problem.
6. $\therefore$ $\frac{7}{10}+7$ years$=\frac{3}{5}+10$ years. Whence
7. $\frac{7}{10}-\frac{3}{5}=10$ years-7 years, or
8. $\frac{1}{10}=3$ years.
9. $\frac{10}{10}=10$ times 3 years$=30$ years$=$husband's age at time of marriage.
10. $\frac{3}{5}$, or $\frac{6}{10}$,$=6$ times 3 years$=18$ years$=$wife's age at the time of marriage.

III. $\therefore$ { 30 years$=$husband's age at time of marriage, and
18 years$=$wife's age at time of marriage.

(*White's Comp. A., p. 241, prob. 35.*)

I. Ten years ago the age of A was $\frac{3}{4}$ of the age of B, and ten years hence the age of A will be $\frac{5}{6}$ of the age of B; find the age of each.

II.
1. $\frac{4}{4}=$B's age 10 years ago. Then
2. $\frac{3}{4}=$A's age 10 years ago.
3. $\frac{4}{4}+10$ years$=$B's age now, and
4. $\frac{3}{4}+10$ years$=$A's age now.
5. $\frac{4}{4}+20$ years$=$B's age 10 years hence, and
6. $\frac{3}{4}+20$ years$=$A's age 10 years hence.
7. $\frac{5}{6}$ of $(\frac{4}{4}+20$ years$)=\frac{5}{6}+16\frac{2}{3}$ years$=$A's age 10 years hence.
8. $\therefore$ $\frac{5}{6}+16\frac{2}{3}$ years$=\frac{3}{4}+20$ years; whence
9. $\frac{5}{6}-\frac{3}{4}=20$ years$-16\frac{2}{3}$ years, or
10. $\frac{1}{12}=3\frac{1}{3}$ years, and
11. $\frac{12}{12}=12$ times $3\frac{1}{3}$ years$=40$ years$=$B's age 10 years ago.
12. $\frac{3}{4}=\frac{9}{12}=9$ times $3\frac{1}{3}$ years$=30$ years$=$A's age 10 years ago.
13. $\therefore$ $\frac{12}{12}+10$ years$=50$ years$=$B's age now, and
14. $\frac{9}{12}+10$ years$=40$ years$=$A's age now.

III. $\therefore$ 50 years$=$B's age, and 40 years$=$A's age.

2. FOX AND HOUND PROBLEMS.

Under this head comes a class of problems which may be variously named as "Hare and Hound Problems," "Step Problems," "Hare and Tortoise Problems," etc.

I. A fox is 50 leaps ahead of a hound, and takes 4 leaps in the same time that the hound takes 3; but 2 of the hound's leaps equal 3 of the fox's leaps. How many leaps must the hound take to catch the fox?

II.
1. 2 leaps of hound's$=$3 leaps of fox's.
2. 1 leap of hound's$=\frac{1}{2}$ of 3 leaps$=1\frac{1}{2}$ leaps of the fox's.
3. 3 leaps of hound's$=3$ times $1\frac{1}{2}$ leaps$=4\frac{1}{2}$ leaps of fox's.
4. $\therefore$ $4\frac{1}{2}$ leaps-4 leaps$=\frac{1}{2}$ leap$=$what the hound gains in taking 3 leaps.
5. $\therefore$ 1 leap$=2$ times $\frac{1}{2}$ leap$=$what the hound gains in taking 6 leaps.
6. $\therefore$ 50 leaps$=$what the hound gains in taking 50×6 leaps, or 300 leaps.

III. ∴ The hound must take 300 leaps to catch the fox.

Remark —We see that 3 of the hound's leaps equals $4\frac{1}{2}$ leaps of the fox's. But while the hound takes 3 leaps, the fox takes 4 leaps; hence the hound gains $4\frac{1}{2}-4$, or $\frac{1}{2}$, leap of the fox's. But he has 50 leaps of the fox's to gain, and since he gains $\frac{1}{2}$ leap of the fox's in 3 leaps, he must take 300 leaps to gain 50 leaps.

I. A thief is 20 steps before an officer, and takes 6 steps while the officer takes 5, but 5 of the officer's steps equal 8 of the thief's; how far will the thief run before he is overtaken?

II.
1. 5 steps of the officer=8 steps of the thief,
2. ∴ 8 steps—6 steps=2 steps=the distance the officer gains on the thief every time he takes 5 steps.
3. $\frac{2}{5}$ step=distance the officer gains on thief in taking 1 step.
4. ∴ 20 steps=distance the officer gains in taking $20\div\frac{2}{5}$, or 50 steps. But, since
5. 5 steps of the officer=8 steps of the thief,
6. 50 steps of the officer=80 steps of the thief.
7. ∴ 80 steps—20 steps=60 steps=distance thief runs before he is overtaken.

III. ∴ The thief will run 60 steps.

Remark.—It should be observed that the thief is 20 of his own steps ahead of the officer, and thus his step becomes the unit of measure. It requires 50 of the officer's own steps to overtake the thief. But these are equal to 80 of the thief's. Since the officer took 50 steps equal to 80 of the thief's steps to overtake the thief and the thief was 20 steps ahead, it follows that the thief took 60 steps before he was overtaken.

I. *Achilles* is 1 mile behind a tortoise and runs 10 times as fast as the tortoise. Will Achilles overtake the tortoise?

Proposed by *Zeno, the Eliatic.*

Zeno's argument:

II.
1. While Achilles is running the 1 mile, the tortoise will have run $\frac{1}{10}$ of a mile farther;
2. While Achilles is running the $\frac{1}{10}$ mile, the tortoise will have advanced $\frac{1}{100}$ of a mile still farther;
3. While Achilles is running the $\frac{1}{100}$ of a mile, the tortoise will have advanced $\frac{1}{1000}$ of a mile still farther; and so on, *ad infinitum.*

III. ∴ Achilles will never overtake the tortoise.

Remark.—This problem was proposed by Zeno, one of the most famous members of the *Eliatic School* of philosophers. He was born 495 B. C. and was executed at Elea, a town on the island of Sicily, in 435 B. C.

This paradox and the one to the effect that an arrow cannot move where it is not, and since also it cannot move where it is, that is, in the space it exactly fills, and hence cannot move at all, were put forth by Zeno to disprove mition.

We bring this problem up for discussion not because we have anything particularly new to add to it; for it has been very thoroughly discussed by some of the greatest thinkers that the world has ever produced, and in its complete solution are involved subtile metaphysical questions that have not yet been satisfactorily explained, and perhaps may never be.

The fallacy in Zeno's argument concerning the motion of the arrow is that he assumes that at every instant the arrow must be resting in a definite point. But if it is resting, it is not moving and cannot move. Only that rests at a point which remains in it for some consecutive instants. Zeno confounds being in a point in the sense of resting in it, with being in a point, with the sense of passing through it.* On the fundamental notion of motion, the reader should read articles 33, 34, 35, and 36 of W. B. Smith's *Infinitesimal Analysis.*

In the case of *Achilles* and the tortoise, if it be understood that they traveled with uniform motion, and traveled a finite distance in a finite time, then Zeno's argument is false; for while the time required is divided into an infinite number of parts, these parts diminish in a geometrical progression and the sum of them all is a finite sum.

The distance *Achilles* must run to overtake the tortoise is $1\frac{1}{9}$ miles. For, in running a mile, Achilles gains $\frac{9}{10}$ of a mile. Hence, to gain 1 mile, he will have to run as many miles as $\frac{9}{10}$ is contained in 1 mile or $1\frac{1}{9}$ miles. This would be the distance he would have to run in any case; and whether he would or would not overtake the tortoise depends upon the law of his motion

If the word, *while*, in the three steps of Zeno's argument represent the same period of time, then it would take *Achilles* as long to travel $\frac{1}{10}$ mile as to travel 1 mile, and as long to travel $\frac{1}{100}$ mile as to travel $\frac{1}{10}$ mile, etc. In this case, *Achilles* and the tortoise are not traveling with uniform motion, and *Achilles* would never overtake the tortoise. It is contended by some logicians and metaphysicians, that in the use of the ambiguous term, "while," lies the fallacy of Zeno's argument.

3. FISH PROBLEMS.

I. The head of a trout weighs 2 pounds, the tail weighs 2 pounds more than the head, plus $\frac{1}{3}$ of the body, and the body weighs as much as the head and tail together; required the weight of the fish?

(*Brooks' Int. Arith., p. 143, prob. 10.*)

II.

1. $\frac{2}{2}$=the weight of the tail.
2. $\frac{2}{2}$+2 lbs., the weight of the head and tail,=the weight of the body.
3. $\frac{2}{6}+\frac{2}{3}$ lbs.=$\frac{1}{3}$ of ($\frac{2}{2}$+2 lbs.)=the weight of $\frac{1}{3}$ of the body.
4. ($\frac{2}{6}+\frac{2}{3}$ lbs.)+2 lbs.+2 lbs.=$\frac{2}{6}+4\frac{2}{3}$ lbs.=2 lbs. more than the weight of the head and $\frac{1}{3}$ of the body.
5. ∴ $\frac{2}{2}$ or $\frac{6}{6}=\frac{2}{6}+4\frac{2}{3}$ lbs., since the tail weighs as much as the head and $\frac{1}{3}$ the body+2 lbs.

   ∴ $\frac{4}{6}=4\frac{2}{3}$ lbs.

7. $\frac{1}{6}=\frac{1}{4}$ of $4\frac{2}{3}$ lbs.=$1\frac{1}{6}$ lbs.
8. $\frac{6}{6}$ or $\frac{2}{2}$=6 times $1\frac{1}{6}$ lbs.=7 lbs., the weight of the tail.
9. 7 lbs.+2 lbs.=9 lbs.=weight of the body.
10. 7 lbs.+2 lbs.+2 lbs.=11 lbs.=the weight of the fish.

III. ∴ The weight of the fish is 11 lbs.

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* Bowen's *Metaphysics*, page 81. The entire chapter on motion in this book is very interesting.

Remark.—In the solution of "fish problems" and those that are similar, we should always take that number as the base from which the other numbers are readily obtained when the base is once known. Thus, in the above problem, when the weight of the tail is known, the weight of the body is easily obtained. Hence, we let $\frac{2}{2}$ represent the weight of the tail. We might as well have taken $\frac{4}{4}$ or $\frac{6}{6}$ to represent the weight of the tail.

I. **The head of a fish is 8 inches long, the tail is as long as the head and $\frac{1}{2}$ of the body+10 inches, and the body is as long as the head and tail; what is the length of the fish?**

II.
1. $\frac{2}{2}$=length of body.
2. 8 in.=length of head.
3. $\frac{1}{2}$ l. of b.+10 in.+8 in.=$\frac{1}{2}$ l. of b.+18 in.=length of tail.
4. $\frac{2}{2}$ l. of b.=length of head+length of tail.
5. ∴ $\frac{2}{2}$ l. of b.=($\frac{1}{2}$ l. of b.+18 in.)+8 in.=$\frac{1}{2}$ l. of b.+26 in. Whence
6. $\frac{2}{2}$ l. of b.—$\frac{1}{2}$ l. of b.=$\frac{1}{2}$ l. of b.=26 in.
7. ∴ $\frac{2}{2}$ l. of b., or length of body,=2 times 26 in.=52 in.
8. $\frac{1}{2}$ l. of b.+18 in.=26 in.+18 in.=44 in.=length of tail.
9. ∴ 52 in.+44 in.+8 in.=104 in.=length of the fish.

III. ∴ The length of the fish is 104 inches.

4. ANIMAL PROBLEMS.

I. A man bought a certain number of sheep for \$80; if he then buys twice as many more, at \$2 less each, they will cost \$180; how many did he buy?

(*Brooks' Int. Arith., p. 162, prob. 11.*)

II.
1. $\frac{2}{2}$=number of sheep bought, and for which he paid \$80.
2. $\frac{4}{2}$=2 times $\frac{2}{2}$=number he bought at \$2 less per head and for which he paid \$180—\$80, or \$100.
3. ∴ $\frac{2}{2}$ at \$2 less per head would have cost $\frac{1}{2}$ of \$100, or \$50.
4. ∴ The same number of sheep that he bought for \$80 at \$2 less per head would cost \$50.
5. \$2=the reduction on the price of 1 sheep.
6. \$30=\$80—\$50=reduction on \$30÷\$2, or 15 sheep.

III. ∴ He bought 15 sheep.

I. Henry Adams bought a number of pigs for \$48; and losing 3 of them, he sold $\frac{2}{3}$ of the remainder, minus 2, for cost, receiving \$32 less than all cost him; required the number purchased.

1. $\frac{3}{3}$=remainder after losing 3. Then
2. $\frac{3}{3}$+3=number at first.
3. $\frac{2}{3}$ of r.—2=number sold.

II.
4. $48—$32=$16=what was received for $\frac{2}{3}$ of r.—2.
5. $8=$\frac{1}{2}$ of $16=what was received for $\frac{1}{2}$ of ($\frac{2}{3}$ of r.—2), or $\frac{1}{3}$ of r.—1.
6. $24=3 times $8=what was received for 3 times ($\frac{1}{3}$ of r.—1)=$\frac{3}{3}$ of r.—3.
7. ∴ $48—$24=$24=what ($\frac{3}{3}$ of r.+3)—($\frac{3}{3}$ of r.—3), or 6 pigs cost.
8. $4=$\frac{1}{6}$ of $24=what 1 pig cost.
9. ∴ $48=what 48÷4, or 12, pigs cost.

III. ∴ He bought 12 pigs.

(*Brooks' Int. A., p. 164, prob. 9.*)

I. A bought some calves for $80; and having lost 10, he sold 4 more than $\frac{2}{3}$ of the remainder for cost and received $32 less than all cost; required the number purchased.

II.
1. $\frac{3}{3}$=remainder after losing 10. Then
2. $\frac{3}{3}$+10=number purchased.
3. $\frac{2}{3}$ of r.+4=number sold.
4. $80—$32=$48=cost of $\frac{2}{3}$ of r.+4, since they sold at cost.
5. $24=$\frac{1}{2}$ of $48=cost of $\frac{1}{2}$ of ($\frac{2}{3}$ of r.+4)=$\frac{1}{3}$ of r.+2.
6. $72=3 times $24=cost of 3 times ($\frac{1}{3}$ of r.+2)=$\frac{3}{3}$ of r.+6.
7. ∴ $80—$72=$8=what ($\frac{3}{3}$ of r.+10)—($\frac{3}{3}$ of r.+6), or 4 cost.
8. $2=$\frac{1}{4}$ of $8=what 1 cost.
9. $80=what 80÷2, or 40 cost.

III. ∴ He bought 40 calves.

(*Brook's Int. A., p. 164, prob. 10.*)

I. A lost $\frac{3}{5}$ of his sheep; now if he finds 5 and sells $\frac{3}{5}$ of what he then has for cost price, he will receive $18; but if he loses 5 and sells $\frac{3}{5}$ of the remainder for cost price, he will receive $6; how many sheep had he at first? (*Brook's Int. A., p. 165, prob. 15.*)

II.
1. $\frac{5}{5}$= the number of sheep he had at first.
2. $\frac{3}{5}$= the number he lost.
3. $\frac{5}{5}$—$\frac{3}{5}$=$\frac{2}{5}$, the number he had after losing $\frac{3}{5}$.
4. $\frac{2}{5}$+5= the number he had after finding 5.
5. $\frac{3}{5}$ of ($\frac{2}{5}$+5)=$\frac{6}{25}$+3, the number he sold.
6. $\frac{2}{5}$—5= the number, had he lost 5.
7. $\frac{3}{5}$ of($\frac{2}{5}$—5)=$\frac{6}{25}$—3, the number he would have sold.
8. $18=what ($\frac{6}{25}$+3) sheep cost.
9. $6= what ($\frac{6}{25}$—3) sheep cost.
10. ∴ $12=$18—$6=what ($\frac{6}{25}$+3) sheep—($\frac{6}{25}$—3) sheep, or 6 sheep cost.
11. $2=$\frac{1}{6}$ of $12= what 1 sheep cost.
12. $18= what 18÷2, or 9 sheep cost. But

13. \$18=what $(\frac{6}{25}+3)$ sheep cost.
14. ∴ $\frac{6}{25}+3$ sheep = 9 sheep, or
15. $\frac{6}{25}=6$ sheep.
16. $\frac{1}{25}=\frac{1}{6}$ of 6 sheep=1 sheep, and
17. $\frac{25}{25}=25$ times 1 sheep =25 sheep.

III. ∴ He had 25 sheep at first.

I. A man bought a certain number of cows for \$200; had he bought 2 more at \$2 less each, they would have cost him \$216; how many did he buy?

II.
1. \$200=cost of cows.
2. \$216=cost of original number of cows+2 more.
3. \$216—\$200=\$16=cost of 2 cows at \$2 less per head.
4. ∴ \$8=$\frac{1}{2}$ of \$16=cost of 1 cow at \$2 less per head. Then
5. \$8+\$2=\$10=cost of each cow purchased.
6. \$200=cost of 200÷10, or 20 cows.

III. ∴ He bought 20 cows. (*Brook's Int. A., p. 162, prob. 8.*)

5. LABOR PROBLEMS.

I. A laborer agreed to work for \$2 a day, on condition that for every day he was idle he should forfeit 50 ¢; how many days did he labor, if at the end of 25 days he received \$35?

II.
1. \$2=amount received for 1 day's labor.
2. \$50=25×\$2=amount he would have received for 25 days' labor.
3. \$50—\$35=\$15=amount he forfeited by his idleness.
4. \$2, his wages,+\$$\frac{1}{2}$, his forfeit,=\$$2\frac{1}{2}$=amount he lost each day by his idleness.
5. ∴ \$15=amount he lost for as many days' idleness as \$$2\frac{1}{2}$ is contained in \$15, or 6 days.
6. ∴ 25 days—6 days=19 days, the time he labored.

III. ∴ He labored 19 days.

I. A man was engaged for one year at \$80 and a suit of clothes; he served 7 months, and received for his wages the clothes and \$35; what was the value of the clothes?

II.
1. $\frac{12}{12}$=value of the suit of clothes.
2. $\frac{12}{12}$+\$80=wages for 1 year or 12 months.
3. $\frac{1}{12}$+\$$6\frac{2}{3}$=$\frac{1}{12}$ of ($\frac{12}{12}$+\$80)=wages for 1 month.
4. $\frac{7}{12}$+\$$46\frac{2}{3}$=7 times ($\frac{1}{12}$+\$$6\frac{2}{3}$)=wages for 7 months.
5. $\frac{12}{12}$+\$35=wages for 7 months.
6. ∴ $\frac{12}{12}$+\$35=$\frac{7}{12}$+\$$46\frac{2}{3}$.
7. $\frac{5}{12}$=\$$11\frac{2}{3}$,
8. $\frac{1}{12}=\frac{1}{5}$ of \$$11\frac{2}{3}$=\$$2\frac{1}{3}$, and
9. $\frac{12}{12}$=12 times \$$2\frac{1}{3}$=\$28=value of suit of clothes.

III. ∴ The suit of clothes is worth \$28.

6. WORK PROBLEMS AND PIPE PROBLEMS.

I. Three men, A, B, C, can do a piece of work in 60 days. After working together 10 days, A withdraws and B and C work together at the same rate for 20 days, then B withdraws and C completes the work in 96 days, working $\frac{1}{3}$ longer each day. Working at his former rate, C could alone do the work in 222 days; find how long it would take A and B each separately to do the work.

II.
1. 60 days=time it would take A, B, and C to do the work.
2. $\frac{1}{60}$=part of the work they do in 1 day, and
3. $\frac{1}{6}$=part of the work they do in 10 days.
4. $\frac{6}{6}-\frac{1}{6}=\frac{5}{6}$=part left for B and C to do.
5. 1 day of C's after B withdrew=$\frac{4}{3}$ of one of his days before B withdrew.
6. 96 days of C's after B withdrew=$96\times\frac{4}{3}$ of one day =128 days of C's before B withdrew.
7. ∴ 20 days+128 days=148 days=time C worked after A withdrew.
8. 222 days=time C could do the work alone.
9. $\frac{1}{222}$=part he could do in 1 day.
10. $\frac{148}{222}=\frac{4}{6}$=part he did in 148 days.
11. ∴ $\frac{5}{6}-\frac{4}{6}=\frac{1}{6}$=part B did in 20 days.
12. ∴ $\frac{1}{120}=\frac{1}{20}$ of $\frac{1}{6}$=part B does in 1 day.
13. $\frac{120}{120}$, or the work,=what he can do in $\frac{120}{120}\div\frac{1}{120}$, or 120 days.
14. $\frac{1}{60}-(\frac{1}{120}+\frac{1}{222})=\frac{17}{4440}$=part A does in 1 day.
15. $\frac{4440}{4440}$, or the work,=what he can do in $\frac{4440}{4440}\div\frac{17}{4440}$, or $261\frac{3}{17}$ days.

III. ∴ A can do the work in $261\frac{3}{17}$ days, and B can do it in 120 days.

I. A cistern has three pipes: the 1st can fill the cistern in $1\frac{1}{3}$ hr.; the 2d, in $3\frac{1}{3}$ hr.; and the 3d can empty it in 5 hours. How long will it take to fill the cistern if all pipes are left running?

II.
1. $1\frac{1}{3}$ hr.=time it takes the first pipe to fill the cistern.
2. ∴ $\frac{3}{4}$=part of cistern filled by 1st pipe in 1 hr.
3. $3\frac{1}{3}$ hr.=time it takes the 2d to fill it.
4. ∴ $\frac{3}{10}$=part filled by the 2d in 1 hr.
5. 5 hr.=time it takes the 3d to empty the cistern.
6. ∴ $\frac{1}{5}$=part emptied by the 3d in 1 hr.
7. ∴ $\frac{3}{4}+\frac{3}{10}-\frac{1}{5}=\frac{17}{20}$=part filled when the pipes are all running.
8. ∴ $\frac{20}{20}$, or the whole cistern,=what is filled in $\frac{20}{20}\div\frac{17}{20}$, or $1\frac{3}{17}$ hr.

III. ∴ The cistern can be filled in $1\frac{3}{17}$ hr.

I. There is coal now on the dock, and coal is running on also from a shoot at a uniform rate. Six men can clear the dock in 1 hour, but 11 men can clear it in 20 minutes; how long would it take 4 men?

II.
1. $\frac{2}{2}$=what one man removes in 1 hour. Then
2. $\frac{12}{2}=6$ times $\frac{2}{2}$=what 6 men remove in 1 hour.
3. $\frac{2}{6}=\frac{1}{3}$ of $\frac{2}{2}$=what 1 man removes in 20 min., or $\frac{1}{3}$ hour.
4. $\frac{22}{6}=11$ times $\frac{2}{6}$=what 11 men remove in $\frac{1}{3}$ hour.
5. ∴ $\frac{12}{2}-\frac{22}{6}=\frac{14}{6}$= what runs on in 1 hr.—$\frac{1}{3}$ hr.=$\frac{2}{3}$ hr. Then
6. $\frac{7}{2}=\frac{14}{6}\div\frac{2}{3}$=what runs on in 1 hour.
7. ∴ $\frac{12}{2}-\frac{7}{2}=\frac{5}{2}$=what was on the dock when the work commenced.
8. $\frac{8}{2}$=what 4 men remove in 1 hour.
9. ∴ $\frac{8}{2}-\frac{7}{2}=\frac{1}{2}$=part of coal removed every hour, that was on the dock at first.
10. $\frac{5}{2}$=coal to be removed in $\frac{5}{2}\div\frac{1}{2}=5$ hours.

(*R. H. A., p. 406, prob. 90.*)

III. ∴ It will take 4 men, 5 hours to clear the dock.

Explanation.—$\frac{12}{2}$=what 6 men remove in 1 hr. and $\frac{22}{6}$=what 11 men removed in $\frac{1}{3}$ hr. In either case the dock was cleared. ∴ $\frac{12}{2}-\frac{22}{6}=\frac{14}{6}$= amount of coal that ran on the dock from the shoot in 1 hr.—$\frac{1}{3}$ hr., or $\frac{2}{3}$ hr. Hence in 1 hr. there will run on, $\frac{14}{6}\div\frac{2}{3}=\frac{7}{2}$. Since $\frac{7}{2}$ run on in 1 hr. and $\frac{12}{2}$ =the whole amount of coal removed in 1 hr., $\frac{12}{2}-\frac{7}{2}$, or $\frac{5}{2}$ must be the amount of coal on the dock when the work began. Since $\frac{8}{2}$=the amount 4 men remove in 1 hr. and $\frac{7}{2}$=the amount that runs on the dock in 1 hr., $\frac{8}{2}-\frac{7}{2}$, or $\frac{1}{2}$ is the part of the original quantity removed each hour. Hence, if $\frac{1}{2}$ is removed in 1 hour $\frac{5}{2}$ would be removed in $\frac{5}{2}\div\frac{1}{2}$, or 5 hours.

I. A and B perform $\frac{9}{10}$ of a piece of work in 2 days, when, B leaving, A completes it in $\frac{1}{2}$ day; in what time can each complete it alone?

II.
1. $\frac{9}{10}$=part A and B do in 2 days.
2. $\frac{9}{20}=\frac{1}{2}$ of $\frac{9}{10}$=part A and B do in 1 day.
3. $\frac{10}{10}-\frac{9}{10}=\frac{1}{10}$=part left after B quits, and which A completes in $\frac{1}{2}$ day.
4. $\frac{2}{10}=\frac{1}{5}$=part A can do in 1 day.
5. ∴ $\frac{5}{5}$=part A can do in $\frac{5}{5}\div\frac{1}{5}=5$ days.
6. $\frac{9}{20}-\frac{1}{5}=\frac{5}{20}=\frac{1}{4}$=part B can do in 1 day.
7. ∴ $\frac{4}{4}$=part B can do in $\frac{4}{4}\div\frac{1}{4}$, or 4, days.

III. ∴ { A can do the work in 5 days, and
B can do the work in 4 days.

(*White's Comp. A., p. 280, prob. 193.*)

I. A and B can do a piece of work in 12 days, B and C in 9 days, and A and C in 6 days; how long will it take each alone to do the work?

II.
1. 12 days=time it takes A and B to do the work.
2. ∴ $\frac{1}{12}$=part they do in 1 day.
3. 9 days=time it takes B and C to do the work.
4. ∴ $\frac{1}{9}$=part they do in 1 day.
5. 6 days=time it takes A and C to do the work.
6. ∴ $\frac{1}{6}$=part they do in 1 day.
7. ∴ $\frac{1}{12}+\frac{1}{9}+\frac{1}{6}=\frac{13}{36}$=part A and B, B and C, and A and C do in 1 day=twice the work A, B, and C do in 1 day.
8. ∴ $\frac{13}{72}=\frac{1}{2}$ of $\frac{13}{36}$=part A, B, and C do in 1 day.
9. $\frac{13}{72}-\frac{1}{12}=\frac{7}{72}$=part A, B, and C do in 1 day—part B and C do in 1 day=part C does in 1 day.
10. $\frac{72}{72}$=part C does in $\frac{72}{72}\div\frac{7}{72}$, or $10\frac{2}{7}$ days.
11. $\frac{13}{72}-\frac{1}{9}=\frac{5}{72}$=part A, B, and C do in 1 day—part B and C do in 1 day=part A does in 1 day.
12. $\frac{72}{72}$=part A does in $\frac{72}{72}\div\frac{5}{72}=14\frac{2}{5}$ days.
13. $\frac{13}{72}-\frac{1}{6}=\frac{1}{72}$=part A, B, and C do in 1 day—part A and C do in 1 day=part B does in 1 day.
14. $\frac{72}{72}$=part B does in $\frac{72}{72}\div\frac{1}{72}$=72 days

III. ∴ $14\frac{2}{5}$ days=time it takes A, 72 days=time it takes B, and $10\frac{2}{7}$ days=time it takes C.

(*White's Comp. A., p. 194, prob. 280.*)

I. A man and a boy can mow a certain field in 8 hours, if the boy rests $3\frac{3}{4}$ hours, it takes them $9\frac{1}{2}$ hours. In what time can each do it?

II.
1. $9\frac{1}{2}$ hr.—$3\frac{3}{4}$ hr.=$5\frac{3}{4}$ hr.=time they both work together in the second case.
2. 8 hr.=time it takes them to do the work.
3. ∴ $\frac{1}{8}$=part they do in 1 hour.
4. $\frac{5\frac{3}{4}}{8}=\frac{23}{32}=5\frac{3}{4}$ times $\frac{1}{8}$=part they do in $5\frac{3}{4}$ hours.
5. ∴ $\frac{32}{32}-\frac{23}{32}=\frac{9}{32}$=part the man did in $3\frac{3}{4}$ hours, while the boy rested.
6. ∴ $\frac{3}{40}=\frac{1}{3\frac{3}{4}}$ of $\frac{9}{32}$=part the man did in 1 hour.
7. ∴ $\frac{40}{40}$=part the man can do in $\frac{40}{40}\div\frac{3}{40}$ or $13\frac{1}{3}$ hours.
8. $\frac{1}{8}-\frac{3}{40}=\frac{1}{20}$=part the boy does in one hour.
9. ∴ $\frac{20}{20}$=part the boy can do in $\frac{20}{20}\div\frac{1}{20}$ or 20 hours.

III. ∴ It will take the man $13\frac{1}{3}$ hours, and The boy 20 hours. (*R. H. A., p. 402, prob. 30.*)

I. Six men can do a work in $4\frac{1}{3}$ days; after working 2 days, how many must join them so as to complete it in $3\frac{2}{5}$ days?

II.
1. $4\frac{1}{3}$ days=time it takes 6 men.
2. 26 days=6 times $4\frac{1}{3}$ days=time it takes 1 man.
3. ∴ $\frac{1}{26}$=part 1 man does in 1 day.
4. $\frac{3}{13}$=6 times $\frac{1}{26}$=part 6 men do in 1 day.
5. $\frac{6}{13}$=2 times $\frac{3}{13}$=part 6 men do in 2 days.
6. $\frac{13}{13}-\frac{6}{13}=\frac{7}{13}$=part to be done in $3\frac{2}{5}$ days—2 days, or $1\frac{2}{5}$ [days.
7. $\dfrac{1\frac{2}{5}}{26}=\frac{7}{130}$=part 1 man does in $1\frac{2}{5}$ days.
8. ∴ $\frac{7}{13}$=part $\frac{7}{13}\div\frac{7}{130}$, or 10 men can do in $1\frac{2}{5}$ days.
9. ∴ 10 men—6 men=4 men, the number that must join them.

III. ∴ They must be joined by 4 more men that they may complete the work in $3\frac{2}{5}$ days. *R. H. A., p. 402, prob. 34.*

7. WINE AND WATER PROBLEMS.

I. How much water is there in a mixture of 100 gal. of wine and water, worth $1 per gal., if 100 gal. of the wine costs $120?

(*Robinson's New Higher Arithmetic, p. 502.*)

II.
1. $1.20=$120÷100=cost of 1 gal. of wine.
2. $1=cost of wine in 1 gal. of the mixture.
3. ∴ $\frac{5}{6}$ gal.=$1.00÷$1.20=quantity of wine in each gal. of the mixture.
4. ∴ $\frac{1}{6}$ gal=1 gal.—$\frac{5}{6}$ gal.=quantity of water in each gal. of the mixture.
5. ∴ $16\frac{2}{3}$ gal.=$\frac{1}{6}$ of 100 gal.=quantity of water in 100 gal. of the mixture.

III. ∴ In 100 gal. of the mixture there are $16\frac{2}{3}$ gal. of water.

I. From a ten-gallon keg of wine, one gallon is drawn off and the keg filled with water; if this is repeated 4 times, what will be the quantity of wine in the keg?

II.
1. $\frac{1}{10}$=part drawn out each time.
2. $\frac{9}{10}$=part that was pure wine after the first draught.
3. $\frac{1}{10}$ of $\frac{9}{10}=\frac{9}{100}$=part wine drawn off the second draught.
4. $\frac{9}{10}-\frac{9}{100}=\frac{81}{100}$= part pure wine left after the second [draught.
5. $\frac{1}{10}$ of $\frac{81}{100}=\frac{81}{1000}$= part wine drawn off at the third
6. $\frac{81}{100}-\frac{81}{1000}=\frac{729}{1000}$=part pure wine left after the third [draught.
7. $\frac{1}{10}$ of $\frac{729}{1000}=\frac{729}{10000}$=part wine drawn off at the fourth
8. $\frac{729}{1000}-\frac{729}{10000}=\frac{6561}{10000}$= part pure wine left after fourth draught. [fourth draught.
9. ∴ $\frac{6561}{10000}$ of 10 gal.=6.561 gal.=pure wine left after the

III. ∴ There will be 6.561 gal. of pure wine in the keg after the fourth draught.

I. In the above problem, how many draughts are necessary to draw off half the wine?

II.
1. $\frac{1}{10}$=part wine drawn off at the first draught.
2. $\frac{10}{10}-\frac{1}{10}=\frac{9}{10}$=part wine left after the first draught.
3. $\frac{1}{10}$ of $\frac{9}{10}=\frac{9}{100}=\frac{9}{10^2}$=part wine drawn off at the second draught.
4. $\frac{9}{10}-\frac{9^2}{10^2}=\frac{81}{100}=(\frac{9}{10})^2$=part wine left after the second draught.
5. $\frac{1}{10}$ of $(\frac{9}{10})^2=\frac{9^2}{10^3}$=part wine drawn off at the third draught.
6. $(\frac{9}{10})^2-\frac{9^2}{10^3}=(\frac{9}{10})^3$=part wine left after the third draught. By induction,
7. $(\frac{9}{10})^n$=part wine left after the *n*th draught.
8. ∴ $10(\frac{9}{10})^n$=number of gal. left after the *n*th draught.
9. 5=number of gal. left after the *n*th draught.
10. ∴ $10(\frac{9}{10})^n=5$, whence
11. $(\frac{9}{10})^n=\frac{1}{2}$. Applying logarithms,
12. $n \log. \frac{9}{10}=\log. \frac{1}{2}$.
13. ∴ $n=\log. \frac{1}{2}\div\log. \frac{9}{10}=.30103\div.\bar{1}.954243=.301030\div.045757=6+$.

III. ∴ In 7 draughts, half and a little more than half of the wine will be drawn off.

8. SHEEP AND COW PROBLEMS.

I. A farmer keeps 60 cows on his farm. For every 4 cows, he plows 1 acre of ground and for every 5 cows, he pastures 1 acre. How many acres in the farm?

II.
1. 1 A.=what he plows for 4 cows.
2. $\frac{1}{4}$ A.=what he plows for 1 cow.
3. 1 A.=what he pastures for 5 cows.
4. $\frac{1}{5}$ A.=what he pastures for 1 cow.
5. ∴ $\frac{1}{4}$ A.+$\frac{1}{5}$ A.=$\frac{9}{20}$ A.=land required for each cow.
6. ∴ 60 cows require as many acres as $\frac{9}{20}$ is contained in 60, or $133\frac{1}{3}$ acres.

III. ∴ There are $133\frac{1}{3}$ acres in the farm.

Remark.—The "Coach Problems," page 187 are solved on the same principle as these problems and might have been grouped with this set.

I. For every 10 sheep I keep I plow an acre of land, and allow one acre of pasture for every 4 sheep; how many sheep can I keep on 161 acres?

II.
1. 1A.=what I plow for every 10 sheep I keep.
2. $\frac{1}{10}$A.=what I plow for each sheep I keep.
3. 1A.=what I allow for pasture for every 4 sheep I keep.
4. $\frac{1}{4}$A.=what I allow for pasture for each sheep I keep.
5. ∴ $\frac{1}{10}$A.+$\frac{1}{4}$A.=$\frac{7}{20}$A.=land required for every sheep.
6. ∴ 161A.=land required for 161÷$\frac{7}{20}$, or 460 sheep.

III. ∴ I can keep 460 sheep on 161 acres.

(*R. Alg. I., p. 112, prob. 64.*)

Complete analysis.

If for every 10 sheep I plow 1 acre, for 1 sheep I plow $\frac{1}{10}$ of an acre; and if for every 4 sheep I pasture 1 acre, for 1 sheep, I pasture $\frac{1}{4}$ of an acre; hence 1 sheep requires $\frac{1}{10}$A.+$\frac{1}{4}$A., or $\frac{7}{20}$A., and on 161 A. I could keep as many sheep as $\frac{7}{20}$A. is contained in 161 A., which are 460 sheep.

9. WITH AND AGAINST THE CURRENT PROBLEMS.

I. A man can row down stream at the rate of 10 miles per hour, and up stream at the rate of 6 miles per hour; find the rate of rowing in still water and the rate of the current.

II.
1. Rate of rowing+rate of current=10 miles per hour.
2. Rate of rowing—rate of current=6 miles per hour.
3. ∴ 2 times rate of rowing=16 miles per hour, by adding steps 1 and 2.
4. ∴ The rate of rowing=$\frac{1}{2}$ of 16 miles per hour=8 miles per hour.
5. 2 times the rate of the current=4 miles per hour, by subtracting step 2 from step 1.
6. ∴ The rate of the current=$\frac{1}{2}$ of 4 miles per hour=2 miles per hour.

III. ∴ The rate of rowing is 8 miles per hour, and the rate of the current is 2 miles per hour.

Note.—This problem was sent to the author, for solution, by a teacher in Michigan where considerable discussion arose as to the correct result.

I. How far can a man row up stream and return in 14 hours, if the rate of the current is 5 miles per hour and his rate of rowing in still water, 7 miles per hour?

II.
1. 7 miles—5 miles=2 miles=his rate per hour in going up stream.
2. ∴ $\frac{1}{2}$ hour=time it will take him to go 1 mile up stream.
3. 7 miles+5 miles=12 miles=his rate per hour in going down stream.

4. ∴ $\frac{1}{12}$ hour=time it takes him to go 1 mile down stream.
5. $\frac{1}{2}$ hour+$\frac{1}{12}$ hour=$\frac{7}{12}$ hour=time it takes him to go 1 mile up stream and return.
6. ∴ 14 hours=the time it will take him to go $14\div\frac{7}{12}$, or 24 miles up stream and return.

III. ∴ He can go 24 miles up stream and return in 14 hours.

10. TIME PROBLEMS.

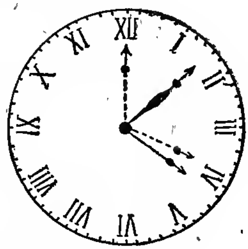
I. A person being asked the hour of day, said, "the time past noon is $\frac{1}{3}$ of the time past midnight;" what was the hour?

II.
1. $\frac{3}{3}$=time past midnight.
2. $\frac{1}{3}$=time past noon.
3. ∴ $\frac{3}{3}-\frac{1}{3}=\frac{2}{3}$=time from midnight to noon.
4. 12 hours=time from midnight to noon.
5. ∴ $\frac{2}{3}$=12 hours.
6. $\frac{1}{3}=\frac{1}{2}$ of 12 hours=6 hours=time past noon.

III. It was 6 o'clock, P. M.

I. At what time between 4 and 5 o'clock is the hour hand $\frac{1}{5}$ as far from 4 as the minute hand is from 2?

II.
1. $\frac{2}{2}$=distance hour hand moves past 4. Then
2. $\frac{24}{2}$=distance minute hand moves past 12 in same time.
3. 10 minutes$-\frac{24}{2}$ or $\frac{24}{2}-$ 10 minutes = distance the minute hand is from 2.
4. ∴ 10 minutes $-\frac{24}{2}$ or $\frac{24}{2}-10$ minutes$=5\times\frac{2}{2}=\frac{10}{2}$.
5. ∴ $\frac{34}{2}=10$ minutes or $\frac{14}{2}=10$ minutes,
6. $\frac{1}{2}=\frac{1}{34}$ of 10 minutes or $\frac{1}{14}$ of 10 minutes $=\frac{5}{17}$ minute or $\frac{5}{7}$ minute.
7. $\frac{24}{2}=24$ times $\frac{5}{17}$ minute or 24 times $\frac{5}{7}$ minute,$=7\frac{1}{17}$ minutes or $17\frac{1}{7}$ minutes past 4 o'clock.



III. ∴ At $7\frac{1}{17}$ minutes or $17\frac{1}{7}$ minutes past 4 o'clock, the hour hand is $\frac{1}{5}$ as far from 4 as the minute hand is from 2.

Explanation.—Locate the hour hand at 4 and the minute hand at 12. Now if the hour hand remained stationary at 4, the minute hand would have to travel on to 2, in order that it may be 5 times as far from 2 as the hour hand is from 4. This is so, because the hour hand is at four, by hypothesis, and is, therefore, at a distance 0 minutes from it. Hence, that the minute hand be 5 times as far from 2, it must be *at* 2, because 5 times 0 is 0. But as the minute hand moves past 12, the hour hand moves past 4. The distance the minute hand is to stop short of 2 or move past 2 is shown in the 3d step. The analysis needs no further detailed explanation.

I. Provided the time past 10 o'clock, A. M., equals $\frac{3}{4}$ of the time to midnight; what o'clock is it?

II.
1. $\frac{4}{4}$=time to midnight. Then
2. $\frac{3}{4}$=time past 10 o'clock.
3. $\frac{4}{4}+\frac{3}{4}=\frac{7}{4}$=time from 10 o'clock to midnight.
4. 14 hours=time from 10 o'clock to midnight.
5. $\therefore \frac{7}{4}$=14 hours.
6. $\frac{1}{4}=\frac{1}{7}$ of 14 hours=2 hours, and
7. $\frac{3}{4}$=3 times 2 hours=6 hours, time past 10 o'clock=4 o'clock P. M.

III. ∴ It is 4 o'clock, P. M.

I. At what time between 3 and 4 o'clock will the hour and minute hands of a watch be together?

II.
1. $\frac{2}{2}$=distance the h. h. moves past 3. Then
2. $\frac{24}{2}=12\times\frac{2}{2}$=distance the m. h. moves past 12.
3. $\frac{24}{2}-\frac{2}{2}=\frac{22}{2}$=distance the m. h. gains on the h. h.
4. 15 min.=distance the m. h. gains on the h. h
5. $\therefore \frac{22}{2}$=15 min.
6. $\frac{1}{2}=\frac{1}{22}$ of 15 min.=$\frac{15}{22}$ min.
7. $\frac{24}{2}$=24 times $\frac{15}{22}$ min.=$16\frac{4}{11}$ min.=distance m. h. moves past 12.

III. ∴ It is $16\frac{4}{11}$ min. past 3 o'clock.

Remark.—In problems of this kind, locate the minute hand at 12 and the hour hand at the first of the two numbers between which the conditions of the problem are to be satisfied. Thus in the above problem, at 3 o'clock the minute hand is at 12 and the hour hand at 3.

The minute hand moves over 60 minute spaces while the hour hand moves over 5 minute spaces. Hence the minute hand moves 12 times as fast as the hour hand. Since at 3 o'clock the minute hand is at 12 and the hour hand at 3, and the minute hand moves 12 times as fast as the hour hand, it is evident that the minute hand will overtake the hour hand between 3 and 4. So we let $\frac{2}{2}$=distance the hour hand moves past 3 until it is overtaken by the minute hand. But since the minute hand moves 12 times as fast as the hour hand, while the hour move $\frac{2}{2}$, the minute hand moves 12 times $\frac{2}{2}$, or $\frac{24}{2}$. Now the minute hand has moved from 12 to $3+\frac{2}{2}$, or 15 minutes$+\frac{2}{2}$. Hence the minute hand has gained 15 minutes on the hour hand. It has also gained $\frac{24}{2}-\frac{2}{2}$, or $\frac{22}{2}$. $\therefore \frac{22}{2}$=15 minutes.

In solving any problem of this nature, first locate the hands as previously stated, and then ask yourself how far the *minute hand must move* to meet the conditions of the problem, if the *hour hand should remain stationary*.

I. At what time betweeu 6 and 7 o'clock will the minute hand be at right angles with the hour hand?

II.
1. $\frac{2}{2}$=distance h. h. moves past 6.
2. $\frac{24}{2}$=12 times $\frac{2}{2}$=distance m. h. moves past 12.
3. $\therefore \frac{24}{2}-\frac{2}{2}=\frac{22}{2}$=distance m. h. gains on h. h.
4. 15 min. or 45 min.=distance m. h. gains on the h. h.
5. $\therefore \frac{22}{2}$=15 min. or 45 min.
6. $\frac{1}{2}=\frac{1}{22}$ of 15 min. or $\frac{1}{22}$ of 45 min.=$\frac{15}{22}$ min. or $2\frac{1}{22}$ min.
7. $\frac{24}{2}$=24 times $\frac{15}{22}$ min. or 24 times $2\frac{1}{22}$ min.=$16\frac{4}{11}$ min. or $49\frac{1}{11}$ min.

III. ∴ The minute hand will be at right angles with the hour hand at $16\frac{4}{11}$ min. or $49\frac{1}{11}$ min. past 6 o'clock.

Explanation.—Locate the minute hand at 12 and the hour hand at 6. Now if the hour hand had remained stationary at 6, the minute hand would have to move to 3 or 9, *i. e.*, it would have to gain 15 min. or 45 min. While the minute hand is moving to 3 the hour hand is moving from 6. So the minute hand must move as far past 3 as the hour hand moves past 6. Or while the minute hand is moving to 9 the hour hand is moving past 6. So the minute hand must move as far past 9 as the hour hand is past 6. ∴ The minute hand must gain 15 minutes in the first case and 45 minutes in the second.

I. At what time between 2 and 3 o'clock are the hour and minute hands opposite?

II.
1. $\frac{2}{2}$=distance hour hand moves past 2. Then
2. $\frac{24}{2}$=distance the minute hand moves past 12, in the same time.
3. ∴ $\frac{24}{2}-\frac{2}{2}=\frac{22}{2}$=distance minute hand gained on the hour hand.
4. 40 min.=distance the minute hand gained on the hour hand.
5. ∴ $\frac{22}{2}$=40 min.
6. $\frac{1}{2}=\frac{1}{22}$ of 40 mim.=$1\frac{9}{11}$ min., and
7. $\frac{24}{2}$=24 times $1\frac{9}{11}$ min.=$43\frac{7}{11}$ min.

III. ∴ It is $43\frac{7}{11}$ min. past 2 o'clock when the hands are opposite.

Explanation.—Locate the minute hand at 12 and the hour hand at 2. Now if the hour hand remained stationary at 2, the minute hand would have to move to 8 or over 40 minutes in order to be opposite the hour hand. But while the minute hand is moving to 8, the hour hand is moving from 2. So the minute hand must move as far past 8 as the hour hand is past 2. Since $\frac{2}{2}$ is the distance the hour hand moves past 2, $\frac{2}{2}$ must be the distance the minute hand must move past 8. Hence the distance the minute hand moves is $\frac{2}{2}$+40 min. But $\frac{24}{2}$=distance the minute hand moves. ∴ $\frac{24}{2}=\frac{2}{2}+$ 40 min. or $\frac{22}{2}$=40 min. as shown in step 5.

I. At what time between 3 and 4 o'clock will the minute hand be 5 minutes ahead of the hour hand?

II.
1. $\frac{2}{2}$=distance hour hand moves while the m. h. is moving to be 5 min. ahead.
2. $\frac{24}{2}=12\times\frac{2}{2}$=distance minute hand moves while the h. h. moves $\frac{2}{2}$.
3. ∴ $\frac{24}{2}-\frac{2}{2}=\frac{22}{2}$=distance gained by the minute hand.
4. 15 min.+5 mim.=20 min.=distance gained by the m. h.
5. ∴ $\frac{22}{2}$=20 min.
6. $\frac{1}{2}=\frac{1}{22}$ of 20 min.=$\frac{10}{11}$ min.
7. $\frac{24}{2}$=24 times $\frac{10}{11}$ min.=$21\frac{9}{11}$ min.

III. ∴It is $21\frac{9}{11}$ min. past 3 o'clock.

Explanation.—Locate the minute hand at 12 and the hour hand at 3. Now if the hour hand remained stationary at 3, the minute hand would have to move to 4 in order to be 5 min. ahead. But while the minute hand is moving to 4 the hour hand is moving from 3. Hence the minute hand must move as far past 4 as the hour hand moves past 3. But the hour hand moves $\frac{2}{2}$ past 3; hence, the minute hand must move $\frac{2}{2}$+5 min. past 4, in all, $\frac{2}{2}$+20 min. Hence, the minute hand gains ($\frac{2}{2}$+20 min.)—$\frac{2}{2}$=20 min. on the hour hand.

Remark.—We always find $\frac{24}{2}$, the distance the minute hand moves, for it indicates the time between any two consecutive hours. The hour hand indicates the hour.

I. At what time between 4 and 5 o'clock do the hands of a clock make with each other an angle of 45°?

II.
1. $\frac{2}{2}$=distance the hour hand moves past 4.
2. $\frac{24}{2}$=distance the minute hand moves past 12.
3. ∴ $\frac{24}{2}-\frac{2}{2}=\frac{22}{2}$= distance the minute hand gains on the hour hand.
4. $12\frac{1}{2}$ min. or $27\frac{1}{2}$ min.=distance gained by minute hand.
5. ∴ $\frac{22}{2}=12\frac{1}{2}$ min. or $27\frac{1}{2}$ min.
6. $\frac{1}{2}=\frac{1}{22}$ of $12\frac{1}{2}$ min. or $\frac{1}{22}$ of $27\frac{1}{2}$ min.=$\frac{25}{44}$ min. or $1\frac{1}{4}$ [min.
7. $\frac{24}{2}$=24 times $\frac{25}{44}$ min. or 24 times $1\frac{1}{4}$ min.=$13\frac{7}{11}$ min. or 30 min.

III. ∴ At $13\frac{7}{11}$ min. past 4 or 30 min. past 4, the hands make an angle of 45° with each other.

Explanation.—Locate the minute hand at 12 and the hour hand at 4. 45° =$\frac{1}{8}$ of 360°. $\frac{1}{8}$ of 60 min.=$7\frac{1}{2}$ min. Hence, that the hands make an angle of 45°, the minute hand must be either $7\frac{1}{2}$ minutes behind the hour hand or $7\frac{1}{2}$ min. ahead. Now if the hour hand remained stationary at 4, the minute hand would have to move over $12\frac{1}{2}$ min. or $2\frac{1}{2}$ min. past 2. But while the minute hand is moving this distance, the hour hand is moving past 4. Hence, the minute hand must move as far past $2\frac{1}{2}$ min. past 2 as the hour hand moves past 4, *i. e.*, the minute hand moves $\frac{2}{2}+12\frac{1}{2}$ min. Hence, it gains ($\frac{2}{2}+12\frac{1}{2}$ min.)—$\frac{2}{2}=12\frac{1}{2}$ min. The reasoning for the second result is the same as for the first.

I. At what time between 4 and 5 o'clock is the minute hand as far from 8 as the hour hand is from 3?

II.

A.
1. $\frac{2}{2}$=distance the hour hand moves past 4.
2. $\frac{24}{2}$=12 times $\frac{2}{2}$=distance minute hand moves past 12 in the same time.
3. ∴ $\frac{24}{2}+\frac{2}{2}=\frac{26}{2}$=distance both move.
4. 35 min.=distance both move.
5. ∴ $\frac{26}{2}$=35 min.
6. $\frac{1}{2}=\frac{1}{26}$ of 35 min.=$1\frac{9}{26}$ min.
7. $\frac{24}{2}$=24 times $1\frac{9}{26}$ min.=$32\frac{4}{13}$ min.

B.
1. $\frac{2}{2}$=distance the h. h. moves past 4.
2. $\frac{24}{2}$=distance minute hand moves past 12.
3. ∴ $\frac{24}{2}-\frac{2}{2}=\frac{22}{2}$=distance the minute hand gains.
4. 45 min.=distance the minute hand gains.
5. ∴ $\frac{22}{2}$=45 min.
6. $\frac{1}{2}=\frac{1}{22}$ of 45 min.=$2\frac{1}{22}$ min.
7. $\frac{24}{2}$=24 times $2\frac{1}{22}$ min.=$49\frac{1}{11}$ min.

III. ∴ It is $32\frac{4}{13}$ min. or $49\frac{1}{11}$ min. past 4 o'clock.

(*R. H. A., p. 403, prob. 40.*)

Explanation.—This problem requires two different solutions. Locate the minute hand at 12 and the hour hand at 4. The hour hand is now 5 minutes from 3. If the hour hand remained stationary, the minute hand would have to move to 7 to be 5 minutes from 8. But while the minute hand is moving to 7, the hour hand is moving past 4. Hence the minute hand must stop as far from 7 as the hour hand moves past 4; *i. e.*, if the hour hand moves $\frac{2}{2}$ past 4 the minute hand must stop $\frac{2}{2}$ from 7. Then the hour hand will be 5 minutes$+\frac{2}{2}$ from 3 and the minute hand will be $\frac{2}{2}+5$ minutes from 8. While the hour hand moved $\frac{2}{2}$, the minute hand moved 35 min $-\frac{2}{2}$ ∴ $\frac{24}{2}=35$ min.$-\frac{2}{2}$, whence $\frac{26}{2}=35$ min. ∴ 35 min.=distance they both move. The second part has been explained in previous problems.

I. At what time between 5 and 6 o'clock is the minute hand midway between 12 and the hour hand? When is the hour hand midway between 4 and the minute hand?

II. A.

1. $\frac{2}{2}$=distance the hour hand moves past 5.
2. $\frac{24}{2}$=distance the minute hand moves in the same time.
3. $\frac{2}{2}+25$ min.=distance from 12 to the hour hand.
4. $\frac{1}{2}$ of $(\frac{2}{2}+25$ min.$)=\frac{1}{2}+12\frac{1}{2}$ min.=distance minute hand moves.
5. ∴ $\frac{24}{2}=\frac{1}{2}+12\frac{1}{2}$ min.
6. $\frac{24}{2}-\frac{1}{2}=\frac{23}{2}=12\frac{1}{2}$ min.
7. $\frac{1}{2}=\frac{1}{23}$ of $12\frac{1}{2}$ min.$=\frac{25}{46}$ min.
8. $\frac{24}{2}=24$ times $\frac{25}{46}$ min.$=13\frac{1}{23}$ min.

B.

1. $\frac{2}{2}$=distance the hour hand moves past 5.
2. $\frac{24}{2}$=distance the minute hand moves in the same time
3. $\frac{2}{2}+5$ min.=distance the hour hand is from 4.
4. $\frac{4}{2}+10$ min.$=2$ times $(\frac{2}{2}+5$ min.$)$=distance the minute hand is from 4, since the hour hand is midway between it and 4.
5. 20 min.$+(\frac{4}{2}+10$ min.$)=\frac{4}{2}+30$ min.=distance the the minute hand is from 12.
6. ∴ $\frac{24}{2}=\frac{4}{2}+30$ min., or
7. $\frac{24}{2}-\frac{4}{2}=\frac{20}{2}=30$ min.
8. $\frac{1}{2}=\frac{1}{20}$ of 30 min.$=1\frac{1}{2}$ min.
9. $\frac{24}{2}=24$ times $1\frac{1}{2}$ min.$=36$ min.

III. ∴ A. It is $13\frac{1}{23}$ min. past 5 o'clock.
B. It is 36 min. past 5 o'clock.

(*R. H. A., p. 403, prob. 41.*)

Explanation.—Locate the minute hand at 12 and the hour hand at 5. If the hour hand remained stationary, the minute hand would have to move over $\frac{1}{2}$ of 25 minutes, or $12\frac{1}{2}$ minutes. But while it is moving over $12\frac{1}{2}$ minutes, the hour hand is moving past 4. Hence, the minute hand will have to move $12\frac{1}{2}$ minutes$+\frac{1}{2}$ of the distance the hour hand moves past 4. Hence $\frac{24}{2}=\frac{1}{2}+12\frac{1}{2}$ minutes, as shown by step 5 of A. In B, if the hour hand remained stationary, the minute hand would have to move over 30 minutes, *i. e.*, to 6, that the hour hand may be midway between it and 4. But while the minute hand is moving to 6 the hour hand is moving past 4. Hence the minute hand must move twice as far past 6 as the hour hand moves past

4. But $\frac{2}{2}$=distance the hour hand moves past 4; hence, $\frac{4}{2}$=distance the minute hand moves past 6. Hence, $\frac{4}{2}+30$ minutes=distance the minute hand moves. $\therefore \frac{24}{2}=\frac{4}{2}+30$ minutes, as shown by step 6 of B.

I. At what time between 3 and 4 o'clock will the minute hand be as far from 12 on the left side of the dial plate as the hour hand is from 12 on the right side?

II.
1. $\frac{2}{2}$=distance the hour hand moves past 3.
2. $\frac{24}{2}=12$ times $\frac{2}{2}$=distance the minute hand moves in the same time.
3. $\frac{24}{2}+\frac{2}{2}=\frac{26}{2}$=distance they both move.
4. 45 min.=distance they both move.
5. $\therefore \frac{26}{2}=45$ min.
6. $\frac{1}{2}=\frac{1}{26}$ of 45 min.$=1\frac{19}{26}$ min.
7. $\frac{24}{2}=24$ times $1\frac{19}{26}$ min.$=41\frac{7}{13}$ min.

III. $\therefore$ It is $41\frac{7}{13}$ min. past 3.

Explanation.—Locate the minute hand at 12 and the hour hand at 3. If the hour hand remained stationary, the minute hand would have to move to 9 to be as far from 12 on the left side of the dial plate as the hour hand is from 12 on the right. But while the minute hand is moving to 9, the hour hand is moving past 3. Hence, the minute hand must stop as far from 9 as the hour hand moves past 3. Hence, it is evident, they both move 45 minutes.

I. A man looked at his watch and found the time to be between 5 and 6 o'clock Within an hour he looked again, and found the hands had changed places. What was the exact time when he first looked?

II.
(1.) $\frac{2}{2}$=distance m. h. was ahead of h. h., or the distance the h. h. moved, since it changed place with the m. h. [the two observations.
(2.) $\frac{24}{2}$=distance the m. h. moved in the time between
(3.) $\therefore \frac{24}{2}+\frac{2}{2}=\frac{26}{2}$=distance they both moved.
(4.) 60 min.=distance they both moved.
(5.) $\therefore \frac{26}{2}=60$ min.
(6.) $\frac{1}{2}=\frac{1}{26}$ of 60 min.$=2\frac{4}{13}$ min. [ahead of h. h.
(7.) $\frac{2}{2}=2$ times $2\frac{4}{13}$ min.$=4\frac{8}{13}$ min.=distance m. h. was
(8.)
1. $\frac{2}{2}$=distance h. h. was past 5, at time of first observation. Then [servation.
2. $\frac{24}{2}$=distance m. h. was past 12 at time of first ob-
3. 25 min.$+\frac{2}{2}+4\frac{8}{13}$ min.$=\frac{2}{2}+29\frac{8}{13}$= distance m. h. was past 12 at time of first observation.
4. $\therefore \frac{24}{2}=\frac{2}{2}+29\frac{8}{13}$ min.
5. $\frac{24}{2}-\frac{2}{2}=\frac{22}{2}=29\frac{8}{13}$ min.
6. $\frac{1}{2}=\frac{1}{22}$ of $29\frac{8}{13}$ min.$=1\frac{9}{26}$ min.
7. $\frac{24}{2}=24$ times $1\frac{9}{26}$ min.$=32\frac{4}{13}$ min.

III. $\therefore$ It was $32\frac{4}{13}$ min. past 5 o'clock.

Explanation.—It is clear that the minute hand was ahead of the hour hand at the time of the first observation, or else they could not have exchanged places within an hour. Now, we call the distance from the point where the hour hand was located at first to the point where the minute hand was located first, $\frac{2}{2}$. But in the mean time the hour hand has moved to the position occupied by the minute hand and the minute hand has moved on around the dial to the position occupied by the hour hand, *i. e.*, the hour hand has moved $\frac{2}{2}$ and the minute 12 times $\frac{2}{2}$, or $\frac{24}{2}$. Hence, they both moved $\frac{26}{2}$. They both moved 60 minutes since the hand moved on around the dial to the position occupied by the hour hand and the hour hand moved to the position occupied by the minute hand. ∴ $\frac{26}{2}=60$ min. as shown in step (5.) The remaining part of the solution has been explained in previous problems.

I. At a certain time between 8 and 9 o'clock a boy stepped into the schoolroom, and noticed the minute hand between 9 and 10. He left, and on returning within an hour, he found the hour hand and minute hand had exchanged places. What time was it when he first entered, and how long was he gone?

II.
- A.
  - (1.) $\frac{2}{2}$=distance m. h. was ahead of the h. h. or distance it moved.
  - (2.) $\frac{24}{2}$=distance m. h. moved while the h. h. moved [$\frac{2}{2}$.
  - (3.) $\frac{24}{2}+\frac{2}{2}=\frac{26}{2}$=distance both moved.
  - (4.) 60 min.=distance both moved.
  - (5.) ∴ $\frac{26}{2}=60$ min.
  - (6.) $\frac{1}{2}=\frac{1}{26}$ of 60 min.$=2\frac{4}{13}$ min.
  - (7.) $\frac{2}{2}=2$ times $2\frac{4}{13}$ min.$=4\frac{8}{13}$ min.=distance m. h. [was ahead.
  - (8.)
    1. $\frac{2}{2}$=distance h. h. moved past 8.
    2. $\frac{24}{2}$=distance m. h. moved in same time.
    3. 40 min $+\frac{2}{2}+4\frac{8}{13}$ min.$=\frac{2}{2}+44\frac{8}{13}$ min. = distance m. h. moved to be $4\frac{8}{13}$ min. ahead.
    4. ∴ $\frac{24}{2}=\frac{2}{2}+44\frac{8}{13}$ min.
    5. $\frac{24}{2}-\frac{2}{2}=\frac{22}{2}=44\frac{8}{13}$ min.
    6. $\frac{1}{2}=\frac{1}{22}$ of $44\frac{8}{13}$ min.$=2\frac{4}{143}$ min.
    7. $\frac{24}{2}=24$ times $2\frac{4}{143}$ min. $=48\frac{96}{143}$ min.=time [past 8.
- B.
  1. $\frac{26}{2}$=distance they both moved.
  2. 60 min.=distance they both moved.
  3. ∴ $\frac{26}{2}=60$ min.
  4. $\frac{1}{2}=\frac{1}{26}$ of 60 min.$=2\frac{4}{13}$ min.
  5. $\frac{24}{2}=24$ times $2\frac{4}{13}$ min.$=55\frac{5}{13}$ min.=time he [was gone.

III. ∴
- A. It was $48\frac{96}{143}$ min past 8 o'clock when he first entered school room.
- B. He was gone $55\frac{5}{13}$ min.

I. Suppose the hour, minute, and second hands of a clock turn upon the same center, and are together at 12 o'clock; how long before the second hand, hour hand, and minute hand respectively, will be midway between the other two hands?

II. A.

1. $\frac{2}{2}$=distance the hour hand moves past 12. Then
2. $\frac{24}{2}$=distance the minute hand moves past 12, and
3. $\frac{1440}{2}$=720 times $\frac{2}{2}$=distance the second hand moves past 12.
4. $\frac{1440}{2}-\frac{24}{2}=\frac{1416}{2}$ = distance from the minute hand to the second hand.
5. $\frac{1440}{2}-\frac{24}{2}=\frac{1416}{2}$ = distance from the second hand to the hour hand.
6. $\frac{24}{2}-\frac{2}{2}=\frac{22}{2}$ = distance from the hour hand to the second hand.
7. $\frac{1416}{2}+\frac{1416}{2}+\frac{22}{2}=\frac{2854}{2}$=distance around the dial.
8. 60 seconds=distance around the dial as indicated by one revolution of the s. h.
9. $\therefore \frac{2854}{2}$=60 sec.
10. $\frac{1}{2}=\frac{1}{2854}$ of 60 sec.$=\frac{30}{1427}$ sec.
11. $\frac{1440}{2}$=1440 times $\frac{30}{1427}$ sec.$=30\frac{390}{1427}$ sec. = time when s. h. is midway between the h. h. and m. h.

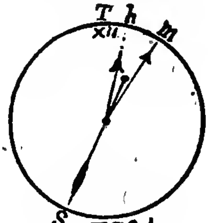


FIG. I.

B.

1. $\frac{2}{2}$=distance the hour hand moves past 12. Then
2. $\frac{24}{2}$=distance the minute hand moves past 12, and
3. $\frac{1440}{2}$=distance the second hand moves past 12.
4. $\frac{24}{2}-\frac{2}{2}=\frac{22}{2}$=distance from h. h. to m. h.
5. $\frac{22}{2}$=distance from s. h. to h. h., because the h. h. is midway between them.
6. $\frac{22}{2}-\frac{2}{2}$=distance from s. h. to 12.
7. $\frac{1440}{2}+\frac{20}{2}=\frac{1460}{2}$ = distance around the dial.
8. 60 sec.=distance around the dial.
9. $\therefore \frac{1460}{2}$=60 sec.
10. $\frac{1}{2}=\frac{1}{1460}$ of 60 sec.$=\frac{3}{73}$ sec.
11. $\frac{1440}{2}$=1440 times $\frac{3}{73}$ sec.$=59\frac{13}{73}$ sec.=time when the h. h. is midway between the s. h. and m. h.

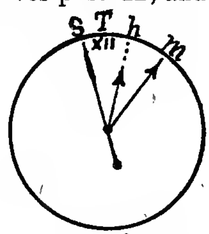


FIG. 2.

C.

1. $\frac{2}{2}$=distance h. h. moves past 12. Then
2. $\frac{24}{2}$=distance m. h. moves past 12, and
3. $\frac{1440}{2}$=distance s. h. moves past 12.
4. $\frac{24}{2}-\frac{2}{2}=\frac{22}{2}$=distance from h. h. to s. h.
5. $\frac{22}{2}$=distance from m. h. to s. h.
6. $\frac{2}{2}+\frac{22}{2}+\frac{22}{2}=\frac{46}{2}$ = distance from 12 to s. h.
7. $\frac{1440}{2}-\frac{46}{2}=\frac{1394}{2}$ = distance around the dial.
8. 60 sec.=distance around the dial.

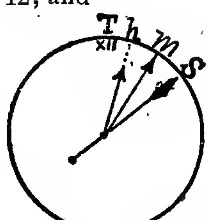


FIG 3.

9. $\therefore \frac{1394}{2}$=60 sec.
10. $\frac{1}{2}=\frac{1}{1394}$ of 60 sec.$=\frac{30}{697}$ sec.
11. $\frac{1440}{2}$=1440 times $\frac{30}{697}$ sec.$=61\frac{683}{697}$ sec.=time past 12 when the m. h. will be midway between the h. h. and s. h.

III. $\therefore$
A. The second hand is midway between h. h. and m. h. at $30\frac{390}{1427}$ sec. past 12.
B. The hour hand is midway between s. h. and m. h. at $59\frac{13}{73}$ sec. past 12.
C. The minute hand is midway between h. h. and s. h. at $61\frac{683}{697}$ sec. past 12.

Explanation.—A. We represent the distance moved by the hour hand by $\frac{2}{2}$, = the space Th. And since the minute hand moves 12 times as fast as the hour hand, it moves $\frac{24}{2}$. The second hand moves 60 times as fast as the minute hand or 720 times as fast as the hour hand. From T to h is $\frac{2}{2}$ and from T to m is $\frac{24}{2}$. $\therefore$ From h to m is $Tm-Th=\frac{24}{2}-\frac{2}{2}=\frac{22}{2}$. From T to s is $\frac{1440}{2}$. $\therefore$ From m to $s=Ts-Tm=\frac{1440}{2}-\frac{24}{2}=\frac{1416}{2}$. And, by the condition of the problem, the distance from m to s = the distance from m to h. $\therefore$ from m to $h=\frac{1416}{2}+\frac{1416}{2}=\frac{2832}{2}$. We have seen, already, that the distance from h to m is $\frac{22}{2}$. $\therefore$ The whole distance around the dial is $\frac{2832}{2}+\frac{22}{2}=\frac{2854}{2}$.

B. From T to h is $\frac{2}{2}$. From T to m is $\frac{24}{2}$. $\therefore$ From h to $m=Tm-Th=\frac{24}{2}-\frac{2}{2}=\frac{22}{2}$. By the condition of the problem, the distance from h to m=the distance from s to h. $\therefore sT-Th=\frac{22}{2}-\frac{2}{2}=\frac{20}{2}$. From T around the dial to the right of s is $\frac{1440}{2}$. $\therefore$ The whole distance around the dial$=\frac{1440}{2}+\frac{20}{2}=\frac{1460}{2}$.

C. From T to h is $\frac{2}{2}$. From T to m is $\frac{24}{2}$. $\therefore$ From h to $m=\frac{24}{2}-\frac{2}{2}=\frac{22}{2}$. By the condition of the problem, the distance from m to s=the distance from h to $m=\frac{22}{2}$. $\therefore$ From T to s is $\frac{2}{2}+\frac{22}{2}+\frac{22}{2}=\frac{46}{2}$. From T around the dial through T to s is $\frac{1440}{2}$. $\therefore$ The whole distance around the dial is $\frac{1440}{2}-\frac{46}{2}=\frac{1394}{2}$.

11. WILL PROBLEMS.

I. A man at his marriage agreed that if at his death he should leave only a daughter, his wife should have $\frac{3}{4}$ of his estate; and if he should leave only a son she should have $\frac{1}{4}$. He left a son and a daughter. What fractional part of the estate should each receive, and what was each one's portion, if his estate was worth $6591?

II.
1. $\frac{1}{4}$=daughter's share.
2. $\frac{3}{4}$=wife's share.
3. $\frac{9}{4}$=3 times $\frac{3}{4}$=son's share.
4. $\frac{1}{4}+\frac{3}{4}+\frac{9}{4}=\frac{13}{4}$=whole estate.
5. $6591=whole estate.
6. $\therefore \frac{13}{4}$=$6591.
7. $\frac{1}{4}=\frac{1}{13}$ of $6591=$507=daughter's share,$=\frac{1}{13}$ of whole estate.
8. $\frac{3}{4}$=3 times $507=$1521=wife's share,$=\frac{3}{13}$ of whole estate.
9. $\frac{9}{4}$=9 times $507=$4563=son's share,$=\frac{9}{13}$ of whole estate.

III. $\therefore$
507=\frac{1}{13}$ of whole estate=daughter's share.
1521=\frac{3}{13}$ of whole estate=wife's share.
4563=\frac{9}{13}$ of whole estate=son's share.

(*Milne's Prac. A., p. 362, prob. 74.*)

Note. — This class of problems originated under the Roman laws of inheritance.

The following problem is quoted in Cajori's *A History of Mathematics*, p. 79: A dying man wills that, if his wife, being with child, gives birth to a son, the son shall receive $\frac{2}{3}$ and she $\frac{1}{3}$ of his estate; but if a daughter is born, she shall receive $\frac{1}{3}$ and his wife $\frac{2}{3}$. It happens that twins are born, a boy and a girl. How shall the estate be divided so as to satisfy the will?

It is said that the celebrated Roman Jurist, Salvianus Julianus, decided that the estate shall be divided into seven equal parts, of which the son receives four, the wife two, and the daughter one. It is according to this decision that the above solution is made. Should such a will come into court in this age, it would very likely be set aside.

For a valuable critique, by Marcus Baker, U. S. Coast Survey, on this class of problems, see *School Visitor, Vol IX, p.* 186.

I. A gentleman, dying, divided \$5,100 among his three sons, whose ages were 9, 11, and 17 respectively, so that the different shares, being on interest at 5%, would amount to equal sums when they became of age; what were the shares?

(*Brooks' Int. Arith., p. 167, prob. 3.*)

II.

1. $\frac{1}{20}=5\%$, the amount any principal increases per year at 5%.
2. $\frac{3}{5}$=the amount any principal increases in 12 years at 5%.
3. $\frac{5}{5}+\frac{3}{5}=\frac{8}{5}$=the amount of the youngest son's share when he becomes of age. In like manner,
4. $\frac{2}{2}+\frac{1}{2}=\frac{3}{2}$=the amount of the next youngest son's share, and
5. $\frac{5}{5}+\frac{1}{5}=\frac{6}{5}$=the amount of the oldest son's share.
6. ∴ $\frac{8}{5}$ of youngest son's principal=$\frac{3}{2}$ of the next youngest son's principal=$\frac{6}{5}$ of oldest son's principal, since the amounts are all equal.
7. $\frac{1}{5}$ of youngest son's principal=$\frac{1}{8}$ of $\frac{3}{2}$, or $\frac{3}{16}$, of the next youngest son's principal=$\frac{1}{8}$ of $\frac{6}{5}$, or $\frac{3}{20}$, of the oldest son's principal.
8. $\frac{5}{5}$, or the youngest son's principal,=$5\times\frac{3}{16}$, or $\frac{15}{16}$, of the next youngest son's principal=$5\times\frac{3}{20}=\frac{3}{4}$ of the oldest son's principal.
9. ∴ $\frac{15}{16}$ of the principal of the next youngest son=the principal of the youngest son.
10. ∴ The principal of the next youngest son=$\frac{16}{15}$ of the principal of the youngest son.
11. $\frac{3}{4}$ of the principal of the oldest son=the principal of the youngest son.
12. ∴ The principal of the oldest son=$\frac{4}{3}$ of the principal of the youngest son.
13. $\frac{15}{15}$=the principal of the youngest son,

14. $\frac{16}{15}$=the principal of the next youngest son, and
15. $\frac{20}{15}=\frac{4}{3}$=the principal of the oldest son.
16. ∴ $\frac{15}{15}+\frac{16}{15}+\frac{20}{15}=\frac{51}{15}$=the sum of their principals.
17. \$5,100=the sum of their principals.
18. ∴ $\frac{51}{15}$=\$5100.
19. $\frac{1}{15}=\frac{1}{51}$ of \$5100=\$1000.
20. $\frac{15}{15}=15\times\$100=\$1500$, the youngest son's share.
21. $\frac{16}{15}=16\times\$100=\$1600$, the next youngest son's share, and
22. $\frac{20}{15}=20\times\$100=\$2000$, the oldest son's share.

III. ∴ { \$1,500=the youngest son's share, \$1,600=the next youngest son's share, and \$2,000=the oldest son's share.

12. COACH PROBLEMS.

I. How far may a person ride in a coach, going at the rate of 15 miles an hour, provided he is gone only 10 hours and walks back at the rate of 12 miles per hour?

II.
1. 15 miles=the distance he can ride in 1 hour.
2. 1 mile =the distance he can ride in $\frac{1}{15}$ hour.
3. 12 miles=the distance he can walk in 1 hour.
4. 1 mile =the distance he can walk in $\frac{1}{12}$ hour.
5. ∴ $\frac{1}{15}$ hour+$\frac{1}{12}$hour=$\frac{3}{20}$hour=the time it takes him to ride 1 mile and walk back.
6. ∴ 10 hours=the time it takes him to ride $10\div\frac{3}{20}$, or $66\frac{2}{3}$ miles, and walk back.

III. ∴ He can ride $66\frac{2}{3}$ miles.

I. Ten men hire a coach to ride to Columbus, but by taking in 5 more persons the expense of each is diminished by \$$\frac{1}{5}$; what did the coach cost them?

II.
1. $\frac{10}{10}$=the amount paid for the coach.
2. $\frac{1}{10}$=the amount each man would have paid had only 10 men paid for it.
3. $\frac{1}{15}$=the amount each man paid for it since there are 15 men.
4. ∴ $\frac{1}{10}-\frac{1}{15}=\frac{1}{30}$=the amount each man saved.
5. \$$\frac{1}{5}$=the amount each man saved.
6. ∴ $\frac{1}{30}$ of the cost of the coach=\$$\frac{1}{5}$.
7. ∴ $\frac{30}{30}$, or the cost of the coach,=30×\$$\frac{1}{5}$=\$6.00.

III. ∴ The coach cost \$6.

I. Two men, A and B, in Circleville, Ohio, hire a coach for \$10, to go to Columbus and back, the distance being 30 miles, with the privilege of taking in three more persons. Having gone 10 miles, they take in C; at Columbus they take in D; and when within 10 miles of

Circleville they take in E. How much shall each man pay?

II.
1. 60 miles=distance A rode,
2. 60 miles=distance B rode,
3. 50 miles=distance C rode,
4. 30 miles=distance D rode, and
5. 10 miles=distance E rode.
6. 210 miles=total distance ridden by the 5 men.
7. ∴ $\frac{60}{210}$ of \$10=\$$2\frac{6}{7}$=amount A must pay,
8. $\frac{60}{210}$ of \$10=\$$2\frac{6}{7}$=amount B must pay,
9. $\frac{50}{210}$ of \$10=\$$2\frac{8}{21}$=amount C must pay,
10. $\frac{30}{210}$ of \$10=\$$1\frac{3}{7}$=amount D must pay, and
11. $\frac{10}{210}$ of \$10=\$$\frac{10}{21}$=amount E must pay.

III. ∴
\$$2\frac{6}{7}$ =amount A must pay,
\$$2\frac{6}{7}$ =amount B must pay,
\$$2\frac{8}{21}$=amount C must pay,
\$$1\frac{3}{7}$ =amount D must pay, and
\$$\frac{10}{21}$ =amount E must pay.

Remark.—The above solution is based upon the principle that a passenger pays in proportion to the distance he rides, and this it seems to me is the proper view to take of the problem. The following solution based on the principle that the expenses should be borne by those passengers who are responsible for the coach at any time during the journey has its advocates.

Second solution.

II.
1. 60 miles=the whole distance traveled.
2. \$10=the whole cost.
3. \$$\frac{1}{6}$=\$10÷60=cost per mile.
4. \$$\frac{10}{6}$=10×\$$\frac{1}{6}$=cost for the 10 miles, which A and B should share equally.
5. \$$\frac{20}{6}$=20×\$$\frac{1}{6}$=cost for the next 20 miles, and this should be shared by A, B, and C equally.
6. \$$\frac{20}{6}$=20×\$$\frac{1}{6}$=cost for the first 20 miles on the return trip, and should be shared equally by A, B, C, and D.
7. \$$\frac{10}{6}$=10×\$$\frac{1}{6}$=cost for the last 10 miles, and should be shared equally by A, B, C, D, and E.
8. \$$3\frac{1}{9}$=\$$\frac{5}{6}$+\$$\frac{10}{9}$+\$$\frac{5}{6}$+\$$\frac{2}{6}$=$\frac{1}{2}$ of \$$\frac{10}{6}$+$\frac{1}{3}$ of \$$\frac{20}{6}$+$\frac{1}{4}$ of \$$\frac{20}{6}$+$\frac{1}{5}$ of \$$\frac{10}{6}$=the amount A should pay.
9. \$$3\frac{1}{9}$=the amount B should pay.
10. \$$2\frac{5}{18}$=\$$\frac{10}{9}$+\$$\frac{5}{6}$+\$$\frac{2}{6}$=$\frac{1}{3}$ of \$$\frac{20}{6}$+$\frac{1}{4}$ of \$$\frac{20}{6}$+$\frac{1}{5}$ of \$$\frac{10}{6}$=amount C should pay.
11. \$$1\frac{1}{6}$=\$$\frac{5}{6}$+\$$\frac{2}{6}$=$\frac{1}{4}$ of \$$\frac{20}{6}$+$\frac{1}{5}$ of \$$\frac{10}{6}$=amount D should pay.
12. \$$\frac{1}{3}$=$\frac{1}{5}$ of \$$\frac{10}{6}$=amount E should pay.

III. ∴ A should pay \$$3\frac{1}{9}$; B, \$$3\frac{1}{9}$; C, \$$2\frac{5}{18}$; D, \$$1\frac{1}{6}$; and E, \$$\frac{1}{3}$.

I. Eight men hire a coach; by getting 6 more passengers, the expenses of each were diminished $\$1\frac{3}{4}$; what do they pay for the coach?

II.
1. $\frac{8}{8}$=amount paid for the coach. [been only 8 men.
2. $\frac{1}{8}$=amount 1 man would have had to pay, had there
3. $\frac{1}{14}$=amount 1 man paid since there were 8 men+6 men, or 14 men.
4. $\therefore \frac{1}{8}-\frac{1}{14}=\frac{7}{56}-\frac{4}{56}=\frac{3}{56}$=what each saved.
5. $\$1\frac{3}{4}$=what each saved.
6. $\therefore \frac{3}{56}=\$1\frac{3}{4}$,
7. $\frac{1}{56}=\frac{1}{3}$ of $\$1\frac{3}{4}=\$\frac{7}{12}$, and
8. $\frac{56}{56}$=56 times $\$\frac{7}{12}=\$32\frac{2}{3}$=amount paid for the coach.

III. $\therefore \$32\frac{2}{3}$=amount paid for the coach.

(R. H. A., p. 403, prob. 46.)

Second solution.

II.
1. $\$1\frac{3}{4}$=amount saved by each man. [the six men.
2. $\$14=8\times\$1\frac{3}{4}$=amount saved by the 8 men and paid by
3. $\therefore \$2\frac{1}{3}=\frac{1}{6}$ of $14=amount paid by each of the 14 men.
4. $\therefore \$32\frac{2}{3}$=14 times $\$2\frac{1}{3}$=amount they paid for the coach.

III. $\therefore$ They paid $\$32\frac{2}{3}$ for the coach.

13. CUP AND COVER PROBLEMS.

Under this head comes a class of problems that may be called, "Watch and Chain Problems," "Horse and Saddle Problems," etc.

I. A lady has two silver cups, and only one cover. The first cup weighs 8 ounces. The first cup and cover weighs 3 times as much as the second cup; and the second cup and cover 4 times as much as the first cup. What is the weight of the second cup and the cover?

II.
1. 3 times weight of second cup=weight of cover+weight of first cup, or 8 oz. [$2\frac{2}{3}$ oz.
2. 1 times weight of second cup=$\frac{1}{3}$ of weight of cover+
3. $\frac{3}{3}$=weight of cover. Then
4. $\frac{1}{3}+2\frac{2}{3}$ oz.=weight of second cup. [cover.
5. $\frac{3}{3}+\frac{1}{3}+2\frac{2}{3}$ oz.$=\frac{4}{3}+2\frac{2}{3}$ oz. = weight of second cup and
6. 32 oz.=4 times 8 oz.=weight of second cup and cover, by the conditions of the problem.
7. $\therefore \frac{4}{3}+2\frac{2}{3}$ oz.=32 oz.
8. $\frac{4}{3}$=32 oz.$-2\frac{2}{3}$ oz.$=29\frac{1}{3}$ oz.
9. $\frac{1}{3}=\frac{1}{4}$ of $29\frac{1}{3}$ oz.$=7\frac{1}{3}$ oz.
10. $\frac{3}{3}$=3 times $7\frac{1}{3}$ oz.=22 oz.=weight of cover. [cup.
11. $\frac{1}{3}+2\frac{2}{3}$ oz.$=7\frac{1}{3}$ oz.$+2\frac{2}{3}$ oz.= 10 oz. = weight of second

III. $\therefore$ { 22 oz.=weight of cover, and
10 oz.=weight of second cup. }

I. A man has two watches, and a chain worth \$20; if he put the chain on the first watch it will be worth $\frac{2}{3}$ as much as the second watch, but if he put the chain on the second watch it will be worth $2\frac{3}{4}$ times the first watch what is the value of each watch?

II.
1. $\frac{2}{3}$ s.$=\frac{2}{2}$ f.+\$20.
2. $\frac{1}{3}$ s.$=\frac{1}{2}$ of ($\frac{2}{2}$ f.+\$20)$=\frac{1}{2}$ f.+\$10.
3. $\frac{3}{3}$ s.=3 times ($\frac{1}{2}$ f.+\$10)$=\frac{3}{2}$ f.+\$30.
4. $\frac{3}{3}$ s.$=\frac{11}{4}$ f.—\$20, by the second condition of the problem.
5. ∴ $\frac{11}{4}$ f.—\$20$=\frac{3}{2}$ f.+\$30, whence
6. $\frac{11}{4}$ f.—$\frac{3}{2}$ f.=\$30+\$20, or
7. $\frac{5}{4}$ f.=\$50.
8. $\frac{1}{4}$ f.$=\frac{1}{5}$ of \$50=\$10, and
9. $\frac{4}{4}$ f.=4 times \$10=\$40=value of first watch.
10. $\frac{3}{3}$ s.$=\frac{3}{2}$ f.+\$30$=\frac{3}{2}$ of \$40+\$30=\$90=value of the second watch.

III. ∴ { \$40=value of first watch, and
\$90=value of second watch.

(*White's Comp. Arith., p. 243, prob. 60.*)

14. DINING, AND CHESS PROBLEMS.

I. A, B, and C dine on 8 loaves of bread; A furnishes 5 loaves; B, 3 loaves; C pays the others 8d. for his share; how must A and B divide the money?

II.
1. 8 loaves=what they all eat.
2. $2\frac{2}{3}$ loaves=what each eats.
3. ∴ 5 loaves—$2\frac{2}{3}$ loaves$=2\frac{1}{3}$ loaves=what A furnished towards C's dinner.
4. ∴ 3 loaves—$2\frac{2}{3}$ loaves$=\frac{1}{3}$ loaf=what B furnished towards C's dinner.
5. ∴ $\dfrac{2\frac{1}{3}}{2\frac{2}{3}}=\frac{7}{8}$=A's share, and
6. ∴ $\dfrac{\frac{1}{3}}{2\frac{2}{3}}=\frac{1}{8}$=B's share.
7. $\frac{7}{8}$ of 8d.=7d.=what A should receive, and
8. $\frac{1}{8}$ of 8d.=1d.=what B should receive.

III. ∴ { A should receive 7d., and
B should receive 1d. (*R. H. A., p. 403, prob. 42.*)

I. B at a game of chess lost \$18, and then won $\frac{1}{3}$ as much as he had remaining, and then had $\frac{1}{2}$ as much as he had at first; how much had he at first?

1. $\frac{6}{6}$=his money at first.
2. $\frac{6}{6}$—\$18=his money after losing \$18.

II.
3. $\frac{2}{6}-\$6=\frac{1}{3}$ of $(\frac{6}{6}-\$18)$=amount won.
4. $\frac{6}{6}-\$18+\frac{2}{6}-\$6=\frac{8}{6}-\$24$=amount he had after winning.
5. $\frac{3}{6}$=amount he had after winning.
6. $\therefore\ \frac{8}{6}-\$24=\frac{3}{6}$,
7. $\frac{5}{6}=\$24$,
8. $\frac{1}{6}=\frac{1}{5}$ of $\$24=\$4\frac{4}{5}$,
9. $\frac{6}{6}=6\times\$4\frac{4}{5}=\$28\frac{4}{5}$=amount he had at first.

III. $\therefore$ B had $\$28\frac{4}{5}$ at first.

15. PARTNERSHIP PROBLEMS.

I. A and B enter into partnership and gain \$240. A owns $\frac{3}{4}$ of the stock, lacking \$10, and gains \$175; required the whole stock and share of each.

(*Brooks' Int. Arithmetic, p. 161.*)

II.
1. $\frac{4}{4}$=whole stock.
2. $\frac{3}{4}-\$10$=what A owns.
3. $\therefore\ \frac{4}{4}-(\frac{3}{4}-\$10)$, or $\frac{1}{4}+\$10$=what B owns. Now,
4. whole stock : whole gain=A's stock : A's gain; or
5. $\frac{4}{4}:\$240=\frac{3}{4}-\$10:\$175$.
6. $\therefore\ 240(\frac{3}{4}-\$10)=175\times\frac{4}{4}$, or
7. $\frac{720}{4}-\$2400=\frac{700}{4}$,
8. $\frac{20}{4}=\$2400$,
9. $\frac{1}{4}=\frac{1}{20}$ of $\$2400=\$120$,
10. $\frac{4}{4}=4\times\$120=\$480$=whole stock.
11. $\frac{3}{4}-\$10=\$360-\$10=\$350$=A's stock.
12. $\frac{1}{4}+\$10=\$120+\$10=\$130$=B's stock.

III.
\$480=whole stock,
\$350=A's stock, and
\$130=B's stock.

I. A and B hired a pasture for 4 months for \$39. At the beginning of the first month A put in 5 cows and 3 sheep and B put in 3 cows and 5 sheep; at the beginning of the second month A put in 5 sheep and took out 2 cows and B put in 5 cows and took out 2 sheep. What part of the cost of the pasture should each pay, if a cow eats as much as two sheep?

1. 5 cows+3 sheep is equivalent to 10 sheep+3 sheep, or 13 sheep.
2. 3 cows+5 sheep is equivalent to 6 sheep+5 sheep, or 11 sheep.
3. 5 cows—2 cows+3 sheep+5 sheep is equivalent to 14 sheep for 3 months.
4. 14 sheep for 3 months is equivalent to 42 sheep for 1 month.

II.
5. 3 cows+5 cows+5 sheep—2 sheep is equivalent to 19 sheep for 3 months.
6. 19 sheep for 3 months is equivalent to 51 sheep for 1 month.
7. 13 sheep for 1 month+42 sheep for 1 month is equivalent to 55 sheep for 1 month.
8. 11 sheep for 1 month+51 sheep for 1 month is equivalent to 62 sheep for 1 month.
9. 55 sheep for 1 month+62 sheep for 1 month is equivalent to 117 sheep for 1 month.
10. ∴ $\$18\frac{1}{3}=\frac{55}{117}$ of \$39=amount A should pay, and
11. $\$20\frac{2}{3}=\frac{62}{117}$ of \$39=amount B should pay.

III. ∴ A should pay $\$18\frac{1}{3}$, and B should pay $\$20\frac{2}{3}$.

I. A, B, and C hire a pasture for \$63. A puts in 6 cows, B puts in 18 horses, and C 48 sheep; how much should each pay, if a cow eats as much as 2 horses, and a horse as much as 4 sheep?

II.
1. 1 horse eats as much as 4 sheep.
2. 2 horses eat as much as 8 sheep.
3. ∴ 1 cow eats as much as 8 sheep, since 1 cow eats as much as 2 horses.
4. 6 cows eat as much as 6×8 sheep, or 48 sheep.
5. 18 horses eat as much as 18×4 sheep, or 72 sheep, since 1 horse eats as much as 4 sheep.
6. ∴ They all put in the equivalent of 48 sheep+48 sheep +72 sheep, or 168 sheep.
7. ∴ A, who put in the equivalent of 48 sheep, should pay $\frac{48}{168}$ of \$63=\$18,
8. B, who put in the equivalent of 72 sheep, should pay $\frac{72}{168}$ of \$63=\$27, and
9. C, who put in 48 sheep, should pay $\frac{48}{168}$ of \$63=\$18.

III. ∴ \$18=amount A should pay, \$27=amount B should pay, and \$18=amount C should pay.

16. COMBINATION PROBLEMS.

I. If 62 lb. of sea-water contain 2 lb. of salt, how much salt must be added so that 42 lb. of sea-water will contain 2 lb. of salt? (*Brooks' Int. Arith., p. 144.*)

1. 62 lb.—2 lb.=60 lb.=the quantity of water in the first mixture.
2. 42 lb.—2 lb.=40 lb.=quantity of water in second mixture.

II.
3. ∴ 2 lb.=quantity of salt to be added to 40 lb. of water to make a mixture of 42 lb.
4. 1 lb.=quantity to be added to 20 lb. of water.
5. 3 lb.=quantity to be added to 60 lb. of water.
6. ∴ 3 lb.—2 lb.=1 lb.=quantity of salt that must be added to the 62 lb. of sea-water so that 42 lb. will contain 2 lb. of salt.

III. ∴ 1 lb. of salt must be added.

I. In a mixture of silver and copper, consisting of 60 oz., there are 4 oz. of copper; how much silver must be added that there may be $\frac{1}{3}$ oz. copper in 6 oz. of the mixture?

II.
1. $\frac{1}{3}$ oz.=what 6 oz. of the new mixture contains.
2. 1 oz.=what 18 oz. of the new mixture contains.
3. 4 oz.=what 72 oz. of the new mixture contains.
4. ∴ 72 oz—4 oz.=68 oz.=the quantity of silver in 72 oz. of the new mixture.
5. ∴ 68 oz.—(60 oz.— 4 oz.)=12 oz.=the quantity of silver that must be added in order that 6 oz. of the mixture will contain $\frac{1}{3}$ oz. of copper.

III. ∴ There must be added 12 oz. of silver.

17. DITCH PROBLEMS.

I. A and B dig a ditch 100 rods long for $100; how many rods does each dig, if they each receive $50, and A digs at $.75 per rod, and B at $1.25?

There has been a vast amount of quibbling about this problem; but a few moments consideration should suffice to settle all dispute, and pronounce upon it the sentence of absurdity.

We have given, the whole amount each received and the amount each received per rod. Hence, if we divide the whole amount each received by the cost per rod, it must give the number of rods he digs. But by doing this we receive 50÷.75, or $66\frac{2}{3}$ rods, what A digs and 50÷1.25, or 40 rods, what B digs, or $106\frac{2}{3}$ rods which is the length of the ditch, and not 100 rods as stated in the problem. The length of the ditch is a function of the cost per rod and the whole cost, and when they are given the length of the ditch is determined. We might propose a problem just as absurd by requiring the circumference of a circle whose area is 1 acre, and diameter 20 rods. Since the area and circumference are functions of the diameter, when either of these are given, the other is determined and should not be limited to an inaccurate statement.

If, in the original problem, A's price per rod increases at a constant ratio so that when the ditch is completed he is receiving $1 per rod, and B's price constantly decreases until when the ditch is completed he is receiving $1 per rod, then the problem is solvable, and the result is 50 rods each.

I. Two men, A and B, undertake to dig a ditch 100 rods long for $100. A's end of the ditch being more difficult to excavate than B's, it was agreed that A should receive 25¢ per rod more than B. How many rods must each dig in order that each may receive $50?

II.
1. Let $2s=\$100$, $n=100$, the number of rods, and $d=.25$, the number of dollars in the difference of the price each is to receive per rod.
2. Let x=number of rods A must dig. Then
3. $n-x$=number of rods B must dig.
4. Then $\frac{s}{x}$=price A receives per rod, and
5. $\frac{s}{n-x}$=price B receives per rod.
6. $\therefore \frac{s}{x}-\frac{s}{n-x}=d.$
7. $\therefore x=\frac{2s+nd}{2d}\pm\sqrt{\frac{4s^2+n^2d^2}{4d^2}}=\frac{n}{2}\pm\sqrt{\left(\frac{s}{d}\right)^2+\left(\frac{n}{2}\right)^2}+\frac{s}{d}$ $=\frac{100}{2}-\sqrt{\left(\frac{50}{.25}\right)^2+\left(\frac{100}{2}\right)^2}+\frac{50}{.25}=50(5-\sqrt{17})$, the number of rods A must dig, and
8. $n-x=n-\left[\frac{n}{2}\pm\sqrt{\left(\frac{s}{d}\right)^2+\left(\frac{n}{2}\right)^2}+\frac{s}{d}\right]=\frac{n}{2}\mp\sqrt{\left(\frac{s}{d}\right)^2+\left(\frac{n}{2}\right)^2}-\frac{s}{d}=\frac{100}{2}+\sqrt{\left(\frac{50}{.25}\right)^2+\left(\frac{100}{2}\right)^2}-\frac{50}{.25}=50(\sqrt{17}-3)$, the number of rods B must dig.

III. $\therefore$ A must dig $50(5-\sqrt{17})$ rods and B must dig $50(\sqrt{17}-3)$ rods.

Note.—From this algebraic solution, we can derive a rule by which all problems of this class may be solved by arithmetic.

Rule.—1. *Divide half the whole cost by the difference of the price per object and to the square of the quotient add the square of half the whole number of objects.*

2. *Subtract the square root of the sum from half the whole number of objects* **increased** *by the quotient of half the whole cost divided by the difference of the price per object, for the number of objects at the higher price per object; and the square root of the sum subtracted from half the number of objects* **diminished** *by the quotient of half the whole cost gives the number of objects at the lesser price per object.*

No arithmetical solution of such problems without the aid of Algebra is possible.

18. PASTURE PROBLEM.

I. If 12 oxen eat up $3\frac{1}{3}$ acres of pasture in 4 weeks, and 21 oxen eat up 10 acres of like pasture in 9 weeks; to find how many oxen will eat up 24 acres in 18 weeks.

II.

1. 10 parts (say)=what one ox eats in a week. Then
2. 120 parts=12×10 parts=what 12 oxen eat in 1 week,
3. 480 parts=4×120 parts=what 12 oxen eat in 4 weeks.
4. ∴ 480 parts=original grass+growth of grass on $3\frac{1}{3}$ A. in 4 weeks.
5. 144 parts=$\frac{1}{3\frac{1}{3}}$ of 480 parts=original grass+growth of grass on 1 A. in 4 weeks.
6. 210 parts=21×10 parts=what 21 oxen eat in 1 week,
7. 1890 parts=9×210 parts=what 21 oxen eat in 9 weeks.
8. ∴ 1890 parts=original grass+growth of grass on 10 A. in 9 weeks.
9. 189 parts=$\frac{1}{10}$ of 1890 parts=original grass+growth on 1 A in 9 weeks
10. ∴ 189 parts—144 parts=45 parts=growth on 1 A. in 9 weeks—4 weeks, or 5 weeks.
11. 9 parts=$\frac{1}{5}$ of 45 parts=growth on 1 A. in 1 week.
12. 36 parts=4×9 parts=growth on 1 A. in 4 weeks.
13. ∴ 144 parts—36 parts=108 parts=original quantity of grass on 1 A.
14. 2592 parts=24×108 parts=original quantity on 24 A.
15. 216 parts=24×9 parts=growth on 24 A. in 1 week.
16. 3888 parts=18×216 parts=growth on 24 A. in 18 weeks.
17. ∴ 2592 parts+3888 parts=6480 parts=quantity of grass to be eaten by the required oxen.
18. 180 parts=18×10 parts=what 1 ox eats in 18 weeks.
19. ∴ 6480 parts=what 6480÷180, or 36 oxen eat in 18 weeks.

III. ∴ It will require 36 oxen to eat the grass on 24 A. in 18 weeks.

Note.—This celebrated problem was, very probably, proposed by Sir Isaac Newton and published in his *Arithmetica Universalis* in 1704. Dr. Artemas Martin says, "I have not been able to trace it to any earlier work." For a full treatment of this problem see *Mathematical Magazine*, Vol. 1, No. 2.

19. INVOLUTION AND EVOLUTION PROBLEMS.

I. If $\frac{2}{5}$ of the number of trees in an orchard be squared, the result will be 144; how many trees in the orchard?

II.
1. $\frac{2}{2}$=number of trees in the orchard.
2. $(\frac{2}{5}\times\frac{2}{2})^2=\frac{4}{25}\times(\frac{2}{2})^2$=the square of $\frac{2}{5}$ of the number.
3. 144=the square of $\frac{2}{5}$ of the number.
4. $\therefore \frac{4}{25}\times(\frac{2}{2})^2=144$,
5. $(\frac{2}{2})^2=144\div\frac{4}{25}=900$,
6. $\frac{2}{2}=\sqrt{900}=30$, the number of trees in the orchard.

III. ∴ 30=number of trees in the orchard.

I. Two-thirds of the cube of a number is 10 more than the cube of $\frac{2}{3}$ of the number; what is the number?

II.
1. $\frac{2}{2}$=the number.
2. $\frac{2}{3}(\frac{2}{2})^3=\frac{2}{3}$ of the cube of the number.
3. $(\frac{2}{3}\times\frac{2}{2})^3$=the cube of $\frac{2}{3}$ of the number.
4. $\therefore \frac{2}{3}(\frac{2}{2})^3-(\frac{2}{3}\times\frac{2}{2})^3=10$, or $\frac{2}{3}(\frac{2}{2})^3-\frac{8}{27}(\frac{2}{2})^3=10$.
5. $\frac{10}{27}(\frac{2}{2})^3=10$.
7. $(\frac{2}{2})^3=10\div\frac{10}{27}=27$.
7. $\therefore (\frac{2}{2})=\sqrt[3]{27}=3$, the number.

III. ∴ The number is 3.

20. SOLUTIONS OF MISCELLANEOUS PROBLEMS.

I. Two men start from two places 495 miles apart, and travel toward each other; one travels 20 miles a day, and the other 25 miles a day; in how many days will they meet?

II.
1. $\frac{2}{2}$=number of days.
2. 20 mi.=distance first travels in 1 day.
3. $\frac{2}{2}\times20$ mi.=distance first travels in $\frac{2}{2}$ days.
4. 25 mi.=distance second travels in 1 day.
5. $\frac{2}{2}\times25$ mi.=distance second travels in $\frac{2}{2}$ days.
6. $\therefore \frac{2}{2}\times20$ mi.$+\frac{2}{2}\times25$ mi.$=\frac{2}{2}\times(20$ mi.$+25$ mi.$)$=distance both travel.
7. 495 mi.=distance both travel.
8. ∴ (20 mi.+25 mi.)$\times\frac{2}{2}$=(45 mi.)$\times\frac{2}{2}$=495 mi. Whence
9. $\frac{2}{2}=495\div45=11$=number of days.

III. ∴ They will meet in 11 days.

Second solution.

II.
1. 20 miles=distance first travels in a day.
2. 25 miles=distance second travels in a day.
3. ∴ 45 miles=distance both travel in a day.
4. ∴ 495 miles=distance both travel in 495÷45, or 11, days.

III. ∴ They will meet in 11 days.

Third solution—the one usually given in the schoolroom.

$$\begin{array}{r} 20+25=45)495(11 \text{ days.} \\ 45\phantom{(11 \text{ days.}} \\ \hline 45\phantom{(11 \text{ days.}} \\ 45\phantom{(11 \text{ days.}} \\ \hline \end{array}$$

I. Find a number whose square root is 5 times its cube root.

II.
1. $\frac{2}{2}$=square root of the number. Then
2. $\frac{2}{2}\times\frac{2}{2}$=the number, because the square root×the square root equals the number.
3. $\frac{3}{3}$=the cube root of the number. Then
4. $\frac{3}{3}\times\frac{3}{3}\times\frac{3}{3}$=the number. But
5. $\frac{2}{2}=5\times(\frac{3}{3})$. Hence, squaring both sides,
6. $\frac{2}{2}\times\frac{2}{2}=25\times(\frac{3}{3}\times\frac{3}{3})$. But
7. $\frac{2}{2}\times\frac{2}{2}$=the number, and
8. $\frac{3}{3}\times\frac{3}{3}\times\frac{3}{3}$=the number.
9. ∴ $\frac{3}{3}\times\frac{3}{3}\times\frac{3}{3}=25\times(\frac{3}{3}\times\frac{3}{3})$. Dividing by $(\frac{3}{3}\times\frac{3}{3})$,
10. $\frac{3}{3}=25$.
11. ∴ $(\frac{3}{3})^3=25^3=15625$.

III. ∴ The number is 15625. *(R. H. A., p. 367, prob. 14.)*

I. A man bought a horse, saddle and bridle for $150; the cost of the saddle was $\frac{1}{6}$ of the cost of the horse, and the cost of the bridle was $\frac{1}{2}$ the cost of the saddle; what was the cost of each?

II.
1. $\frac{12}{12}$=cost of the horse. Then
2. $\frac{2}{12}=\frac{1}{6}$ of $\frac{12}{12}$=cost of the saddle, and
3. $\frac{1}{12}=\frac{1}{2}$ of $\frac{2}{12}$=cost of the bridle.
4. $\frac{15}{12}=\frac{12}{12}+\frac{2}{12}+\frac{1}{12}$=cost of all.
5. $150=cost of all.
6. ∴ $\frac{15}{12}$=$150, and
7. $\frac{1}{12}=\frac{1}{15}$ of $150=$10=cost of bridle.
8. $\frac{12}{12}$=12 times $10=$120=cost of horse.
9. $\frac{2}{12}$=2 times $10=$20=cost of saddle.

III. ∴ $10=cost of the bridle,
$20=cost of the saddle, and
$120=cost of the horse.

(White's Comp. A., p. 241, prob. 39.)

I. A boat is worth \$900; a merchant owns $\frac{5}{8}$ of it, and sells $\frac{1}{3}$ of his share; what part has he left, and what is it worth?

II.
- A.
  1. $\frac{5}{8}$=part the merchant owned.
  2. $\frac{1}{3}$ of $\frac{5}{8}=\frac{5}{24}$=part he sold.
  3. ∴ $\frac{5}{8}-\frac{5}{24}=\frac{15}{24}-\frac{5}{24}=\frac{10}{24}=\frac{5}{12}$=part he had left.
- B.
  1. \$900=value of $\frac{12}{12}$, or the whole ship.
  2. \$75=$\frac{1}{12}$ of \$900=value of $\frac{1}{12}$ of the ship.
  3. \$375=5 times \$75=value of $\frac{5}{12}$ of the ship, *i. e.* part he had left.

III. ∴ $\frac{5}{12}$=part he had left, and \$375=value of it.

I. A and B were playing cards. B lost \$14, which was $\frac{7}{10}$ times $\frac{2}{3}$ as much as A then had; and when they commenced, $\frac{5}{8}$ of A's money equaled $\frac{2}{7}$ of B's. How much had each when they began to play?

II.
- (1.) $\frac{5}{8}$ of A's money=$\frac{2}{7}$ of B's.
- (2.) $\frac{1}{8}$ of A's money=$\frac{1}{5}$ of $\frac{2}{7}=\frac{2}{35}$ of B's.
- (3.) $\frac{8}{8}$ of A's money=8 times $\frac{2}{35}=\frac{16}{35}$ of B's.
- (4.) $\frac{35}{35}$=B's money when they began to play. Then
- (5.) $\frac{16}{35}$=A's money when they began.
- (6.)
  1. $\frac{15}{15}$=A's money after winning \$14 from B.
  2. \$14=what B lost.
  3. $\frac{7}{10}$ times $\frac{2}{3}=\frac{7}{15}$=part A's money is of \$14.
  4. ∴ $\frac{7}{15}$=\$14,
  5. $\frac{1}{15}=\frac{1}{7}$ of \$14=\$2, and
  6. $\frac{15}{15}$=15 times \$2=\$30=A's money after winning [\$14 from B.
- (7.) ∴ \$30—\$14=\$16=A's money at first.
- (8.) ∴ $\frac{16}{35}$=\$16, from (5),
- (9.) $\frac{1}{35}=\frac{1}{16}$ of \$16=\$1, and
- (10.) $\frac{35}{35}$=35 times \$1=\$35=B's money at first.

III. ∴ \$16=A's money at first, and \$35=B's money at first.

(*Stod. Int. A., p. 111, prob. 30.*)

I. A drover being asked how many sheep he had, said, if to $\frac{1}{3}$ of my flock you add the number $9\frac{1}{2}$, the sum will be $99\frac{1}{2}$; how many sheep had he?

II.
1. $\frac{3}{3}$=the number of sheep.
2. $\frac{1}{3}+9\frac{1}{2}=\frac{1}{3}$ of the number$+9\frac{1}{2}$.
3. $99\frac{1}{2}=\frac{1}{3}$ of the number$+9\frac{1}{2}$.
4. ∴ $\frac{1}{3}+9\frac{1}{2}=99\frac{1}{2}$ or
5. $\frac{1}{3}=99\frac{1}{2}-9\frac{1}{2}=90$, and
6. $\frac{3}{3}$=3 times 90=270=number of sheep.

III. ∴ He had 270 sheep.

I. Heman has 6 books more than Handford, and both have 26; how many have each?

II.
1. $\frac{2}{2}$=number Handford has. Then
2. $\frac{2}{2}+6$=Heman's number.
3. $\frac{2}{2}+\frac{2}{2}+6=\frac{4}{2}+6$=number both have.
4. 26=number both have.
6. ∴ $\frac{4}{2}+6=26$ or
5. $\frac{4}{2}=26-6=20$.
7. $\frac{1}{2}=\frac{1}{4}$ of $20=5$, and
8. $\frac{2}{2}=2$ times $5=10$=Handford's number.
9. $\frac{2}{2}+6=16$=Heman's number.

III. ∴ Handford had 10 books, and Heman had 16 books. (*Stod. Int. A., p. 116, prob. 2.*)

I. A man and his wife can drink a keg of wine in 6 days, and the man alone in 10 days; how many days will it last the woman?

II.
1. 6 days=time it takes both to drink it.
2. $\frac{1}{6}$=part they drink in one day.
3. 10 days=time it takes the man to drink it.
4. $\frac{1}{10}$=part he drinks in one day.
5. ∴ $\frac{1}{6}-\frac{1}{10}=\frac{5}{30}-\frac{3}{30}=\frac{1}{15}$=part the woman drinks in one day.
6. $\frac{15}{15}$=what the woman drinks in $\frac{15}{15}\div\frac{1}{15}=15$ days.

III. ∴ It will take the woman 15 days. (*R. Alg. I., p. 112, prob. 59.*)

I. A man was hired for 80 days, on this condition: that for every day he worked he should receive 60 cents, and for every day he was idle he should forfeit 40 cents. At the expiration of the time, he received $40. How many days did he work?

II.
1. $.60=what he receives a day.
2. $48=80×$.60=what he would have received had he worked the whole time.
3. $40=what he received.
4. ∴ $48—$40=$8=what he lost by his idleness.
5. $1=$.60, his wages,+$.40, what he had to forfeit,=what he lost a day.
6. ∴ $8=what he lost in 8÷1, or 8 days.
7. 80 days—8 days=72 days, the time he worked.

III. ∴ He worked 72 days.

I. A ship-mast 51 feet high, was broken off in a storm, and $\frac{2}{3}$ of the length broken off, equaled $\frac{3}{4}$ of the length remaining; how much was broken off, and how much remained?

II.
1. $\frac{2}{3}$ of length broken off $=\frac{3}{4}$ of length remaining,
2. $\frac{1}{3}$ of length broken off $=\frac{1}{2}$ of $\frac{3}{4}=\frac{3}{8}$ of length remaining,
3. $\frac{3}{3}$ of length broken off $=3$ times $\frac{3}{8}=\frac{9}{8}$ of length remaining.
4. $\frac{8}{8}$ = length remaining.
5. $\frac{9}{8}$ = length broken off.
6. $\frac{9}{8}+\frac{8}{8}=\frac{17}{8}$ = whole length.
7. 51 feet = whole length.
8. $\therefore\ \frac{17}{8}=51$ feet,
9. $\frac{1}{8}=\frac{1}{17}$ of 51 feet $=3$ feet, and
10. $\frac{8}{8}=8$ times 3 feet $=24$ feet, length remaining.
11. $\frac{9}{8}=9$ times 3 feet $=27$ feet, length broken off.

III. $\therefore$ { 24 feet = length remaining, and
27 feet = length broken off.

I. A boy being asked his age, said, "4 times my age is 24 years more than 2 times my age;" how old was he?

II.
1. $\frac{2}{2}$ = his age.
2. $4\times\frac{2}{2}=\frac{8}{2}=4$ times his age.
3. $2\times\frac{2}{2}=\frac{4}{2}=2$ times his age.
4. $\therefore\ \frac{8}{2}=\frac{4}{2}+24$ years or
5. $\frac{8}{2}-\frac{4}{2}=\frac{4}{2}=24$ years.
6. $\frac{1}{2}=\frac{1}{4}$ of 24 years $=6$ years, and
7. $\frac{2}{2}=2$ times 6 years $=12$ years, his age.

III. $\therefore$ He is 12 years old. (*Stod. Int. A., p. 116, prob. 16.*)

I. If 10 men or 18 boys can dig 1 acre in 11 days, find the number of boys whose assistance will enable 5 men to dig 6 acres in 6 days.

II.
1. 1 A. = what 10 men dig in 11 days.
2. $\frac{1}{10}$ A. = what 1 man digs in 11 days.
3. $\frac{1}{110}$ A. $=\frac{1}{11}$ of $\frac{1}{10}$ A. = what 1 man digs in 1 day.
4. $\frac{1}{22}$ A. $=\frac{5}{110}$ A. $=5$ times $\frac{1}{110}$ A. = what 5 men dig in 1 day.
5. $\frac{3}{11}$ A. $=\frac{6}{22}$ A. $=6$ times $\frac{1}{22}$ A. = what 5 men dig in 6 days.
6. $\therefore$ 6 A. $-\frac{3}{11}$ A. $=5\frac{8}{11}$ A. = what is to be dug by the boys in 6 days.
7. 1 A. = what 18 boys dig in 11 days.
8. $\frac{1}{18}$ A. = what 1 boy digs in 11 days.
9. $\frac{1}{198}$ A. $=\frac{1}{11}$ of $\frac{1}{18}$ A. = what 1 boy digs in 1 day.
10. $\frac{1}{33}$ A. $=\frac{6}{198}$ A. $=6$ times $\frac{1}{198}$ A. = what 1 boy digs in 6 days.
11. $5\frac{8}{11}$ A. = what $5\frac{8}{11}\div\frac{1}{33}$, or 189, boys dig in 6 days.

III. $\therefore$ It will take 198 boys.

(*R. 3d p., O. E., p. 318, prob. 66.*)

I. A man after doing $\frac{3}{5}$ of a piece of work in 30 days, calls an assistant; both together complete it in 6 days. In what time could the assistant complete it alone?

II.
1. $\frac{3}{5}$=part the man does in 30 days.
2. $\frac{1}{50}=\frac{1}{30}$ of $\frac{3}{5}$=part he does in 1 day.
3. $\frac{2}{5}=\frac{5}{5}-\frac{3}{5}$=part he and the assistant do in 6 days.
4. $\frac{1}{15}=\frac{1}{6}$ of $\frac{2}{5}$=part he and the assistant do in 1 day.
5. $\therefore \frac{1}{15}-\frac{1}{50}=\frac{10}{150}-\frac{3}{150}=\frac{7}{150}$=part the assistant does in 1 day.
6. $\frac{150}{150}$=part the assistant does in $\frac{150}{150}\div\frac{7}{150}=21\frac{3}{7}$ days.

III. ∴ It will take the assistant $21\frac{3}{7}$ days.

(*R. 3d p., O. E., p. 318, prob. 71.*)

Explanation.—Since the man does $\frac{3}{5}$ of the work before he called on the assistant, there remains $\frac{5}{5}-\frac{3}{5}=\frac{2}{5}$, which he and the assistant do in 6 days. Hence they do $\frac{1}{6}$ of $\frac{2}{5}$, or $\frac{1}{15}$ of the work in one day. If the man and his assistant do $\frac{1}{15}$ of the work in 1 day and the man does $\frac{1}{50}$ of the work in 1 day, the assistant does the difference between $\frac{1}{15}$ and $\frac{1}{50}$ which is $\frac{7}{150}$ of the work in 1 day. Hence it will take $\frac{150}{150}\div\frac{7}{150}$, or $21\frac{3}{7}$ days, to do the work.

I. A person being asked the time of day, replied that it was past noon, and that $\frac{3}{4}$ of the time past noon was equal to $\frac{3}{5}$ of the time to midnight. What was the time of day?

II.
1. $\frac{3}{4}$ of the time past noon=$\frac{3}{5}$ of the time to midnight.
2. $\frac{1}{4}$ of the time past noon=$\frac{1}{3}$ of $\frac{3}{5}=\frac{1}{5}$ of the time to midnight.
3. $\frac{4}{4}$, or the time past noon,=4 times $\frac{1}{5}=\frac{4}{5}$ of the time to midnight.
4. $\frac{5}{5}$=time to midnight. Then
5. $\frac{4}{5}$=time past noon.
6. $\frac{4}{5}+\frac{5}{5}=\frac{9}{5}$=time from noon to midnight.
7. 12 hours=time from noon to midnight.
8. $\therefore \frac{9}{5}$=12 hours,
9. $\frac{1}{5}=\frac{1}{9}$ of 12 hours=$1\frac{1}{3}$ hours, and
10. $\frac{4}{5}$=4 times $1\frac{1}{3}$ hours=$5\frac{1}{3}$ hours=5 hr. 20 min., time past noon.

III. ∴ It is 20 min. past 5 o'clock, P. M.

(*Milne's Prac. A., p. 360, prob. 47.*)

Note.—From 3, we have the statement that the time past noon is $\frac{4}{5}$ of the time to midnight. Hence, if $\frac{5}{5}$ is the time to midnight, $\frac{4}{5}$ is the time past noon or if $\frac{10}{10}$ is the time to midnight, $\frac{8}{10}$ is the time past noon.

I. A person being asked the time of day, said that $\frac{5}{7}$ of the time past noon equals the time to midnight. What is the time of day?

II.
1. $\frac{7}{7}$=time past noon. Then
2. $\frac{5}{7}$=time to midnight.
3. $\frac{5}{7}+\frac{7}{7}=\frac{12}{7}$=time from noon to midnight.
4. 12 hours=time from noon to midnight.
5. ∴ $\frac{12}{7}$=12 hours.
6. $\frac{1}{7}=\frac{1}{12}$ of 12 hours=1 hour, and
7. $\frac{7}{7}$=7 times 1 hour=7 hours=time past noon.

III. ∴ It is 7 o'clock P. M.

I. A man being asked the hour of day, replied that $\frac{1}{4}$ of the time past 3 o'clock equaled $\frac{1}{2}$ of the time to midnight; what was the hour?

II.
1. $\frac{1}{4}$ of the time past 3 o'clock=$\frac{1}{2}$ of the time to midnight.
2. $\frac{4}{4}$, or the time past 3 o'clock,=4 times $\frac{1}{2}=\frac{4}{2}$ of the time to midnight.
3. $\frac{2}{2}$=time to midnight.
4. $\frac{4}{2}$=time past 3 o'clock.
5. $\frac{4}{2}+\frac{2}{2}=\frac{6}{2}$=time from 3 o'clock to midnight.
6. 9 hours=time from 3 o'clock to midnight.
7. ∴ $\frac{6}{2}$=9 hours.
8. $\frac{1}{2}=\frac{1}{6}$ of 9 hours=$1\frac{1}{2}$ hours, and
9. $\frac{4}{2}$=4 times $1\frac{1}{2}$ hours=6 hours=time past 3 o'clock.
10. $\frac{4}{2}$+3 hours=9 hours, time past noon.

III. ∴ It is 9 o'clock, P. M.

(*Brooks' Int. A., p. 156, prob. 17.*)

I. A person being asked the hour of day, replied, $\frac{2}{3}$ of the time past noon equals $\frac{2}{9}$ of time from now to midnight $+2\frac{2}{3}$ hours; what was the time?

II.
1. $\frac{2}{3}$ of time past noon=$\frac{2}{9}$ of time to midnight$+2\frac{2}{3}$ hours.
2. $\frac{1}{3}$ of time past noon=$\frac{1}{2}$ of ($\frac{2}{9}+2\frac{2}{3}$ hours)=$\frac{1}{9}$ of time to midnight$+1\frac{1}{3}$ hours.
3. $\frac{3}{3}$, or time past noon,=3 times ($\frac{1}{9}+1\frac{1}{3}$ hours)=$\frac{1}{3}$ of time [to midnight+4 hours.
4. $\frac{3}{3}$=time to midnight.
5. $\frac{1}{3}$+4 hours=time past noon.
6. $\frac{3}{3}+\frac{1}{3}$+4 hours=$\frac{4}{3}$+4 hours=time from noon to mid- [night.
7. 12 hours=time from noon to midnight.
8. ∴ $\frac{4}{3}$+4 hours=12 hours.
9. $\frac{4}{3}$=12 hours—4 hours=8 hours,
10. $\frac{1}{3}=\frac{1}{4}$ of 8 hours=2 hours, and
11. $\frac{1}{3}$+4 hours=6 hours=time past noon.

III. ∴ It is 6 o'clock, P. M.

(*Stod. Int. A., p. 128, Prob. 29.*)

I. A father gave to each of his sons \$5 and had \$30 remaining; had he given them \$8 each, it would have taken all his money; required the number of sons.

II. 1. \$8=amount each received by the second condition.
2. \$5=amount each received by the first condition.
3. \$3=\$8—\$5= excess of second condition over first, on each son.
4. ∴ \$30=excess of second condition over first, on 30÷3, or [10 sons.

III. ∴ There were 10 sons.

I. If 50 lb. of sea water contain 2 lb. of salt, how much fresh water must be added to the 50 lb. so that 10 lb. of the new mixture may contain $\frac{1}{3}$ lb. of salt.

II. 1. $\frac{1}{3}$ lb. of salt=what 10 lb. of the new mixture contains.
2. $\frac{3}{3}$, or 1, lb. of salt=what 3 times 10 lb., or 30 lb., of the new mixture contain.
3. 2 lb. of salt=what 2 times 30 lb., or 60 lb., of the new [mixture contain.
4. ∴ 60 lb.—50 lb.=10 lb.=quantity of fresh water that must be added.

III. ∴ 10 lb. of fresh water must be added that 10 lb. of the new mixture may contain $\frac{1}{3}$ lb. of salt.

I. A farmer had his sheep in three fields. $\frac{2}{3}$ of the number in the first field equals $\frac{3}{4}$ of the number in the second field, and $\frac{2}{3}$ of the number in the second field equals $\frac{3}{4}$ of the number in the third field. If the entire number was 434, how many were in each field?

II.
(1.) 1. $\frac{2}{3}$ of number in first field=$\frac{3}{4}$ of number in second field.
2. $\frac{1}{3}$ of number in first field=$\frac{1}{2}$ of $\frac{3}{4}$=$\frac{3}{8}$ of number in [second field.
3. $\frac{3}{3}$, or number in first field,=3 times $\frac{3}{8}$=$\frac{9}{8}$ of number in second field.
(2.) 1. $\frac{2}{3}$ of number in second field=$\frac{3}{4}$ of number in third field.
2. $\frac{1}{3}$ of number in second field=$\frac{1}{2}$ of $\frac{3}{4}$=$\frac{3}{8}$ of number [in third field.
3. $\frac{3}{3}$, or number in second field,=3 times $\frac{3}{8}$=$\frac{9}{8}$ of number in third field.
(3.) $\frac{8}{8}$=number in third field. Then
(4.) $\frac{9}{8}$=number in second field, and
(5.) $\frac{81}{64}$=$\frac{9}{8}$ of number in second field=number in first field in terms of number in third field.
(6.) ∴ $\frac{8}{8}+\frac{9}{8}+\frac{81}{64}=\frac{64}{64}+\frac{72}{64}+\frac{81}{64}=\frac{217}{64}$=number in the three fields.
(7.) 434=number in the three fields.
(8.) ∴ $\frac{217}{64}$=434,
(9.) $\frac{1}{64}=\frac{1}{217}$ of 434=2, and
(10.) $\frac{64}{64}$=64 times 2=128=number of sheep in third [field.
(11.) $\frac{72}{64}$=72 times 2=144=number of sheep in second field.
(12.) $\frac{81}{64}$=81 times 2=162=number of sheep in first [field.

III. ∴ { 162=number of sheep in first field,
144=number of sheep in second field, and
128=number of sheep in third field.

(*Milne's Prac. A., p. 362, prob. 68.*)

I. In a certain school of 80 pupils there are 32 girls; how many boys must leave that there may be 5 boys to 4 girls?

II. {
1. 80=whole number of pupils.
2. 32=number of girls.
3. 80—32=48=number of boys.
4. $\frac{4}{4}$=number of girls. Then, since the number of boys are to be to the number of girls as 5 : 4,
5. $\frac{5}{4}$=number of boys. But
6. $\frac{4}{4}$=32.
7. $\frac{1}{4}$=$\frac{1}{4}$ of 32=8, and
8. $\frac{5}{4}$=5 times 8=40=number of boys.
9. ∴ 48—40=8=number that must leave that there may be 5 boys to 4 girls.

III. ∴ 8 boys must leave that there may be 5 boys to 4 girls.

I. How far may a person ride in a coach, going at the rate of 9 miles per hour, provided he is gone only 10 hours, and walks back at the rate of 6 miles per hour?

II. {
1. 9 mi.=distance he can *ride* in 1 hour.
2. 1 mi.=distance he can *ride* in $\frac{1}{9}$ hour.
3. 6 mi.=distance he can *walk* in 1 hour.
4. 1 mi.=distance he can *walk* in $\frac{1}{6}$ hour.
5. ∴ $\frac{1}{9}$ hr.+$\frac{1}{6}$ hr.=$\frac{5}{18}$ hr.=time it takes him to ride 1 mi. and walk back.
6. ∴ 10 hours=time it takes him to ride 10÷$\frac{5}{18}$, or 36, mi. [and walk back.

III. ∴ He can ride 36 miles.

I. A hound ran 60 rods before he caught the fox, and $\frac{2}{3}$ of the distance the fox ran before he was caught, equaled the distance he was ahead when they started. How far did the fox run, and how far in advance of the hound was he when the chase commenced?

II. {
1. $\frac{3}{3}$=distance the fox ran before he was caught. Then
2. $\frac{2}{3}$=distance he was ahead.
3. $\frac{3}{3}$+$\frac{2}{3}$=$\frac{5}{3}$=distance the hound ran to catch the fox.
4. 60 rods=distance the hound ran to catch the fox.
5. ∴ $\frac{5}{3}$=60 rods,
6. $\frac{1}{3}$=$\frac{1}{5}$ of 60 rods=12 rods, and
7. $\frac{2}{3}$=2 times 12 rods=24 rods=distance the fox was [ahead.
8. $\frac{3}{3}$=3 times 12 rods=36 rods=distance the fox ran before he was caught.

III. $\therefore$ { 24 rods=distance the fox was ahead, and
36 rods=distance he ran before he was caught.

I. If $\frac{1}{3}$ of 12 be 3, what will $\frac{1}{4}$ of 40 be?

II. {
1. $\frac{1}{3}$ of $12=4$.
2. $\frac{1}{4}$ of $40=10$. By supposition
3. $4=3$. Then
4. $1=\frac{1}{4}$ of $3=\frac{3}{4}$, and
5. $10=10$ times $\frac{3}{4}=7\frac{1}{2}$.

III. $\therefore$ $\frac{1}{4}$ of $40=7\frac{1}{2}$, on the supposition that $\frac{1}{3}$ of 12 is 3.

I. A lady spent in one store $\frac{1}{2}$ of her money and \$1 more; in another, $\frac{1}{2}$ of the remainder and $\$\frac{1}{2}$ more; in another, $\frac{1}{2}$ of the remainder and \$1 more; and in another, $\frac{1}{2}$ of the remainder and $\$\frac{1}{2}$ more; she then had nothing left. What sum had she at first?

First Solution.

II. {
1. Let $\frac{2}{2}$=her money at first. Then,
2. $\frac{1}{2}+\$1$=amount she spent in the 1st store.
3. $\frac{2}{2}-(\frac{1}{2}+\$1)=\frac{1}{2}-\$1$,=amount she had left.
4. $\frac{1}{4}=\frac{1}{2}$ of $(\frac{1}{2}-\$1)+\$\frac{1}{2}$,=amount she spent in the 2d store.
5. $\frac{1}{2}-\$1-\frac{1}{4}=\frac{1}{4}-\$1$,=the amount she had left after visiting the 2d store.
6. $\frac{1}{8}+\$\frac{1}{2}=\frac{1}{2}$ of $(\frac{1}{4}-\$1)+\$1$,=amount she spent in the 3d store.
7. $\frac{1}{8}-\$\frac{3}{2}=(\frac{1}{4}-\$1)-(\frac{1}{8}+\$\frac{1}{2})$,=amount she had left after visiting the 3d store.
8. $\frac{1}{16}-\$\frac{1}{4}=\frac{1}{2}$ of $(\frac{1}{8}-\$\frac{3}{2})+\$\frac{1}{2}$,=amount she spent in the 4th store.
9. $\frac{1}{16}-\$\frac{5}{4}=(\frac{1}{8}-\$\frac{3}{2})-(\frac{1}{16}-\$\frac{1}{4})$,=amount she had left after visiting the 4th store.
10. \$0=what she had left after visiting the 4th store.
11. $\therefore$ $\frac{1}{16}-\$\frac{5}{4}=0$, whence
12. $\frac{1}{16}=\$\frac{5}{4}$, and
13. $\frac{16}{16}=16\times\$\frac{5}{4}=\$20$, the amount she had at first.

III. $\therefore$ She had \$20 at first.

Second Solution.

II. {
1. $\$1=\$\frac{1}{2}+\$\frac{1}{2}$,=amount she had on entering the 4th store.
2. $\$4=2(\$1+\$1)$,=amount she had on entering the 3d store.
3. $\$10=2(\$4+\$\frac{1}{2})+\$1$,=amount she had on entering the 2d store.
4. $\$20=2\times\$10$,=amount she had on entering the first store.

III. $\therefore$ She had \$20 at first.

I. A gold and silver watch were bought for $160; the silver watch cost only $\frac{1}{7}$ as much as the gold one; how much was the cost of each?

II.
1. $\frac{7}{7}$=cost of the gold watch. Then
2. $\frac{1}{7}$=cost of the silver watch.
3. $\frac{7}{7}+\frac{1}{7}=\frac{8}{7}$=cost of both.
4. $160=cost of both.
5. ∴ $\frac{8}{7}$=$160,
6. $\frac{1}{7}=\frac{1}{8}$ of $160=$20=cost of the silver watch, and
7. $\frac{7}{7}$=7 times $20=$140=cost of the gold watch.

III. ∴ { $20=cost of the silver watch, and
$140=cost of gold the watch. }

I. If 6 sheep are worth 2 cows, and 10 cows are worth 5 horses; how many sheep can you buy for 3 horses?

II.
1. Value of 2 cows=value of 6 sheep.
2. Value of 1 cow=value of 3 sheep.
3. Value of 10 cows=value of 30 sheep. But 10 cows are worth 5 horses,
4. ∴ Value of 5 horses=value of 30 sheep.
5. Value of 1 horse=value of 6 sheep.
6. Value of 3 horses=value of 18 sheep.

III. ∴ 3 horses are worth 18 sheep.

I. A teacher agreed to teach a certain time upon these conditions: if he had 20 scholars he was to receive $25; but if he had 30 scholars, he was to receive but $30. He had 29 scholars. Required his wages.

II.
1. $25=his rate of wages for 20 pupils.
2. $1.25=$\frac{1}{20}$ of $25=his rate of wages for 1 pupil.
3. $30=his rate of wages for 30 pupils.
4. $1=$\frac{1}{30}$ of $30=his rate of wages for 1 pupil.
5. ∴ $1.25—$1.00=$.25=reduction per pupil by the addition of 10 pupils.
6. $.025=$.25÷10=reduction per pupil by the addition of 1 pupil.
7. $.225=9 times $.025=reduction per pupil by the addition of 9 pupils.
8. ∴ $1.25—$.225=$1.025=his rate of wage per pupil.
9. $29.725=29 times $1.025=his wages for 29 pupils.

III. ∴ His wages were $29.725.

(*Mattoon's Arith., p. 385, prob. 200.*)

Note.—This problem is really indeterminate, because there is no definite rate of increase of wages given for each additional scholar. We might say, since the wages were increased $5 by the addition of 10 scholars, they would be increased $.50 by the addition of one scholar and, consequently, $4.50 by the addition of 9 scholars. Hence, his wages should be $25+$4.50, or $29.50. By assuming different relations between the increase of wages and additional scholars, other results may be obtained. The above solution seems to be the most satisfactory.

I. **A steamboat that can run 15 mi. per hr. with the current and 10 mi. per hr. against it, requires 25 hr. to go from Cincinnati to Louisville and return; what is the distance between the cities?**

II.
1. 15 mi.=distance the boat can travel down stream in 1 hour.
2. 1 mi.=distance the boat can travel down stream in $\frac{1}{15}$ hour.
3. 10 mi.=distance the boat can travel up stream in 1 hr.
4. 1 mi.=distance the boat can travel up stream in $\frac{1}{10}$ hr.
5. ∴ $\frac{1}{15}$ hr.$+\frac{1}{10}$ hr.$=\frac{1}{6}$ hr.=time required for the boat to travel 1 mi. down and return.
6. ∴ 25 hr.=time required for the boat to travel $25\div\frac{1}{6}$, or 150, miles down and return.

III. ∴ The distance between the two places is **150 miles.**

I. A sold to B 9 horses and 7 cows for $300; to C, at the same price, 6 horses and 13 cows, for the same sum: what was the price of each?

II.
1. Cost of 9 horses+cost of 7 cows=$300. Then the
2. Cost of 36 horses+cost of 28 cows=$1200, by taking 4 times the number of each.
3. Cost of 6 horses+cost of 13 cows=$300. Then the
4. Cost of 36 horses+cost of 78 cows=$1800, by taking 6 times the number of each. But
5. Cost of 36 horses+cost of 28 cows=$1200.
6. ∴ Cost of 50 cows=$600, by subtracting; and
7. Cost of 1 cow=$\frac{1}{50}$ of $600=$12. The
8. Cost of 7 cows=7 times $12=$84.
9. ∴ Cost of 9 horses=$300—cost of 7 cows=$300—$84 =$216. The
10. Cost of 1 horse=$\frac{1}{9}$ of $216=$24.

III. ∴ { The cows cost $12 apiece, and The horses $24 apiece.

PROBLEMS.

1. **A man bought a horse and a cow for $100, and the cow cost $\frac{2}{3}$ as much as the horse; what was the cost of each?**
Ans. horse, $60; cow, $40.

2. Stephen has 10 cents more than Marthia, and they together have 40 cents; how many have each?
Ans. Stephen, 25¢; Marthia, 15¢.

3. A's fortune added to $\frac{1}{2}$ of B's fortune, equals $2000; what is the fortune of each, provided A's fortune is to B's as 3 to 4?
Ans. A's, $1200; B's, $1600.

4. If 10 oxen eat 4 acres of grass in 6 days, in how many days will 30 oxen eat 8 acres? *Ans.* 4 days.

5. If a 5-cent loaf weighs 7 oz. when flour is worth $6 a barrel, how much ought it weigh when flour is worth $7 per barrel?
Ans.——

6. A lady gave 80 cents to some poor children; to each boy she gave 2 cents, and to each girl 4 cents; how many were there of each, provided there were three times as many boys as girls?
Ans. 8 girls; 24 boys.

7. Two men or three boys can plow an acre in $\frac{1}{6}$ of a day; how long will it take 3 men and 2 boys to plow it?
Ans. $\frac{1}{13}$ da.

8. A agreed to labor a certain time for $60, on the condition that for each day he was idle he should forfeit $2, at the expiration of the time he received $30; how many days did he labor, supposing he received $2 per day for his labor? *Ans.* $22\frac{1}{2}$ days.

9. The head of a fish is 4 inches long, the tail is as long as the head, plus $\frac{1}{2}$ of the body, and the body is as long as the head and tail; what is the length of the fish? *Ans.* 32 inches.

10. In a school of 80 pupils there are 30 girls; how many boys must leave that there may be 3 boys to 5 girls? *Ans.* 32.

11. A steamboat, whose rate of sailing in still water is 12 miles an hour, descends a river whose current is 4 miles an hour and is gone 6 hours; how far did it go? *Ans.* 32 miles.

12. A man keeps 72 cows on his farm, and for every 4 cows he plows 1 acre, and keeps 1 acre of pasture for every 6 cows; how many acres in his farm? *Ans.* 30 acres.

13. A company of 15 persons engaged a dinner at a hotel, but before paying the bill 5 of the company withdrew by which each person's bill was augmented $\$\frac{1}{2}$; what was the bill? *Ans.* $15.

14. A man sold his horse and sleigh for $200, and $\frac{4}{5}$ of this is 8 times what his sleigh cost, and the horse cost 10 times as much as the sleigh; required the cost of each.
Ans. horse, $200; sleigh, $20.

15. A went to a store and borrowed as much as he had, and spent 4 cents; he then went to another store and did the same, and then had 4 cents remaining; how much money had he at first? *Ans.* 4 cents.

16. A lady being asked her age, said that if her age were increased by its $\frac{1}{5}$, the sum would equal 3 times her age 12 years ago; what was her age? *Ans.* 20.

17. A lady being asked the hour of day, replied that $\frac{2}{3}$ of the time past noon equaled $\frac{4}{5}$ of the time to midnight, minus $\frac{4}{5}$ of an hour; what was the time? *Ans* 6 o'clock, P. M.

18. What is the hour of day if $\frac{1}{8}$ of the time to noon equals the time past midnight? *Ans.* 9 o'clock, A. M.

19. A person being asked the time of day, said $\frac{3}{5}$ of the time to midnight equals the time past midnight; what was the time? *Ans.* 9 o'clock, A. M.

20. A traveler on a train notices that $4\frac{1}{2}$ times the number of spaces between the telegraph poles that he passes in a minute is the rate of the train in miles per hour. How far are the poles apart? *Ans.* 198 feet.

21. C's age at A's birth was $5\frac{1}{2}$ times B's age, and now is the sum of A's and B's ages, but if A were now 3 years younger and B 4 years older, A's age would be $\frac{3}{4}$ of B's age. Find their ages. *Ans.* A's, 72 years; B's, 88 years; C's, 160 years.

22. In the above problem change the last *and* to *or*, and what are their ages? *Ans.* A's, 36; B's, 44, and C's, 80.

23. I have four casks, A, B, C, and D respectively. Find the capacity of each, if $\frac{3}{7}$ of A fills B, $\frac{3}{4}$ of B fills C, and C fills $\frac{9}{16}$ of D; but A will fill C and D and 15 quarts remaining. *Ans.* A 35 gal., B 15, C $11\frac{1}{4}$, and D 20.

24. A man and a boy can do a certain work in 20 days: if the boy rests $5\frac{1}{4}$ days it will take them $22\frac{1}{3}$ days; in what time can each do it? *Ans.* The man, 36 da.; the boy, 45 da.

25. A can do a job of work in 40 days, B in 60 days; after both work 3 days, A leaves; when must he return that the work may occupy but 30 days? *Ans.* 10 days.

26. If 8 men or 15 boys plow a field in 15 days of $9\frac{1}{3}$ hr., how many boys must assist 16 men to do the work in 5 days of 10 hr. each? *Ans.* 12 boys.

27. Bought 10 bu. of potatoes and 20 bu. of apples for \$11; at another time 20 bu. of potatoes and 10 bu. of apples for \$13; what did I pay for each per bu.? *Ans.* Apples 30¢, potatoes 50¢.

28. A farmer sold 17 bu. of barley and 13 bu. of wheat for \$31.55, getting 35¢ a bu. more for wheat than for the barley. Find the price of each per bu. *Ans.* Barley 90¢, wheat \$1.25.

29. After losing $\frac{3}{4}$ of my money I earned \$12; I then spent $\frac{2}{3}$ of what I had and found I had \$36 less than I lost; how much money had I at first? *Ans.* \$60.

30. In a company of 87, the children are $\frac{3}{8}$ of the women, and the women $\frac{4}{9}$ of the men; how many are there of each? *Ans.* 54 men, 24 women, and 9 children.

31. If 4 horses or 6 cows can be kept 10 days on a ton of hay, how long will it last 2 horses and 12 cows? *Ans.* 4 days.

32. A, B, and C buy 4 loaves of bread, A paying 5 cents, B 8 cents, and C 11 cents. They eat 3 loaves and sell the fourth to D for 24 cents. Divide the 24 cents equitably.

Ans. A 5 cents, B 8 cents, and C 11 cents.

33. A and B are at opposite points of a field 135 rods in compass, and start to go around in the same direction, A at the rate of 11 rods in 2 minutes and B 17 rods in 3 minutes. In how many rounds will one overtake the other? *Ans.* B 17 rounds.

34. If a piece of work can be finished in 45 days by 35 men and the men drop off 7 at a time every 15 days, how long will it be before the work is completed? *Ans.* 75 days.

35 A watch which loses 5 min a day was set right at 12 M., July 24th. What will be the true time on the 30th, when the hands of that watch point to 12? *Ans.* 12:30$\frac{30}{287}$ P. M.

36. A seed is planted. Suppose at the end of 3 years it produces a seed, and on each year thereafter each of which when 3 years old produce a seed yearly. All the seeds produced, do likewise; how many seeds will be produced in 21 years?

Ans. 1872.

37. The circumference of a circle is 390 rods. A, B, and C start to go around at the same time. A walks 7 rods per minute, B 13 rods per minute in the same direction; C walks 19 rods per minute in the opposite direction. In how many minutes will they meet? *Ans.* 195 min.

38. If 12 men can empty a cistern into which water is running at a uniform rate, in 40 min., and 15 men can empty it in 30 min., how long will it require 18 men to empty it?

Ans. 24 min.

39. Four men A, B, C, and D, agree to do a piece of work in 130 days. A gets 42d., B 45d., C 48d., and D 51d., for every day they worked, and when they were paid each man has the same amount. How many days did each work? [da.
Ans. A 35$\frac{5885}{7409}$ da., B 33$\frac{3023}{7409}$ da., C 31$\frac{2371}{7409}$ da., and D 29$\frac{3539}{7409}$

40. A fountain has four receiving pipes, A, B, C, and D; A, B, and C will fill it in 6 hours; B, C, and D in 8 hours; C, D, and A in 10 hours; and D, A, and B in 12 hr.: it also has four discharging pipes, W, X, Y, and Z; W, X, and Y will empty it in 6 hours; X, Y, Z in 5 hours; Y, Z, and W in 4 hours; and Z, W, and X in 3 hours. Suppose the pipes all open, and the fountain full, in what time will it be emptied? *Ans.* 6$\frac{6}{19}$ hours.

CHAPTER XVII.

ALLIGATION.

1. ***Alligation*** is the process employed in the solution of problems relating to the compounding of articles of different values or qualities.

2. ***Alligation*** { 1. Alligation Medial.
2. Alligation Alternate.

I. ALLIGATION MEDIAL.

1. ***Alligation Medial*** is the process of finding the mean, or average, rate of a mixture composed of articles of different values or qualities, the quantity and rate of each being given.

I. A grocer mixed 120 lb. of sugar at 5¢ a pound, 150 lb. at 6¢., and 130 lb. at 10¢.; what is the value of a pound of the mixture?

II. {
1. 120 lb. @5¢=$6.00,
2. 150 lb. @6¢=$9.00, and
3. *130* lb. @10¢=$*13.00.*
4. 400 lb. is worth $28.00.
5. 1 lb. is worth $28÷400=$.07=7 cents.

III. ∴ One pound of the mixture is worth 7 cents.

(*Stod. Comp. A., p. 244, prob. 3.*)

II. ALLIGATION ALTERNATE.

1. ***Alligation Alternate*** is the process of finding in what ratio, one to another, articles of different rates of quality or value must be taken to compose a mixture of a given mean, or average, rate of quality or value.

CASE I.

Given the value of several ingredients, to make a compound of a given value.

I. What relative quantities of tea, worth 25, 27, 30, 32, and 45 cents per lb. must be taken for a mixture worth 28 cents per lb.

		Diff.		*Bal.*		
	25¢	3¢	2 lb.	17 lb.		19 lb.
	27¢	1¢			4 lb.	4 lb.
28¢	30¢	2¢	3 lb.			3 lb.
	32¢	4¢			1 lb.	1 lb.
	45¢	17¢		3 lb.		3 lb.

Solution. — In average, the principle is, that the gains and loses are equal. We write the average price and the particular values 25, 27, 30, 32, and 45 as in the margin. This is only a convenient arrangement of the operation. Now one pound bought for 25¢ and sold in a mixture worth 28¢ there is a gain of 28¢—25¢, or 3¢; one pound bought at 27¢ and sold in a mixture worth 28¢, there is a gain of 28¢—27¢, or 1¢; one pound bought at 30¢ and sold in a mixture worth 28¢ there is a loss of 30¢—28¢, or 2¢; one pound bought at 32¢ and sold in a mixture worth 28¢, there is a loss of 32¢—28¢, or 4¢; and one pound bought at 45¢ and sold in a mixture worth 28¢ there is a loss of 45¢—28¢, or 17¢. Since the gains and losses are equal, we must take the ingredients composing this mixture in such a proportion as to make the gains and losses balance. We will first balance the 25¢ tea and the 30¢ tea. Since we gain 3¢ a pound on the 25¢ tea, and lose 2¢ on the 30¢ tea, how many pounds of each must we take so that the gain and loss on these two kinds may be equal? Evidently, we should gain 6¢ and lose 6¢. To find this, we simply find the L. C. M. of 3 and 2. Now if we gain 3¢ on one pound of the 25¢ tea, to gain 6¢, we must take as many pounds as 3¢ is contained in 6¢, which are 2 lb. If we lose 2¢ on one pound of the 30¢ tea, to lose 6¢, we must take as many pounds as 2¢ is contained in 6¢, which are 3 lb. Next, balance the 25-cent tea and the 45-cent tea. The L. C. M. of 3¢ and 17¢ is 51¢. Now if we gain 3¢ on one pound of the 25-cent tea to gain 51¢, we must take as many pounds as 3¢ is contained in 51¢ which are 17 lb. If we lose 17¢ on one pound of the 45-cent tea, to lose 51¢, we must take as many pounds as 17¢ is contained in 51¢ which are 3 lb. Next, balance the 27-cent tea and the 32-cent tea. The L. C. M. of 1¢ and 4¢ is 4¢. If we gain 1¢ on one pound of the 27-cent tea, to gain 4¢, we must take as many pounds as 1¢ is contained in 4¢, which are 4 lb. If we lose 4¢ on one pound of the 32-cent tea, it balances the gain on the 27-cent tea. Placing the number of pounds to be taken of each kind as shown above, and then adding horizontally, we have 19 lb. at 25¢, 4 lb. at 27¢, 3 lb. at 30¢, 1 lb. at 32¢, and 3 lb. at 45¢. It is not necessary to balance them in any particnlar order. All that must be observed, is that all the ingredients be used in balancing.

Note.—**To prove the problem, use Alligation Medial.**

CASE II.

To proportionate the parts, one or more of **the quantities,** but not the amount of the **combination, being given.**

I. How many bushels of hops, worth respectively 50, 60, and 75¢ per bushel, with 100 bushels at 40¢ per bushel, will make a mixture worth 65¢ a bushel?

A. Mean 65¢.

	Dif.	*Bal.*					
40¢	25¢	2 bu.			2 bu.	×50=	100 bu.
50¢	15¢		2 bu.		2 bu.		100 bu.
60¢	5¢			2 bu.	2 bu.		100 bu.
75¢	10¢	5 bu.	3 bu.	1 bu.	9 bu.		450 bu.

B. Mean 65¢.

	Dif.		*Bal.*			
40¢	25¢	2 bu.			×50=	100 bu.
50¢	15¢		2 bu.			2 bu.
60¢	5¢			2 bu.		2 bu.
75¢	10¢	5 bu.	3 bu.	1 bu.		254 bu.

SOLUTION.—In this solution, we proceed as in Case I. In A, we obtain the relative amounts to be used of each kind, which is 2 bu. at 40¢, 2 bu. at 50¢, 2 bu. at 60¢, and 9 bu. at 75¢. But we are to have 100 bu. of the first kind. Hence, we must multiply these results by 100÷2, or 50. Doing this, we obtain 100 bu. at 40¢, 100 bu. at 50¢, 100 bu. at 60¢, and 450 bu. at 75¢.

Since either or both of the balancing columns, except the first, may be multiplied by any number whatever without affecting the average, it follows that there are an infinite number of results satisfying the conditions of the problem. Since we are to have 100 bu. at 40¢, the first column can be multiplied by only 50.

In B, we have multiplied the first column by 50 and added in the results in the other two columns. This gives us 100 bu. at 40¢, 2 bu. at 50¢, 2 bu. at 60¢, and 254 bu. at 75¢. The second and third columns may be multiplied by any number whatever. But the first must always must be multiplied by 50, because we are to have 100 bu. at 40 cents per bushel.

(R. H. A., p. 338, prob. 2.)

I. How much lead, specific gravity 11, with $\frac{1}{2}$ oz. copper, sp. gr. 9, can be put on 12 oz. of cork, sp. gr. $\frac{1}{4}$, so that the three will just float, that is, have a sp. gr. (1) the same as water?

Mean 1.

$\frac{1}{11}$	$\frac{10}{11}$	3		×$\frac{1}{6}$	×$\frac{352}{27}$=	$39\frac{1}{9}$ oz.=2 lb. $7\frac{1}{9}$ oz.
$\frac{1}{9}$	$\frac{8}{9}$		3			$\frac{1}{2}$ oz.
4	3	$\frac{10}{11}$	$\frac{8}{9}$			12 oz.

SOLUTION.—The specific gravity of any body is the ratio which shows how many times heavier the body is than an equal volume of water. Thus, when we say that the specific gravity of lead is 11, we mean that a cubic inch, a cubic foot, a cubic yard, or any quantity whatever is 11 times as heavy as an equal quantity of water.

Now if a cubic inch (say) of lead be immersed in water, it will displace a cubic inch of water; and since it weighs 11 times as much as a cubic inch of water, it displaces $\frac{1}{11}$ of its own weight. Hence, to have equal weights of water and lead we must take only $\frac{1}{11}$ as much lead as water. Now since a volume of water and $\frac{1}{11}$ as much lead have the same weight, and in the proper combination have a volume of 1, since the sp. gr. of the combination is 1, there is a loss of $1-\frac{1}{11}$, or $\frac{10}{11}$, in volume on the part of the lead. For the same reason, there is a loss of $\frac{8}{9}$ in volume on the part of the copper, and 3 on the part of the cork. Balancing, we see that we must take 3 volumes of lead with $\frac{10}{11}$ volumes of cork, a unit volume of water being the basis, in order that the two substances will just float, *i. e.*, have a specific gravity (1). In like manner, we must take 3 volumes of copper with $\frac{8}{9}$ volumes of cork. Now since we must always take 3 volumes of lead for every $\frac{10}{11}$ volumes of cork, it is evident that the weights of the substances are in the same proportion. Hence, we may say, we must take 3 oz. of lead with every $\frac{10}{11}$ oz. of cork, and 3 oz. of copper with every $\frac{8}{9}$ oz. of cork.

But we are to have only $\frac{1}{2}$ oz. of copper. Hence, we must multiply the second balancing column by some number that will give us $\frac{1}{2}$ oz. of copper, *i. e.*, we must multiply 3 by some number that will give us $\frac{1}{2}$. The number by which we must multiply is $\frac{1}{2}\div3=\frac{1}{6}$. But multiplying $\frac{8}{9}$ by $\frac{1}{6}$, we get $\frac{4}{27}$ oz. of cork. But we are to have altogether 12 oz. of cork. Hence we must yet have 12 oz.$-\frac{4}{27}$ oz.$=\frac{320}{27}$ oz. To produce this, we must multiply $\frac{10}{11}$ by some number that will give $\frac{320}{27}$ oz. This number is $\frac{320}{27}\div\frac{10}{11}=\frac{352}{27}$. But we must also multiply 3 by $\frac{320}{27}$. This will give us $39\frac{1}{9}$ oz.=2 lb. $7\frac{1}{9}$ oz. of lead. Hence, we must use 2 lb. $7\frac{1}{9}$ oz. of lead, so that the three will just float.

(*R. H. A., p. 339. prob. 7.*)

I. How many shares of stock, at 40%, must A buy, who has bought 120 shares, at 74%, 150 shares, at 68%, and 130 shares, at 54%, so that he may sell the whole at 60%, and gain 20%?

1.
(1.) 100%=the average cost.
(2.) 20%=gain.
(3.) 120%=the average selling price.
(4.) 60%=the average selling price.
(5.) ∴ 120%=60%.
(6.) 1%=$\frac{1}{120}$ of 60%=$\frac{1}{2}$%.
(7.) 100%=100 times $\frac{1}{2}$%=50%, the average cost.

II. 2. (1.) 120 shares @ 74% = 8880%.
(2.) 150 shares @ 68% = 10200%.
(3.) 130 shares @ 54% = 7020%.
(4.) ∴ 400 shares are worth 26100%, and
(5.) 1 share is worth 26100% ÷ 400 = $65\frac{1}{4}$%, the average.

3. 50% | 40 % | 10 % | $15\frac{1}{4}$ shares. | × 40 = 610 shares.
| $65\frac{1}{4}$% | $15\frac{1}{4}$% | 10 shares. | 400 shares.

III. ∴ He must take 610 shares. (*R. H. A., p. 339, prob. 8.*)

Explanation.—Since 60% is the average selling price, and his gain is 20%, it is evident that his average cost is 60% ÷ 1.20, or 50%. In step 3, we find that the average cost of the 400 shares is $65\frac{1}{4}$%. Hence, the problem is the same as to find how many shares at 40%, must A buy who has 400 shares at at an average of $65\frac{1}{4}$% so that his average cost will be 50%. Balancing, we find that he must take $15\frac{1}{4}$ shares at 40% with 10 shares at $65\frac{1}{4}$%. But he has 400 shares at $65\frac{1}{4}$%. Hence, we must multiply the balancing column by 400 ÷ 10, or 40. This gives 610 shares at 40%.

CASE III.

To proportion the parts, the amount of the whole combination being given.

I. How many barrels of flour, at \$8, and \$8.50, with 300 bbl. at \$7.50, 800 bbl. at \$7.80, and 400 bbl. at \$7.65, will make 2000 bbl. at \$7.85 a bbl.?

II.
1. 300 bbl. @ \$7.50 a bbl. = \$2250.
2. 800 bbl. @ \$7.80 a bbl. = \$6240.
3. 400 bbl. @ \$7.65 a bbl. = \$3060.
4. ∴ 1500 bbl. are worth \$11550.
5. \$7.85 = the average price per bbl. of 2000 bbl.
6. ∴ \$15700 = 2000 × \$7.85 = the value of 2000 bbl.
7. ∴ \$15700 — \$11550 = \$4150 = the value of 2000 bbl. — 1500 bbl., or 500 bbl.
8. ∴ \$8.30 = \$4150 ÷ 500 = the average value of 1 bbl.
9. \$8.30 | \$8.00 | \$.30 | 2 bbl. | × (500 ÷ 5) = 200 bbl.
| \$8.50 | \$.20 | 3 bbl. | 300 bbl.
5 bbl.

III. ∴ 1. 200 bbl. at \$8.00 per bbl. must be taken with
2. 300 bbl. at \$8.50 per bbl.

(*R. H. A., p. 339, prob. 2.*)

I. A dealer in stock can buy 100 animals for \$400, at the following rates: calves, \$9; hogs, \$2; lambs, \$1; how many may he take of each kind?

			Bal.										
	$1	$3	5 lambs.		3	10	17	24	31	38	45	62	59
$4	$2	$2		5 hogs.	68	60	52	44	36	28	20	12	4
	$9	$5	3 calves.	2 calves.	29	30	31	32	33	34	35	36	37
			8	7									

Explanation.—A lamb bought for $1 and sold for $4 is a gain of $3; a hog bought for $2 and sold for $4 is a gain of $2; and a calf bought for $9 and sold for $4 is a loss of $5. We must make the gains and loses equal. The L. C. M. of $3 and $5 is $15. If we gain $3 on one lamb to gain $15 we must take as many lambs as $3 is contained in $15, which are 5 lambs. If we lose $5 on one calf, to lose $15, we must take as many calves as $5 is contained in $15, which are 3 calves. The L. C. M. of $2 and $5 is $10. If we gain $2 on one hog, to gain $10, we must take as many hogs as $2 is contained in $10, which are 5 hogs. If we lose $5 on one calf, to lose $10, we must take as many calves as $5 is contained in $10, which are 2 calves. Adding the balancing columns, considering them as abstract numbers, we have 8 and 7. $8+7=15$. $100\div15=6\frac{2}{3}$. $\therefore$ Multiplying each balancing column by $6\frac{2}{3}$, will give $33\frac{1}{3}$ lambs, $33\frac{1}{3}$ hogs, and $33\frac{1}{3}$ calves. But this result is not compatible with the nature of the problem. Hence we must see if we can take a number of 8's and a number of 7's that will make 100. By trial, we find that *two* 8's and *twelve* 7's will make 100. Hence, multiplying the first column by 2 and the second by 12, and adding the columns horizontally, we have for our result, 10 lambs, 60 hogs, and 30 calves. Again, we find, by trying three 8's, four 8's, and so on, that *nine* 8's taken from 100, will leave 28 which is *four* 7's. Hence, nine 8's and four 7's will make 100. Then, multiplying the first column by 9 and the second by 4, and adding the columns horizontally, we have for a second result 45 lambs, 20 hogs, and 35 calves. Now these are the only answers that can be obtained by taking an integral number of 8's and integral number of 7's to make 100. But other answers may be obtained by taking 8 a *fractional* number of times, and 7 a fractional number of times to make 100. Suppose, for illustration, we try to take a number of thirds 8 times. We find that 8 taken 6-third times and 7 taken 36 third times will make 100. Multiplying the first column by $\frac{6}{3}$ and the second by $\frac{36}{3}$, and adding, the columns horizontally, we have, for a result, 10 lambs, 60 hogs, and 30 calves—the same as that obtained by taking 8 twice and 7 twelve times. Again, we find, that 8 taken 13 third times and 7 taken 28-third times will make 100. Multiplying and adding as before we find that our results are fractional. Hence, we can not take a fraction whose denominator is three. It is clear that we must take a fraction whose denominator will reduce to unity when being multiplied by 5. Hence, if we try to take 8 a number of fifths times and 7 a number of fifths times to make 100, our results will all be integral. By trial, we find that 8 taken 3-fifths times and 7 taken 68-fifths times will make 100. Multiplying and adding as before, we have, for our results, 3 lambs, 68 hogs, and 29 calves. Again, we find that 8 taken 10-fifths times and 7 taken 60-fifths times, will make 100. Multiplying and adding as before, we have, for results, 10 lambs, 60 hogs, and 30 calves. Again, by trial, we find that 8 taken 17-fifths and 7 taken 52-fifths times will make 100. Multiplying the first column by $\frac{17}{5}$ and the second by $\frac{52}{5}$, and adding the columns horizontally, we have, for results, 17 lambs, 52 hogs, and 31 calves. Continuing the process, we find nine admissible answers. These are the only answers, satisfying the nature of the problem.

CHAPTER XVIII.

SERIES.

DEFINITIONS.

1. ***A Series*** is any number of quantities, having a fixed order, and related to each other in value according to a fixed law.

Thus, 1, 3, 5, 7, 9, 11; 3, 9, 27, 81; 1, 2, 3, 5, 8, 13, 21; etc., are series.

2. The quantities involved in a series are called ***Terms;*** the first and last are called the *extremes* and the others the *means.*

3. Series
   1. Arithmetical.
   2. Geometrical.
   3. Harmonic.
   4. Binomial.
   5. Logarithmic.
   6. Exponential.
   7. Hypergeometric.
   8. Etc., etc.

4. The series usually treated in arithmetic are the Arithmetical and Geometrical; and these together with the Harmonic are commonly called ***Progressions.***

5. ***An Arithmetical Progression*** is a series of numbers which increase or decrease by a constant quantity, called the ***Common Difference.***

Thus, 5, 8, 11, 14, 17 is an *increasing* arithmetical progression whose common difference is 3; and, 17, 15, 13, 11, 9 is a *decreasing* arithmetical progression whose common difference is 2.

6. ***A Geometrical Progression*** is a series of numbers which increase or decrease by a constant multiplier, called the ***Ratio.***

Thus, 2, 6, 18, 54 is an *increasing* geometrical progression whose ratio is 3; and, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, is a *decreasing* geometrical progression whose ratio is $\frac{1}{2}$.

7. ***A Harmonic Progression*** is a series of numbers, the reciprocals of which are in Arithmetical Progression.

Thus, $\frac{1}{2}$, $\frac{1}{5}$, $\frac{1}{7}$, $\frac{1}{9}$, is a Harmonic Progression.

I. ARITHMETICAL PROGRESSION.

1. Every Arithmetical Progression involves five quantities, viz., the *first term*, represented by a; the *last term*, represented by l; the *number of terms*, represented by n; the *common difference*, represented by d; and *the sum* of all the terms, represented by s.

2. If any three of these five quantities be given, the others may be found. Hence, every formula involves four quantities,

viz., the three given quantities and the required quantity. But 4 things can be selected from 5 things in $\frac{5\times4\times3\times2}{1\times2\times3\times4}$ ways or 5 ways. From every one of these 5 combinations of 4 letters, 4 formulas may be obtained. Hence, for the solution of every problem in arithmetical progression 4×5, or 20, formulas are required.

3. But these 20 formulas may be easily derived from two fundamental formulas which we shall proceed to derive.

Problem.—Given $\left\{\begin{array}{l}\text{the first term, } a, \\ \text{the common difference, } d, \text{ and} \\ \text{the number of terms, } n,\end{array}\right\}$ to find the last term, l.

Consider the series, 2, 5, 8, ... to 15 terms.

Here, first term, $a,=2$; common difference, $d,=3$; and number of terms, $n,=15$.

$2=a$, the first term.

5, the *second* term, $=2+3=a+d$, the first term+the common difference.

8, the *third* term, $=2+2\times3=a+2d$, the first term+*twice* the common difference.

11, the *fourth* term, $=2+3\times3=a+3d$, the first term+*three* times the common difference.

By continuing this, we see that any term is equal to the first term + the common difference multiplied by 1 less than the number of the term. Hence, for an increasing series, we have the

Formula.— $l=a+(n-1)d$.

Rule.—*Multiply the common difference by one less than the number of the term required and add the product to the first term; if the series is decreasing, subtract the product from the first term.*

I. A body falls $16\frac{1}{12}$ feet the first second, and in each succeeding second $32\frac{1}{6}$ feet more than in the next preceding one. How far will it fall in the 16th second?

By formula, $l=a+(n-1)d=16\frac{1}{12}$ ft. $+(16-1)32\frac{1}{6}$ ft. $=498\frac{7}{12}$ feet.

II.
1. $16\frac{1}{12}$ feet=the first term.
2. $32\frac{1}{6}$ feet=the common difference.
3. 16=the number of terms.
4. $15=16-1$, =one less than the number of terms.
5. $482\frac{1}{2}$ feet $=15\times32\frac{1}{6}$ feet, =the common difference multiplied by one less than the number of terms.
6. ∴ $16\frac{1}{12}$ feet $+482\frac{1}{2}$ feet $=498\frac{7}{12}$ feet, =the distance the body falls the 16th second.

III. ∴ The body falls $498\frac{7}{12}$ feet the 16th second.

PROBLEMS.

1. Find the 11th term of the progression 2, 9, 16, etc.
2. Find the 99th term of the series $3\frac{1}{4}$, $3\frac{3}{4}$, $4\frac{1}{4}$, etc.
3. Find the 59th term of the series $\frac{7}{9}$, $\frac{3}{4}$, $\frac{13}{18}$, etc.
4. Insert five arithmetical means between 3 and 15.
5. How many times does a clock that strikes quarter hours and hours strike in 12 hours?

Problem.—Given $\left\{\begin{array}{l}\text{the first term, } a, \\ \text{the last term, } l, \text{ and} \\ \text{the number of terms, } n,\end{array}\right\}$ to find the sum of the terms.

Consider the series 5, 11, 17, 23, 29, 35, 41.

The sum$=5+11+17+23+29+35+41$.

The sum$=41+35+29+23+17+11+5$.

$\therefore$ 2 times the sum$=46+46+46+46+46+46+46=6\times46=6(5+41)$.

$\therefore$ The sum$=\frac{6(5+41)}{2}=\frac{n(a+l)}{2}$.

Hence, we have for the sum of the series, the

Formula.—$s=(a+l)\frac{n}{2}$.

Rule.—*To the first term add the last term and multiply the sum by half the number of terms.*

I. Find the sum of all integers less than 100 which are multiples of 7.

By formula, $s=(a+l)\frac{n}{2}=(7+98)\frac{14}{2}=735$.

II. $\left\{\begin{array}{ll}1. & 7=\text{the first term,} \\ 2. & 7=\text{the common difference,} \\ 3. & 98=\text{the last term.} \\ 4. & 14=\text{the number of terms.} \\ 5. & \therefore 735=\frac{14(7+98)}{2}=\text{the sum.}\end{array}\right.$

III. $\therefore$ The sum$=735$.

Remark.—The two formulas, $l=a+(n-1)d$ and $s=(a+l)\frac{n}{2}$ are all that are needed to solve any problem in arithmetical progression. They should be committed to memory.

I. The first term of a decreasing arithmetical series is 10, the number of terms 10, and the sum of the series 85; required the last term and the common difference.

(*Ray's Elementary Algebra, p. 221, prob. 7.*)

II.
1. $10=$the first term$=a$,
2. $10=$ the number of terms$=n$,
3. $85=$the sum of the terms$=s$.
4. $\therefore$ $l=a-(n-1)d$, the series being decreasing,$=10-(10-1)d=10-9d$, and
5. $s=(a+l)\frac{n}{2}$, or $85=(10+l)\frac{10}{2}=50+5l$.
6. $\therefore$ $85=50+5(10-9d)=100-45d$; whence,
7. $d=(100-85)\div45=\frac{1}{3}$, the common difference.
8. $\therefore$ $l=10-9\times\frac{1}{3}=7$, from step 4.

III. $\therefore$ $\begin{cases}\frac{1}{3}=\text{the common difference, and}\\ 7=\text{the last term.}\end{cases}$

Remark.— It is not necessary to remember the two formulas, $l=a+(n-1)d$ and $l=a-(n-1)d$, for finding the last term. The first of these is sufficient; for, when the series is decreasing and a, l, n are given, the formula gives a negative value for d, the common difference, and adding a negative common difference in forming the terms of a series is the same as subtracting a numerically equal positive common difference.

PROBLEMS.

1. Find the sum of the series 1, 2, 3, etc., to 1000 terms. *Ans.* 500500.
2. Find the sum of the series 1, 3, 5, 7, . . . to 101 terms. *Ans.* 10201.
3. Find the sum of all the multiples of 11, from 110 to 990, inclusive. *Ans.* 44550.
4. If a person saves $100 per year, and puts this sum at simple interest at $4\frac{1}{2}\%$ at the end of each year, to how much will his property amount at the end of 25 years? *Ans.* $3850.

II. GEOMETRICAL PROGRESSION.

1. The five quantities involved in a ***Geometrical Progression*** are the *first term, a;* the *last term, l;* the *number of terms, n;* the *common ratio, r;* and the *sum of the terms, s.*

2. Any *three* of these five quantities being given, the others may be found. In Geometrical Progression as in Arithmetical Progression, 20 formulas are necessary for the solution of all problems. But these 20 formulas are easily obtained from two fundamental formulas which we will now derive.

Problem.—Given $\begin{cases}\text{the first term, } a,\\ \text{the number of terms, } l, \text{ and}\\ \text{the ratio, } r,\end{cases}$ to find the last term.

Consider the series 2, 6, 18, 54, 162, etc.

2=the first term.
6, the *second* term,$=2\times3=ar$, the first term multiplied by the ratio, 3.
18, the *third* term,$=2\times3^2=ar^2$, the first term multiplied by the ratio *squared*.

54, the *fourth* term, $=2\times3^3=ar^3$, the first term multiplied by the ratio *cubed.*

162, the *fifth* term, $=2\times3^4=ar^4$, the first term multiplied by the ratio raised to the *fourth* power.

From this, we see that any term is equal to the first term multiplied by the ratio raised to a power whose exponent is 1 less than the number of the term.

Hence, for finding any term of a geometrical progression, we have the

Formula.—$l=ar^{n-1}$

Rule.—*To find any term of a geometrical progression, raise the ratio to a power whose exponent is* 1 *less than the number of the required term.*

I. Find the 8th term of the series 6, 18, 54, etc.

By formula, $l=ar^{n-1}=6\times3^{8-1}=13122$.

II.
1. 6 = the first term,
2. 3 = the ratio,
3. 8 = the number of the term required.
4. $2187=3^7$, the ratio raised to a power whose exponent is 1 less than the number of the term.
5. ∴ $13122=6\times2187$ = the 8th term of the progression.

III. ∴ The 8th term of the series is 13122.

PROBLEMS.

1. Find the 10th term of the series 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, etc.
2. Find the 9th term of the series 1, $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, etc.
3. Find the 6th term of the progression $\frac{3}{8}$, $\frac{1}{2}$, $\frac{2}{3}$, etc. *Ans.* $\frac{128}{81}$.
4. Find the 9th term of the progression $6\frac{3}{4}$, $4\frac{1}{2}$, 3, etc. *Ans.* $\frac{64}{243}$.

Problem.—Given $\left\{\begin{array}{l}\text{the first term, } a,\\ \text{the last term, } l, \text{ and}\\ \text{the ratio, } r,\end{array}\right\}$ to find the sum of the progression.

Consider the progression 5, 15, 45, 135, 405.

The sum $=5+15+45+135+405$.

3 times the sum $=3\times5+3\times15+3\times45+3\times135+3\times405$,
$=15+45+135+405+3\times405$.

1 times the sum $=5+15+45+135+405$.

∴ 2 times the sum $=3\times405-5$, by subtracting the last equation from the one above.

∴ The sum $=\dfrac{3\times405-5}{2}$. But 3×405 is the last term multiplied by the ratio, $l\times r$; 5 is the first term, a; and, 2 is the ratio, $r-1$.

Hence, for finding the sum of a geometrical progression, we have the

Formula.—$s=\dfrac{lr-a}{r-1}$.

Rule.—*Multiply the last term by the ratio, from the product subtract the first term, and divide the result by the ratio minus* 1.

Remark 1.—The above formula and rule apply whether the series be increasing or decreasing, though when the series is decreasing $lr-a$ is negative and $r-1$ is negative. But a negative quantity divided by a negative quantity gives a positive quantity. In order that the formula applies arithmetically, for decreasing series, we write the formula $s=\dfrac{a-lr}{1-r}$.

Remark 2.—If the series is an infinite decreasing series, the sum, s, approaches the value $\dfrac{a}{1-r}$ as n approaches infinity.

Thus, the sum of the series 1, ½, ¼, ⅛, etc. to an infinite number of terms approaches $\dfrac{1}{1-\frac{1}{2}}$, or 2.

Hence, for the limit of the sum of an Infinite Decreasing Geometrical Progression, we have the limit of $s=\dfrac{a}{1-r}$.

To find the limit of the sum of an infinite decreasing geometrical progression, we have the following

RULE.— Divide the first term by the ratio minus 1.

NOTE.—It must be borne in mind that we cannot find the *sum* of an infinite decreasing geometrical progression. We can only find the limit of the sum as more and more terms are taken. Thus, in the series, 1, ½, ¼, ⅛, etc. to infinity, we can make the sum of the series as nearly equal to 2 as we please, though we can never take enough terms to make it exactly equal to 2. Hence, we should not speak of the sum of an infinite decreasing geometrical progression, but of the limit of the sum.

I. A man agreed to work for 14 days on the condition that he should receive 1 cent for the first day, 2 cents for the second day, 4 cents for the third day, and so on. How much did he receive in all?

By formula, $s=\dfrac{lr-a}{r-1}=\dfrac{ar^{n-1}\times r-1}{r-1}=\dfrac{ar^n-a}{r-1}=\dfrac{\$0.01\times2^{14}-\$0.01}{2-1}$ $=\$163.83$.

II.
1. 1 cent=the first term,
2. 2 =the ratio, and
3. 14 =the number of terms.
4. 8192 cents$=2^{14-1}\times1$ cent, the amount he received for the last day.
5. ∴ $\dfrac{2\times8192\text{ cents}-1\text{ cent}}{2-1}=16383$ cents,=\$163.83, the amount he received.

III. ∴ He received \$163.83.

PROBLEMS.

1. Find the limit of the sum of the series 1, $\frac{1}{10}$, $\frac{1}{100}$, etc., to infinity. *Ans.* $1\frac{1}{9}$.
2. Find the limit of the sum of the series 1, $\frac{1}{3}$, $\frac{1}{9}$, etc., to infinity. *Ans.* $1\frac{1}{2}$.

CHAPTER XIX.

SPECIFIC GRAVITY, OR SPECIFIC DENSITY.

1. ***The Specific Gravity*** of any substance is the ratio of its weight to that of an *equal volume* of some other substance taken as a standard.

In Physics, it is customary to consider specific gravity as the ratio of the weight of a substance to that of an *equal volume* of distilled water at 4° C., if the given substance be either solid or liquid; but if gaseous, it is compared with either air or hydrogen at 0° C. and 76 cm. pressure.

2. ***Density*** is the quantity of matter in a unit of volume.

Thus, if m=mass, v=volume, δ=density, then $\delta=\frac{m}{v}$.

3. ***Specific Density*** is the ratio of the mass of a unit volume of a substance to that of a unit of volume of some other substance taken as a standard.

The standards for *specific density* are the same as those for specific gravity. Thus, in the C. G. S. (centimeter—gram—second) system, 1 gram of water occupies 1 cubic centimeter, and, therefore, the density of water is $\frac{1 \text{ gram}}{1 \text{ cubic cm.}}=1$. More accurately, 1 cu. cm. of water at 3.9° C. weighs 1.000,013 standard grams and, hence, its density is $\frac{1.000,013 \text{ grams}}{1 \text{ cu. cm.}}=1.000,013$.

If m and m_1 be masses of equal volumes of water and of the substance, and v be these equal volumes; the density of water is $\frac{m}{v}$ and of the substance $\frac{m_1}{v}$.

$$\text{Specific density of substance}=\frac{\text{density of substance}}{\text{density of water}}=\frac{\frac{m_1}{v}}{\frac{m}{v}}=\frac{m_1}{m}.$$

Weight of m units of mass of water, W,$=gm$, where g is acceleration due to gravity.

Weight of m_1 units of mass of the substance, W_1,$=gm_1$.

∴ Specific gravity of the substance=

$$\frac{\text{weight of substance}}{\text{weight of equal volume of water}}=\frac{W_1}{W}=\frac{m_1 g}{mg}=\frac{m_1}{m}.$$

From these two equations of specific gravity and specific density, it is seen that *specific gravity* and *specific density* are mathematically equivalent. These two expressions are, therefore, used synonymously.

It is a fact, discovered by Archimedes, and proved by experiment, that

A body submerged in a fluid is buoyed up by a force equal to the weight of the fluid displaced.

4. *True and Apparent Weight.*—It, therefore, follows that when a body is weighed in air it is buoyed up by a force equal to the weight of the air displaced by it. This is the *apparent* weight of the body. The *true* weight would be obtained by weighing the body in a vacuum. The apparent weight is taken as the true weight in practice.

Prob. 1.—To find the Specific Density of a solid insoluble in water.

Find the weight of the body in air, and also its weight as it hangs suspended in ice water. Let w be the weight in air and w' its weight in water. Then by the principle of Archimedes, $w-w'$=the weight of the water displaced.

$$\therefore \text{Density}=\frac{w}{w-w'}.$$

I. A body weighs 62 grams in air and 42 grams in water; find its specific density.

II.
1. 62 grams=weight of substance in air.
2. 42 grams=weight of substance in water.
3. 20 grams=weight of water displaced, by principle of Archimedes.
4. 62 grams÷20 grams=3.1=density, by definition.

III. ∴ The specific density of the body is 3.1.

Prob. 2.—To find the Specific Density of a solid soluble in water.

Find the weight of the body in air, and also its weight in some liquid of known density in which it is not soluble. Let s be the density of the liquid, w the weight of the body in air, and w' the weight of the body in the liquid. Then $w-w'$ equals weight of the liquid displaced. Hence, $\frac{w}{w-w'}$=the specific density of the body as compared with the liquid. But since the density of the liquid as compared with water is s, therefore, the density of the body as compared with water is s times $\frac{w}{w-w'}=\frac{ws}{w-w'}$.

I. A solid weighs 100 grams in air and 64 grams in a liquid of density 1.2; what is its density?

(*Carhart and Chute's Elements of Physics, p. 117.*)

II.
1. 100 grams=weight of body in air.
2. 64 grams=weight of body in the liquid.
3. 36 grams=weight of liquid displaced by the body.
4. ∴ 100 grams÷36 grams=$2\frac{7}{9}$=density of body as compared with the liquid.

5. ∴ $1.2 \times 2\frac{7}{9} = 3\frac{1}{3}$ = density of the body as compared with water.

III. ∴ Specific density of the body $= 3\frac{1}{3}$.

Prob. 3.—To find the Specific Density of a body lighter than water.

Weigh the body in air. Then, to make it sink, attach to it a heavy body whose weight in water is known, and find the weight of the bodies in water. Let w be the weight of the body in air and w' the weight of the body and sinker in water, and w_1 and w_2 the weight of the sinker in air and water, respectively. Then, $w+w_1-w'$ = weight of water displaced by body and sinker, w_1-w_2 = weight of water displaced by sinker alone. Hence, $w+w_1-w'-(w_1-w_2)$, or $w-w'+w_2$ = weight of water displaced by body. Hence, $\frac{w}{w-w'+w_2}$ equals the specific density.

I. Find the density of a solid from the following data:
Weight of solid in air = 0.5 g.
Weight of sinker in water = 3.5 g.
Weight of solid and sinker in water = 3.375 g.
(*Carhart and Chute's Elements of Physics, p. 117.*)

II.
1. Let w = weight of sinker in air.
2. 0.5 g. + w = weight of solid and sinker in air.
3. 3.375 g. = weight of solid and sinker in water.
4. ∴ w—2.875 g. = weight of water displaced by solid and sinker.
5. 3.5 g. = weight of solid in water.
6. w—3.5 = weight of water displaced by sinker alone.
7. (w—2.875 g.)—(w—3.5 g.) = 0.625 g. = weight of water displaced by the solid.
8. ∴ $\frac{0.5\text{ g}}{0.625\text{ g.}} = \frac{1}{125} = .008$ = specific density.

III. ∴ The specific density = .008.

Prob. 4.—To find the Specific Density of a liquid.

(1.) *By means of the Specific Gravity bottle.*

The specific gravity bottle is a bottle constructed to hold a given weight of distilled water at a specified temperature. Its capacity is usually 500 cu. centimeters at 4° C. To use the bottle, find the weight of the bottle when empty and also when filled with the liquid at 4° C. The difference will be the weight of 500 cu. centimeters of the liquid. This divided by 500 grams, the weight of 500 cu. centimeters of water, will give the specific density of the liquid. Expressed algebraically,

$\delta = \frac{w_1 - w}{500}$, where w_1 is the weight of the bottle and liquid and w the weight of the bottle.

Remark.—Any ordinary bottle may be used in the following way: Weigh the bottle; weigh the bottle filled with water; and, after carefully drying the bottle, weigh the bottle filled with air. From the weight of bottle and water, subtract the weight of the bottle, this gives the weight of the water; from the weight of the bottle and the liquid subtract the weight of the bottle, this gives the weight of an equal volume of the liquid. Dividing the weight of the liquid by the weight of the water gives the specific density,—algebraically expressed as follows:

$\delta=\frac{w_2-a}{w_1-a}$, where w_1 is the weight of the bottle when filled with water, w_2 the weight of the bottle when filled with the liquid, and a is the weight of the bottle when empty.

(2.) *By weighing a Solid in the Liquid.*

Weigh the solid in air, then in ice water, and finally in the liquid. From the weight of the solid weighed in the liquid subtract the weight of the solid in air; this gives the weight of the liquid displaced. (Why?) From the weight of the solid weighed in water subtract its weight in air; this gives the weight of the water displaced. Divide the weight of the liquid displaced, by the weight of the water displaced, and the result will be the specific density of the liquid,—algebraically expressed as follows:

$\delta=\frac{w_1-w'_2}{w_1-w_2}$, where w_1 is the weight of the solid in air, w'_2 the weight of the solid in the liquid, and w_2 the weight of the solid in water.

(3.) *By the Areometer, or Hydrometer.*

The Hydrometer consists of a straight stem of wood, glass, or metal weighted at one end so as to float in a liquid in a vertical position. By observing the depths to which it sinks in different liquids, the relative weights of these liquids can be determined. The stem is graduated, the zero being taken at the point to which it sinks in water.

I. A body weighs 24 grams in air and 20 grams in water; what will be its apparent weight in alcohol, specific density 0.8?

II.
1. 24 grams=weight in air.
2. 20 grams=weight in water.
3. 4 grams=24 grams—20 grams=weight of water displaced.
4. ∴ 24 grams÷4 grams=6, the specific density of the body.
5. 6÷0.8=7.5, the specific density of the body as compared with alcohol.
6. ∴ The body when weighed in alcohol loses 1÷7.5, or $\frac{2}{15}$ of its weight.
7. ∴ Its apparent weight in alcohol=$\frac{13}{15}$×24 grams=$20\frac{4}{5}$ grams.

III. The body will weigh $20\frac{4}{5}$ grams in alcohol.

Second solution.

II.
1. Let x=number of grams in the weight of the body in alcohol. Then
2. 24 grams—x grams=weight of alcohol displaced.
3. 24 grams—20 grams=4 g., weight of water displaced.
4. $\therefore \frac{24 \text{ grams} - x \text{ grams}}{4 \text{ grams}}$=density of alcohol=0.8,
5. whence x=20.8, the weight of the body in alcohol.

III. ∴ The weight of the body in alcohol is 20.8 grams.

The method of alligation has practical application in problems of specific density. Thus, if it is desired to show that a liquid unaffected by gravity, assumes the form of a sphere, it may be accomplished by placing a liquid in some other liquid with which it does not mix and having the same specific density.

This experiment is performed by placing, with a pipette, a mass of olive oil in a mixture of alcohol and water.

I. In what proportion must water, specific density 1, and alcohol, specific density 0.8, be mixed to make a mixture specific gravity, 0.9176, the same as olive oil?

II.
1. ∵ 0.9176=specific gravity of olive oil,
2. ∴ weight of 0.9176 vol. of water=weight of 1 vol. of olive oil,
3. weight of 1. vol. of water=weight of 1÷0.9176, or $1\frac{103}{1147}$ vols. of olive oil.
4. 0.8=specific gravity of alcohol,
5. ∴ weight of 0.8 vol. of water=weight of 1 vol. of alcohol,
6. weight of 1 vol. of water=weight of 1÷0.8, or $1\frac{1}{4}$ vols. of alcohol.
7.

	Given Volumes.	Dif. of Volumes.	Ratio of Vol.	Ratio of Mix.
Vol. in mix.	1 vol. of W.	$\frac{103}{1147}$ vol. of W.	412	735
$1\frac{103}{1147}$	$1\frac{1}{4}$ vol. of A.	$\frac{735}{4588}$ vol. of A.	735	412
per lb.				1147

III. Hence, to obtain any quantity of a mixture of alcohol and water, specific gravity 0.9176, 735 parts, by weight, of water must be taken for every 412 parts, by weight, of alcohol.

The same experiment can be made by placing carbon bisulphide (CS_2) colored with iodine in a solution of zinc sulphate ($ZnSO_4$) so that the mixture will have a specific gravity, 1.272, the same as carbon bisulphide.

I. Find how much water, specific gravity 1, must be added to a strong solution of zinc sulphate, specific gravity, 1.400, to make a mixture whose specific gravity is 1.272, the same as carbon bisulphide.

II.

1. ∵ 1.4=specific gravity of $ZnSO_4$,
2. ∴ volume of 1.4 grams of $ZnSO_4$=volume of 1 gram of water,
3. volume of 1 gram of $ZnSO_4$=volume of $\frac{5}{7}$ gram of water.
4. ∵ 1.272=specific gravity of CS_2,
5. ∴ volume of 1.272 grams of CS_2=volume of 1 gram of water,
6. volume of 1 gram of CS_2=volume of $\frac{125}{159}$ gram of water,
7. volume of 1 gram of water=volume of 1 gram of water.
8.

	Given Weights.	Dif. of Weights.	Ratio of Wts.	Ratio of Mix.
Weight in Mix. $\frac{125}{159}$ gram per unit vol.	$\frac{5}{7}$ gram of $ZnSO_4$	$\frac{80}{1113}$ g.	40	119
	1 gram of H_2O	$\frac{34}{159}$ g.	119	40

III. Hence, 119 parts, by weight, of zinc sulphate, must be taken for every 40 parts, by weight, of water.

Remark.—The last two problems are exactly alike so far as the principle involved in their solution is concerned. In the first problem, we used equal weights and comparative volumes; in the last problem we used equal volumes and comparative weights.

I. Hiero's Crown, sp. gr., $14\frac{5}{8}$, was of gold, sp. gr., $19\frac{1}{4}$, and of silver, sp. gr., $10\frac{1}{2}$; it weighed $17\frac{1}{2}$ lbs.: how much gold was in it?

II.

1. ∵ $19\frac{1}{4}$=specific gravity of gold,
2. ∴ volume of $19\frac{1}{4}$ lbs. of gold=volume of 1 lb. of water, and
3. volume of 1 lb. of gold=volume of $\frac{4}{77}$ lbs. of water.
4. ∵ $10\frac{1}{2}$=specific gravity of silver,
5. ∴ volume of $10\frac{1}{2}$ lbs. of silver=volume of 1 lb. of water, and
6. volume of 1 lb. of silver=volume of $\frac{2}{21}$ lbs. of water.
7. ∵ $14\frac{5}{8}$=specific gravity of crown,
8. ∴ volume of $14\frac{5}{8}$ lbs. of the crown=volume of 1 lb. of water, and
9. volume of 1 lb. of the crown=volume of $\frac{8}{117}$ lbs. of water.
10.

	Given Volumes.	Dif. in Vol. per lb.	Ratio of Vol.	Ratio of Mix.
Vol. in crown, $\frac{8}{117}$ per lb.	$\frac{4}{77}$, vol. per lb. for G.	$\frac{444}{27027}$	74	121
	$\frac{2}{21}$, vol. per lb. for S.	$\frac{726}{27027}$	121	74
				195

11. ∴ $\frac{121}{195}$ of $17\frac{1}{2}$ lbs.= $10\frac{67}{78}$ lbs.= weight of gold in the crown, and
12. $\frac{74}{195}$ of $17\frac{1}{2}$ lbs.= $6\frac{25}{78}$ lbs.= weight of silver in the crown.

III. ∴ $10\frac{67}{78}$ lbs.=weight of gold in the crown.

CHAPTER XX.

SYSTEMS OF NOTATION.

1. ***A System of Notation*** is a method of expressing numbers by means of a series of powers of some fixed number called the *Radix*, or *Base* of the scale in which the different numbers are expressed.

2. ***The Radix*** of any system is the number of units of one order which makes one of the next higher.

3.

Names of Systems.	Radix.	Names of Systems.	Radix.
Unitary	1	Nonary	9
Binary	2	Decimal, or Denary	10
Ternary	3	Undenary	11
Quaternary	4	Duodenary,	12
Quinary	5	Vigesimal,	20
Senary	6	Trigesimal,	30
Septenary	7	Sexagesimal,	60
Octonary	8	Centesimal,	100

4. In writing any number in a uniform scale, as many distinct characters, or symbols, are required as there are units in the radix of the given system. Thus, in the decimal system, 10 characters are required; in the ternary, 3; viz., 1, 2, and 0; in the senary, 6; viz., 1, 2, 3, 4, 5, and 0; and so on.

5. Let r be the radix of any system, then any number, N, may be expressed in the form,

$N = ar^n + br^{n-1} + cr^{n-2} + dr^{n-3} + \ldots + pr^2 + qr + s$, in which the co-efficients a, b, c, ... , are each less than r.

To express an integral number in a proposed scale: *Divide the number by the radix, then the quotient by the radix, and so on; the successive remainders taken in order will be the successive digits beginning from units place.*

I. Express the common number, 75432, in the senary system.

II.
1. 6)75432
2. 6)12572+0
3. 6)2095+2
4. 6)349+1
5. 6)58+1
6. 6)9+4
7. 1+3

III. ∴ 75432 in the decimal system=1341120 expressed in the senary system.

I. Transform 3256 from a scale whose radix is 7, to a scale whose radix is 12.

II.
1. 12)3256
2. 12)166+4
3. 12)11+1
4. 0+8

III. ∴ 3256 in the septenary system=814 in the duodenary system.

Explanation.—In the senary system, 7 units of one order make one of the next higher. Hence, 3 units of the fourth order = 7×3, or 21, units of the third order. 21 units +2 units = 23 units. 23÷12 = 1, with a remainder 11. 11 units of the third order = 77 units of the second order. 77 units +5 units=82 units. 82÷12=6, with a remainder 10. 10 units of the second order=70 units of the first order. 70 units +6 units=76 units. 76 ÷12=6, with a remainder 4. Hence, the first quotient is 166, with a remainder 4. Treat this quotient in like manner, and so on, until a quotient is obtained, that is less than 12.

I. What is the sum of 45324502 and 25405534, in the senary system?

```
 45324502
 25405534
115134440
```

Explanation.—4+2=6. 6÷6=1, with no remainder. Write the 0 and carry the 1. 3+1=4. Write the 4. 5+5=10. 10÷6=1, with a remainder 4. Write the 4 and carry the 1. 5+4+1=10. 10÷6=1, with a remainder 4. Write the 4 and carry the 1. 0+2+1=3. Write the 3. 4+3=7. 7÷6=1 with a remainder 1. Write 1 and carry 1 5+5+1=11. 11÷6= 1, with a remainder 5. Write the 5 and carry the 1. 2+4+1=7. 7÷6= 1, with a remainder 1. Write 1 and carry 1. The result is 115134440.

I. What is the difference between 24502 and 5534 in the octonary system?

```
24502
 5534
16746
```

Explanation.—4 cannot be taken from 2. Hence, borrow one unit from a higher denomination. Then (2+8)—4=6 (8—1)—3=4. 5 from (4+8) =7. 5 from (3+8)=6. Hence, the result is 16746.

I. Transform 3413 from the scale of 6 to the scale of 7.

II.
1. 7)3413
2. 7)310+3
3. 7)24+3
   2+2

III. ∴ 3413 in the senary system=2233 in the septenary system.

I. Multiply 24305 by 34120 in the senary system.

```
     24305
     34120
    ------
    530140
   24305
  150032
 121323
----------
1411103040
```

Explanation.—Multiplying 5 by 2 gives 10. 10÷6=1, with a remainder 4. Write 4 and carry 1 to the next order. 2 times 0=0. 0+1=1. Write the 1. 2 times 3=6. 6÷6=1, with a remainder 0. Write the 0 and carry the 1 to the next higher order. 2 times 4=8. 8+1=9. 9÷6=1, with a remainder 3. Write 3 and carry the 1 to the next higher order. 2 times 2=4. 4+1=5. Write 5. Multiply in like manner by 1, 4, and 3. Add the partial products, remembering that 6 units of one order, in the senary system, uniformly make one of the next higher.

I. Multiply 2483 by 589 in the undenary system, or system whose radix is 11.

We must represent 10 by some character. Let it be *t*.

```
   2483
    589
  -----
  1t985
 1t502
11184
-------
13322t5
```

Explanation.—In the undenary system, 11 units of one order uniformly make one of the next higher order. 9 times 3=27. 27÷11=2, with a remainder 5. Write 5 and carry the 2 to the next higher order, or second order. 9 times 8=72. 72+2=74. 74÷11=6, with a remainder 8. Write 8 and carry the 6 to the next higher order, or third order. 9 times 4=36. 36+6=42. 42÷11=3, with a remainder 9. Write 9 and carry the 3 to the next higher order, or the fourth order. 9 times 2=18. 18+3=21. 21÷11 =1, with a remainder *t*. Write *t* and carry the 1 to the next higher order. Multiply in like manner by 8 and 5. Add the partial products, remembering that 11 units of one order equals one of the next higher. Wherever 10 occurs, it must be represented by a single character *t*.

I. Divide 1184323 by 589 in the duodenary system.

In the duodenary system, we must have 12 characters; viz., 1, 2, 3, 4, 5, 6, 7, 8, 9, *t*, *e*, and 0. *t* represents 10 and *e*, 11.

589)1184323(2486
*e*56
22*t*3
1*te*0
3*e*32
39*t*0
1523
1523

Explanation.—In the duodenary system, 12 units of one order make one of the next higher. 1184 will contain 589, 2 times. Then multiply the divisor, 589, by 2 thus: 2 times 9=18. 18÷12=1, with a remainder 6. Write the 6 and carry the 1. 2 times 8=16. 16+1=17. 17÷12=1, with a remainder 5. Write the 5 and carry the 1. 2 times 5=*t*. *t*+1=*e*. Write the *e*. Then subtract. 6 from (12+4)=*t*, 5 from 7=2, and *e* from (12+1)=2.

Hence, the first partial dividend is 22 *t*. Bring down 3. Then 22t3 will contain 589, 4 times. Multiply as before. By continuing the operation we obtain 2483 for a quotient.

I. Divide 95088918 by *tt*4, in the duodenary system.

*tt*4)95088918(*t*4*tee*
9074
4548
3754
9*e*49
9074
*t*951
9*e*58
*te*58
*te*58

I. Extract the square root of 11122441 in the senary system.

11122441(2405
2×2= 4
44 | 312
| 304
2×24=52 0 | 42441
2×240=520 5 | 42441

Explanation.—The greatest square in 11 expressed in the senary system is 4. Subtracting and bringing down the next period, we have 312 for the next partial dividend. Doubling the root already found and finding how many times it is contained in 312 expressed in the senary system, we find it is 4. Continuing the process the same as in the decimal system, the result is 2405.

I. Extract the square root of 11000000100001 in the binary system.

	11000000100001(1101111
	1
101	1000
	101
	110000
11001	11001
	1011110
110101	110101
	10100100
1101101	1101101
	11011101
11011101	11011101

(*Todhunter's Algebra, p. 255, Ex. 23.*)

I. Find in what scale, or system, 95 is denoted by 137.

II.
1. Let r=the radix of the system. Then
2. $r^2+3r+7=95$,
3. $r^2+3r=95-7=88$, and
4. $r^2+3r+\frac{9}{4}=88+\frac{9}{4}=\frac{361}{4}$, by completing the square.
5. $r+\frac{3}{2}=\frac{19}{2}$, by extracting the square root, and
6. $r=\frac{19}{2}-\frac{3}{2}=\frac{16}{2}=8$, the radix of the system.

III. ∴ 95 is denoted by 137 in the octonary system.

(*Todhunter's Alg., p. 255, prob. 26.*)

I. Find in what system 1331 is denoted by 1000.

II.
1. Let r=the radix of the system. Then
2. $r^4+0r^3+0r^2+0r+0=1331$, or
3. $r^4=1331$. Whence
4. $r=\sqrt[4]{1331}=11$, the radix of the system.

III. ∴ 1331 is denoted by 1000 in the undenary system.

(*Todhunter's Alg., p. 255, prob. 28.*)

CHAPTER XXI.

MENSURATION.

1. ***Mensuration*** is that branch of applied mathematics which treats of *geometrical magnitudes*.

2. ***Geometrical Magnitudes*** are *lines*, *surfaces*, and *solids*.

3. ***Geometrical Magnitudes.***

- A. Line.
  - 1. Straight.
  - 2. Curved.
- B. Angles.
  - 1. Right Angle.
  - 2. Oblique
    - Acute
    - Obtuse
  - 3. Straight Angle.
  - 4. Reflex Angle.
- C. Surface.
  - 1. Plane.
    - 1^1. Polygon.
      - 1^2. Triangle.
        - 1^3. Right.
          - 1^4. Isosceles.
          - 2^4. Scalene.
        - 2^3 Oblique.
          - 1^4. Isosceles.
          - 2^4 Scalene.
          - 3^4 Equilateral.
      - 2^2. Quadrilateral.
        - 1^3. Parallelogram.
          - 1^4 Right ‖ Rectangle ‖ Square.
          - 2^4 Oblique ‖ Rhomboid ‖ Rhombus.
        - 2^3. Trapezoid.
        - 3^3. Trapezium.
      - 3^2 Pentagon.
      - 4^2. &c.
    - 2^1. Circle.
    - 3^1. Ellipse.
    - 4^1. &c.
  - 2. Curved.
    - 1^1. Spherical Surface.
    - 2^1. Conical Surface.
    - 3^1. Cylindrical Surface.
    - 4^1. &c.
- D. Solid.
  - 1. Polyhedron.
    - 1^1 Prism.
      - 1^2. As to No. of sides in the base.
        - 1^3. Triangular.
        - 2^3. Quadrangular.
        - 3^3. Pentagonal.
        - 4^3. Hexagonal.
        - 5^3. &c.
      - 2^2 Parallelopipedon.
        - 1^3. Right | Rectangular | Cube.
        - 2^3. Oblique.
    - 2^1 Pyramid.
      - 1^2 As to No. of sides in the base.
        - 1^3. Triangular.
        - 2^3. Quadrangular.
        - 3^3, &c.
      - 2^2. Right.
      - 3^2 Oblique.
      - 4^2 Tetrahedron.
    - 3^1 Octahedron.
    - 4^1 Dodecahedron.
    - 5^1 Icosahedron.
  - 2. Cylinder.
    - 1^1. Right.
    - 2^1. Oblique.
  - 3. Cone.
    - 1^1 Right.
    - 2^1. Oblique.
  - 4. Sphere.
  - 5. &c.

4. ***A Line*** is a geometrical magnitude having length, without breadth or thickness.

5. ***A Straight Line*** is a line which pierces space evenly, so that a piece of space from along one side of it will fit any side of any other portion.

6. ***A Curved Line*** is a line no part of which is straight.

7. ***A Surface*** is the common boundary of two parts of a solid, or of a solid and the remainder of space.

8. ***A Plane Surface,*** or ***Plane,*** is a surface which divides space evenly, so that a piece of space from along one side of it will fit either side of any other portion of it.

9. ***A Curved Surface*** is a surface no part of which is plane.

10. ***A Polygon*** (Πολύγωνος, from Πολύς, many, and γωνία, angle) is a portion of a plane bounded by straight lines.

11. ***A Circle*** (κίρκος, circle, ring) is a portion of a plane bounded by a curved line every point of which is equally distant from a point within called the center,

12. ***An Ellipse*** (ἔλλειψις) is a portion of a plane bounded by a curved line any point from which, if two straight lines are drawn to two points within, called the *foci*, the sum of the two lines will be constant.

13. ***A Triangle*** (Lat. *Triangulum*, from *tres*, *tria*, three, and *angulus*, corner, angle) is a polygon bounded by three straight lines.

14. ***An Angle*** is the opening between two lines which meet in a point.

15. ***Angles***
- 1. Straight Angle.
- 2. Right Angle.
- 3. Oblique
  - 1. Acute.
  - 2. Obtuse.

16. ***A Straight Angle*** has its sides in the same line, and on different sides of the point of meeting, or *vertex*.

17. ***A Right Angle*** is half of a *Straight Angle*, and is formed by one straight line meeting another so as to make the adjacent angles equal.

18. ***An Oblique Angle*** is formed by one line meeting another so as to make the adjacent angles unequal.

19. ***An Acute Angle*** is an angle less than a right angle.

20. ***An Obtuse Angle*** is an angle greater than a right angle.

21. ***A Right Triangle*** is a triangle, one of whose angles is a right angle.

22. ***An Oblique-Angled Triangle*** is one whose angles are all oblique.

23. ***An Isosceles Triangle*** is one which has two of its sides equal.

24. ***A Scalene Triangle*** is one which has no two of its sides equal.

25. ***An Equilateral Triangle*** is one which has all the sides equal.

26. ***A Quadrilateral*** (Lat. *quadrilaterus*, from *quatuor*, four, and *latus*, *lateris*, a side) is a polygon bounded by four straight lines.

27. ***A Parallelogram*** (*Παραλληλόγραμμον*, from *Παράλληλος*, parallel, and *γραμμή*, a stroke in writing, a line) is a quadrilateral having its opposite sides parallel, two and two.

28. ***A Right Parallelogram*** is a parallelogram whose angles are all right angles.

29. ***An Oblique Parallelogram*** is a parallelogram whose angles are oblique.

30. ***A Rectangle*** (Lat. *rectus*, right, and *angulus*, an angle) is a right parallelogram.

31. ***A Square*** is an equilateral rectangle.

32. ***A Rhomboid*** (*ῥομβοειδές*, from *ῥόμβος*, rhomb, and *εἶδος*, shape) is a parallelogram whose angles are oblique.

33. ***A Rhombus*** *ῥόμβος*, from *ῥέμβειν*, to turn or whirl round) is an equilateral rhomboid.

34. ***A Pentagon*** (*Πεντάγωνον*, *Πέντε*, five, and *γωνία*, angle) is a polygon bounded by five sides. Polygons are named in reference to the number of sides that bound them. A *Hexagon* has six sides; *Heptagon*, seven; *Octagon*, eight; *Nonagon*, nine; *Decagon*, ten; *Undecagon*, eleven; *Dodecagon*, twelve; *Tridecagon*, thirteen; *Tetradecagon*, fourteen; *Pentedecagon*, fifteen; *Hexdecagon*, sixteen; *Heptadecagon*, seventeen; *Octadecagon*, eighteen; *Enneadecagon*, nineteen; *Icosagon*, twenty; *Icosaisagon*, twenty-one; *Icosadoagon*, twenty-two; *Icosatriagon*, twenty-three; *Icosatetragon*, twenty-four; *Icosapentegon*, twenty-five; *Icosahexagon*, twenty-six; *Icosaheptagon*, twenty-seven; *Icosaoctagon*, twenty-eight; *Icosaenneagon*, twenty-nine; *Triacontagon*, thirty; *Tricontaisagon*, thirty-one; *Tricontadoagon*, thirty-two; *Tricontatriagon*, thirty-three; and so on to *Tessaracontagon*, forty; *Pentecontagon*, fifty; *Hexacontagon*, sixty;

Hebdomacontagon, seventy; *Ogdoacontagon*, eighty; *Enenacontagon*, ninety; *Hecatonagon*, one hundred; *Diacosiagon*, two hundred; *Triacosiagon*, three hundred; *Tetracosiagon*, four hundred; *Pentecosiagon*, five hundred; *Hexacosiagon*, six hundred; *Heptacosiagon*, seven hundred; *Oktacosiagon*, eight hundred; *Enacosiagon*, nine hundred; *Chiliagon*, one thousand; &c.

35. ***A Spherical Surface*** is the boundary between a sphere and outer space.

36. ***A Conical Surface*** is the boundary between a cone and outer space.

37. ***A Cylindrical Surface*** is the boundary between the cylinder and outer space.

38. ***A Solid*** is a part of space occupied by a physical body, or marked out in any other way.

39. ***A Polyhedron*** (*Πολύεδρος*, from *Πολύς*, many, and *ἕδρα*, seat, base) is a solid bounded by polygons.

40. ***A Prism*** is a polyhedron in which two of the faces are polygons equal in all respects and having their homologous sides parallel.

41. ***The Altitude*** of a prism is the perpendicular distance between the planes of its bases.

42. ***A Triangular Prism*** is one whose bases are triangles.

43. ***A Quadrangular Prism*** is one whose bases are quadrilaterals.

44. ***A Parallelopipedon*** is a prism whose bases are parallelograms.

45. ***A Right Parallelopipedon*** is one whose lateral edges are perpendicular to the planes of the bases.

46. ***A Rectangular Parallelopipedon*** is one whose faces are all rectangles.

47. ***A Cube*** *κύβος*, a cube, a cubical die) is a rectangular parallelopipedon whose faces are squares.

48. ***A Right Prism*** is one whose lateral edges are perpendicular to the planes of the bases.

49. ***An Oblique Prism*** is one whose lateral edges are oblique to the planes of the bases.

50. ***A Pyramid*** (*Πυραμίς*) is a polyhedron bounded by a polygon called the *base*, and by triangles meeting at a common point called the *vertex* of the pyramid.

51. ***The Convex Surface*** of a pyramid is the sum of the triangles which bound it.

52. ***A Right Pyramid*** is one whose base is a regular polygon, and in which the perpendicular, drawn from the vertex to the plane of the base, passes through the center of the base. The perpendicular is called the *axis*.

53. ***A Tetrahedron*** (*τέτρα* four, and *ἕδρα*, seat, base) is a pyramid whose faces are all equilateral triangles.

54. ***The Altitude*** (Lat. *Altitudo*, from *altus*, high, and *ude* denoting state or condition) of a pyramid is the perpendicular distance from the vertex to the plane of the base.

55. ***The Slant Height*** of a pyramid, is the perpendicular distance from the vertex to any side of the base.

56. ***A Triangular Pyramid*** is one whose base is a triangle.

57. ***An Octahedron*** (*ὀκτάεδρος* from *ὀκτώ* eight, and *ἕδρα* seat, base) is a polyhedron bounded by eight equal equilateral triangles.

58. ***A Dodecahedron*** (*δώδεκα*, twelve, and *ἕδρα*, *seat*, *base*) is a polyhedron bounded by twelve equal and regular pentagons.

59. ***An Icosahedron*** (*εἴκοσι*, twenty, and *ἕδρα*, seat, base) is a polyhedron bounded by twenty equal equilateral triangles.

60. ***A Cylinder*** (*κύλινδρος*, from *κυλίνδειν*, *κυλίειν*, to roll) is a solid bounded by a surface generated by a line so moving that every two of its positions are parallel, and two parallel planes.

61. ***The Axis*** (*ἄξων*) of a cylinder is the line joining the centers of its bases.

62. ***A Right Cylinder*** is one whose axis is perpendicular to the planes of the bases.

63. ***A Cone*** (*κῶνος*, from Skr. *ço*, to bring to a point) is a solid bounded by a surface generated by a straight line moving so as always to pass through a fixed point called the *apex*, and a plane.

64. ***A Right Cone*** is a solid generated by revolving a right-angled triangle about one perpendicular.

65. ***An Oblique Cone*** is one in which the line, called the *axis*, drawn from the apex to the center of the base is not perpendicular.

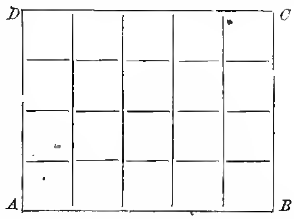
66. ***The Frustum*** (Lat. *frustum*, piece, bit) of a pyramid or a cone is the portion included between the base and a parallel section.

67. ***A Sphere*** (*σφαῖρα*) is a solid bounded by a curved surface, every point of which is equally distant from a point within, called the center.

Before we enter into the solution of problems in Mensuration, it will be necessary first to explain a difficulty which we encounter.

The common way of teaching that *feet* multiplied by *feet* give *square feet* is wrong; for there is no rule in mathematics justifying the multiplication of one denominate number by another. If it is correct to say *feet* multiplied by *feet* give *square feet*, we might, with equal propriety, say *dollars* multiplied by *dollars give square dollars*—a product wholly unintelligible. In all our reasoning, we deal with abstract numbers alone or the symbols of abstract numbers. These do not represent lines, surfaces, or solids, but the relations between these numbers may represent the relations between the magnitudes under consideration.

Suppose, for example, that the line AB contains 5 units, and the the line BC 4 units. Let a denote the abstract number 5, and b the abstract number 4. Then $ab=20$. Now this product ab is not a surface, nor the representation of a surface. It is simply the abstract number 20. But this number is exactly the same as the number of square units contained in the rectangle whose sides are AB and BC, as may be seen by constructing the rectangle $ABCD$. Hence the surface of the rectangle is measured by 20 squares described on the unit of length.



This relation is universal, and we may always pass from the abstract thus obtained by the product of any two letters, to the measure of the corresponding rectangle by simply considering the abstract units as so many concrete or denominate units.

In like manner, the product of three letters abc is not a solid obtained by multiplying lines together, which is an impossible operation. It is simply the product of three abstract numbers represented by the letters a, b, and c, and is consequently an abstract number. But this number contains precisely as many units as there are solid units in the parallelopipedon whose edges correspond to the lines a, b, and c; hence, we may easily pass from the abstract to the concrete. Hence, if we wish to find the area of a rectangle whose width is 4 feet and length 6 feet, we simply say, $6\times4=24$ square feet. We pass at once from the abstract in the first member to the concrete in the second.

It is a question whether pupils should be taught a falsehood in order that they may learn a truth.

(*See Bledsoe's Philosophy of Mathematics, pp. 97-106.*)

I. PARALLELOGRAMS.

Prob. I. To find the area of a parallelogram; whether it be a square, a rectangle, a rhomboid, or a rhombus.

Formula.— $A=l\times b$, where A=area, l=length, and b=breadth; or, $A=b\times a$, where A=area, b=base, and a=altitude.

Rule—*Multiply the length by the breadth; or, the base by the altitude.*

I. What is the area of a parallelogram whose length is 15 feet and breadth 7 feet?

By formula, $A=l\times b$=length$\times$breadth$=15\times7=105$ sq. feet.

II. 1. 15 feet=length.
2. 7 feet=breadth.
3. ∴ $15\times7=105$ sq. ft. =area.

III. ∴ The area is 105 sq. ft.

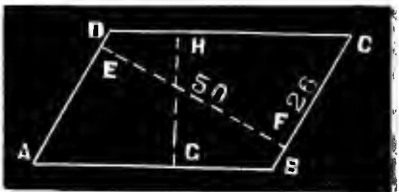


FIG. 4.

Note.—The *base* is not necessarily the side toward the ground. Thus in the parallelogram $ABCD$, BC may be considered the base, in which case, the altitude would be the perpendicular distance EF, between the sides BC and AD. If HG and BC were given, we could not find the area of the parallelogram because we have not the base and altitude given.

I. What is the area of the parallelogram $ABCD$, if BC is 26 feet and EF 50 feet?

By formula, $A=a\times b=EF\times BC=50\times26=1300$ sq. ft.

II. 1. 26 feet=BC=base.
2. 50 feet=EF=altitude.
3. ∴ $26\times50=1300$ sq. ft.=area.

III. ∴ The area of $ABCD$=1300 sq. ft.

I. A floor containing 132 square feet, is 11 feet wide; what is its length?

By formula, $A=l\times b$. ∴ $l=A\div b=132\div11=12$ ft.

II. 1. 132 sq. ft.=area.
2. 11 ft.=breadth.
3. $132\div11=12$ ft.=length.

III. ∴ The floor is 12 ft. long.

Prob. II. The diagonal of a square being given, to find the area.

Formula.—$A=d^2 \div 2$.

Rule.—*Divide the square of the diagonal by 2, and the quotient will be the area.*

I. What is the area of a square whose diagonal is 8 chains?

By formula, $A=d^2 \div 2 = 8^2 \div 2 = 32$ sq. chains.

II.
1. 8 ch.=length of diagonal= BD.
2. 64 sq. ch.=8×8=$EGFH$= square described on the diagonal BD.
3. 32 sq. ch.=64 sq. ch.÷2= area of the square $ABCD$.

III. ∴ 32 sq. ch.=the area of the square.

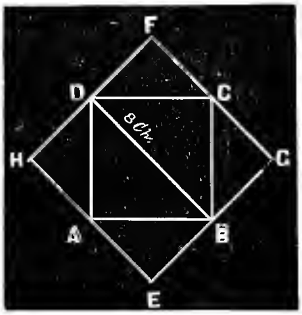


FIG. 5.

Prob. III. The area of a square being given, to find its diagonal.

Formula.—$d=\sqrt{2A}$.

Rule.—*Extract the square root of double the area.*

I. The area of a square is 578 sq. ft.; what is the diagonal?

By formula, $d=\sqrt{2A}=\sqrt{2\times578}=\sqrt{1156}=34$ feet.

II.
1. 578 sq. ft.=area of the square.
2. 1156 sq. ft.=2×578 sq. ft.=double the area.
3. 34 feet=$\sqrt{1156}$=the diagonal.

III. ∴ The diagonal is 34 feet.

Prob. IV. The diagonal of a square being given, to find its side.

Formula.—$S=\sqrt{\frac{1}{2}d^2}$.

Rule.—*Extract the square root of one-half the square of the diagonal.*

I. What is the side of a square whose diagonal is 12 feet?

By formula, $S=\sqrt{\frac{1}{2}d^2}=\sqrt{\frac{1}{2}\times12^2}=\sqrt{72}=6\sqrt{2}=8.4852+$ft.

II.
1. 12 ft.=the diagonal.
2. 144 sq. ft.=12^2=square described on the diagonal.
3. 72 sq. ft.=area of square whose side is required.
4. ∴ 8.4852 ft.=$6\sqrt{2}=\sqrt{72}$=side of the square.

III. ∴ The side of the square is 8.4852+ft.

Prob. V. To find the side of a square having its area given.

Formula.—$S=\sqrt{A}$.

Rule.—*Extract the square root of the number denoting its area.*

I. What is the side of a square field whose area is 2500 square rods?

By formula, $S=\sqrt{A}=\sqrt{2500}=50$ rods.

II. $\begin{cases} 1.\ 2500 \text{ sq. rd.} = \text{area of the field.} \\ 2.\ 50 \text{ rd.} = \sqrt{2500} = \text{side of the square field.} \end{cases}$

III. ∴ The side of the field is 50 rods.

PROBLEMS.

1. Find the area of the parallelogram *ABCD;* given *AC*=7 ft. 2 in., and the perpendicular from *B* on *AC*, 3 feet. [See Fig. 4, on p. 240.]

2. Find the area of a parallelogram in which one side is 4 ft. 3 in., and the perpendicular distance between this and the opposite side is 4 feet.

3. The area of a parallelogram is 17½ acres, and each of two parallel sides is 42 chains; find the perpendicular distance between them.

4. Find the area of a rhombus, a side of which is 10 feet and a diagonal of 12 feet. [The diagonals of a rhombus bisect each other at right angles.]

5. Find the area of a rhombus whose diagonals measure 18 feet and 24 feet.

6. A field is in the form of a rhombus, whose diagonals are 2870 links and 1850 links; find the rent of the field at $5 per acre.

7. The diagonals of a parallelogram are 34 feet and 24 feet, and one side is 25 feet; find its area.

8. Find the cost of carpeting a room, 30 feet long and 21 feet wide, with carpet 2 feet wide at 80 cents per yard.

9. How many square yards are there in a path, 4 feet wide, surrounding a lawn 24 yards long and 22 yards wide?

10. How many yards of paper, 20 inches wide, will be required to paper the walls of a room, 16 feet by 14 feet by 9 feet, allowing 8 inches for a base-board at the floor and 12 inches for border at the ceiling?

11. The perimeter of a rectangle is 56 feet; find its area, if its length is 3 times its breadth.

12. What is the area in acres of a square whose perimeter is such that it takes 12 minutes to run around the square, at the rate of 5½ miles per hour?

13. Cut a rectangular board, 16 feet long and 9 feet wide, into two pieces in such a way that they will form a square.

14. How many feet of framing, 4 inches wide, will it take to frame a picture, 3 feet by 2 feet? *Ans.* 6 ft. 4 in.

15. A sheet of galvanized iron, 50 inches wide, is placed against the top of a wall, 6 feet high, while the lower edge is 5 feet 5 inches from the foot of the wall; find the area of the sheet of iron. *Ans.* 4850 sq. in.

16. Allowing 8 shingles to the square foot, how many shingles will it take to roof a barn which is 40 feet long and 15 feet from the comb to the eaves? *Ans.* 9600 shingles.

17. The area of a square is 169 sq. ft.; find its perimeter, in chains.

18. What is the side of a square, of which the number expressing its area in square feet is equal to the number expressing its perimeter in yards? *Ans.* $1\frac{1}{3}$ feet.

19. What is the area of a path a yard wide, running diagonally across a square lawn whose side is 30 feet? *Ans.* $648[20\sqrt{2}-1]$ sq. in.

20. What is the area of a square whose diagonal is 12 feet? *Ans.* 72 sq. ft.

21. What is the area of a square whose diagonal is 5 feet longer than its side? *Ans.* $25(3+2\sqrt{3})$ sq. ft.

22. In a garden, in the form of a square, there is located a spring whose distances from the three corners, A, B, C, are 30, 40, and 50 feet, respectively. Find the length of a side of the garden. *Ans.* $5\sqrt{(81+52\sqrt{3})}$ ft.

23. Convert a square, whose sides are 20 inches, into an equivalent area bounded by circular arcs only.

Note.—The required figure is called a *pelicoid.*

II. TRIANGLES.

Prob. VI. Given the base and altitude of a right-angled triangle, to find the hypothenuse.

Formula.—$h=\sqrt{a^2+b^2}$.

Rule.—*To the square of the base add the square of the altitude and extract the square root of the sum.*

I. In the right-angled triangle ACB, the base $AC=56$ and the altitude $BC=33$; what is the hypothenuse?

By formula, $h=\sqrt{a^2+b^2}=\sqrt{33^2+56^2}=\sqrt{1089+3136}=\sqrt{4225}=65$.

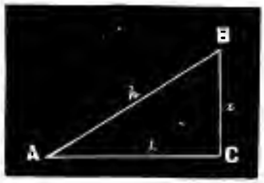


FIG. 6.

II.
1. $56=AC=$the base.
2. $3136=56^2=$ the square of the base.
3. $33=BC=$the altitude.
4. $1089=33^2=$the square of the altitude.
5. $4225=3136+1089=$the sum of the squares of the base and altitude.
6. $65=\sqrt{4225}=$the square root of the sum of the squares of the base and altitude=the hypothenuse.

III. ∴ The hypothenuse=65.

Prob. VII. To find a side, when the hypothenuse and the other side are given.

Formulas.—$\begin{cases} a=\sqrt{h^2-b^2}. \\ b=\sqrt{h^2-a^2}. \end{cases}$

Rule.—*From the square of the hypothenuse subtract the square of the given side and extract the square root of the remainder.*

I. The hypothenuse of a right-angled triangle is 109, and the altitude 60; what is the base?

By formula, $b=\sqrt{h^2-a^2}=\sqrt{109^2-60^2}=\sqrt{8281}=91$.

II.
1. 109=hypothenuse.
2. 11881=109^2=square of the hypothenuse.
3. 60=the altitude.
4. 3600=60^2=the square of the altitude.
5. 8281=11881—3600=difference of the squares of the hypothenuse and altitude.
6. 91=$\sqrt{8281}$=the square root of this difference=the base.

III. ∴ The base is 91.

Remark.—When $a=b$, $h=\sqrt{2a^2}=a\sqrt{2}$. From this, we see that the diagonal of a square is $\sqrt{2}$ times its side.

Prob. VIII. To find the area of a triangle, having given the base and the altitude.

Formula.—$A=\frac{1}{2}a\times b$.

Rule.—*Multiply the base by the altitude and take half the product.*

I. What is the area of a triangle whose base is 24 feet and altitude 16 feet?

By formula, $A=\frac{1}{2}a\times b=\frac{1}{2}\times16\times24=192$ sq. ft.

II.
1. 24 ft.=base.
2. 16 ft.=altitude.
3. 384 sq. ft.=16 × 24 = product of base and altitude.
4. 192 sq. ft.=$\frac{1}{2}$ of 384 sq. ft.= half the product of the base and the altitude=area.

III. ∴ The area of the triangle is 192 sq. ft.

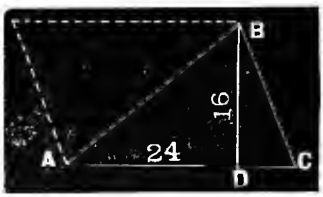


FIG. 7.

Prob. IX. To find the area of a triangle, having given its three sides.

Formula.—$A=\sqrt{s(s-a)(s-b)(s-c)}$, where $s=\frac{1}{2}(a+b+c)$.

***Rule.**—*Add the three sides together and take half the sum; from the half sum, subtract each side separately; multiply the half sum and the three remainders together and extract the square root of the product.*

**Demonstration.*—In Fig. 7, let $AC=b$, $BC=a$, and $AB=c$. In the right-angled triangle ADB, $BD^2=AB^2-AD^2$, and in the right-angled triangle CDB, $BD^2=BC^2-DC^2$. ∴ $AB^2-AD^2=BC^2-DC^2$, or c^2-

I. What is the area of a triangle whose sides are 13, 14, and 15, feet respectively?

By formula, $A=\sqrt{s(s-a)(s-b)(s-c)}=\sqrt{21\times(21-13)\times(21-14)\times(21-15)}=\sqrt{21\times8\times7\times6}=\sqrt{7056}=84$ sq. ft.

II.
1. 42 ft.=13 ft.+14 ft.+15 ft.=sum of the three sides.
2. 21 ft.=½ of 42 ft.=half the sum of the three sides.
3. 21 ft.—13 ft.=8 ft.=first remainder.
4. 21 ft.—14 ft.=7 ft.=second remainder.
5. 21 ft.—15 ft.=6 ft.=third remainder.
6. 7056=21×6×7×8=product of half sum and three remainders
7. 84 sq. ft.= $\sqrt{7056}$=square root of the product of the half sum and three remainders=the area of the triangle.

III. ∴ The area of the triangle is 84 sq. ft.

Prob. X. To find the radius of a circle inscribed in a triangle.

Formula.—$R=2A\div(a+b+c)$.

***Rule.**—*Divide twice the area of the triangle by the sum of the three sides.*

I. Find the radius of a circle inscribed in a triangle whose sides are 3, 4, and 5 feet, respectively.

II.
1. 6 sq. ft.= $\sqrt{s(s-a)(s-b)(s-c)}$=area of the triangle, by formula, Prob. IX.
2. 12 sq. ft.=twice the area of the triangle.
3. 12 ft.=3 ft.+4 ft.+5 ft.=sum of the three sides.
4. ∴ 1 ft.=12÷12=twice the area divided by the sum of the sides=the radius of the inscribed circle.

III. ∴ The radius of the inscribed circle is 1 ft.

$AD^2=a^2-DC^2$, whence $c^2-a^2=AD^2-DC^2$. But $AD^2-DC^2=(AD+DC)(AD-DC)=b(AD-DC)$. ∴ $b(AD-DC)=c^2-a^2$, and $AD-DC=(c^2-a^2)\div b$. But $AD+DC=b$. ∴ By adding the last two equations, we have $2AD=\frac{c^2-a^2}{b}+b=\frac{c^2-a^2+b^2}{b}$; whence $AD=\frac{c^2-a^2+b^2}{2b}$. Since $BD^2=AB^2-AD^2=c^2-AD^2$, if we substitute the value of AD just found, we have $BD^2=c^2-\left(\frac{c^2-a^2+b^2}{2b}\right)^2=\frac{4b^2c^2-(c^2-a^2+b^2)^2}{4b^2}=$
$\frac{(2bc+c^2-a^2+b^2)(2bc-c^2+a^2-b^2)}{4b^2}=\frac{(b^2+2bc+c^2-a^2)[a^2-(b^2-2bc+c^2)]}{4b^2}=$
$\frac{[(b+c)^2-a^2][a^2-(b-c)^2]}{4b^2}$.

∴ $BD=\frac{1}{2b}\sqrt{[(b+c)^2-a^2][a^2-(b-c)^2]},=\frac{2}{b}\sqrt{s(s-a)(s-b)(s-c)}$. Now the area of $ABC=\frac{1}{2}AC\times BD$.

∴ $A=\frac{1}{2}b\times BD=\frac{1}{2}b\times\frac{1}{2b}\sqrt{[(b+c)^2-a^2][a^2-(b-c)^2]}$

$=\frac{1}{4}\sqrt{[(b+c)^2-a^2][a^2-(b-c)^2]}=\frac{1}{4}\sqrt{(b+c+a)(b+c-a)(a+b-c)(a-b+c)}$.

$=\sqrt{s(s-a)(s-b)(s-c)}$, where $2s=(a+b+c)$. *Q. E. D.*

**Note.*—For Demonstration, see any geometry.

Prob. XI. To find the radius of a circle, circumscribed about a triangle whose sides are given.

Formula.—$R=\dfrac{abc}{4A}=\dfrac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$.

***Rule.**—*Divide the product of the three sides by four times the area of the triangle.*

I. What is the radius of a circle circumscribed about a triangle whose sides are 13, 14, and 15 feet, respectively?

II.
1. 2730 cu. ft.$=13\times14\times15=$the product of the three sides.
2. 84 sq. ft.$=\sqrt{s(s-a)(s-b)(s-c)}=$the area of the triangle, by Prob. IX.
3. 336 sq. ft.$=4\times84$ sq. ft.=four times the area of the triangle.
4. $8\frac{1}{8}$ ft.$=2730\div336=$the product of the three sides divided by four times the area of the triangle=the radius of the circumscribed circle

III. ∴ The radius of the circumscribed circle is $8\frac{1}{8}$ ft.

Prob. XII. To find the area of an equilateral triangle, having given the side.

Formula.—$A=\frac{1}{4}\sqrt{3}s^2$, where $s=$side. This is what Prob. IX. becomes, when $a=b=c$.

Rule.—*Multiply the square of a side by* $\frac{1}{4}\sqrt{3}$,=.433013+.

I. What is the area of an equilateral triangle whose sides are 20 feet?

**Demonstration.*—Let ABC be any triangle, and $ABCE$ the circumscribed circle. Draw the diameter BE, and draw EC. Draw the altitude BD of the triangle ABC. The triangles ADB and BCE are similar, because both are right-angled triangles, and the angle $BAD=$the angle BEC. Hence, $AB:EB::BD:BC$. Hence, $AB\times BC=BE\times BD$ or $ac=2R\times BD$. But, in the demonstration of Prob. IX., we found $BD=\frac{2}{b}\sqrt{s(s-a)(s-b)(s-c)}$.

∴ $ac=2R\times\frac{2}{b}\sqrt{s(s-a)(s-b)(s-c)}$. Whence

$R=\dfrac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}=\dfrac{abc}{4A}$.

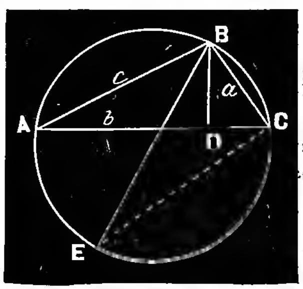


FIG. 8.

By formula, $A=\frac{1}{4}\sqrt{3}\times20^2=100\sqrt{3}=173.205+$sq. ft.

II.
1. 20 ft.=length of a side.
2. 400 sq. ft.$=20^2$=square of a side.
3. 173.205 sq. ft.$=\frac{1}{4}\sqrt{3}\times400=.433013\times400=\frac{1}{4}\sqrt{3}$ times the square of a side,=the area of the triangle.

III. ∴ The area of the equilateral triangle is 173.205+sq. ft.

Prob. XIII. The area and base of a triangle being given, to cut off a triangle containing a given area, by a line running parallel to one of its sides.

Formula.—$b'=b\sqrt{\frac{A'}{A}}$, where A=area of the given triangle; b, the base of the given triangle; and A', the area of the portion to be cut off.

Rule.—*As the area of the given triangle is to the area of the triangle to be cut off, so is the square of the given base to the square of the required base. The square root of the result will be the base of the required triangle.*

I. The area of the triangle ABC is 250 square chains and the base AB, 20 chains; what is the base of the triangle, area equal to 60 sq. chains, cut off by ED parallel to BC?

By formula, $AD=b'=b\sqrt{\frac{A'}{A}}=20\sqrt{\frac{60}{250}}=4\sqrt{6}=9.7979+$ ch.

II.
1. 250 sq. ch.=area of the given triangle ABC.
2. 60 sq. ch.=area of the triangle AED.
3. 20 ch.=base of the triangle ABC.
4. ∴ 250 sq. ch. : 60 sq. ch. :: $20^2 : AD^2$. Whence
5. $AD^2=(400\times60)\div250=96$.
6. ∴ $AD=\sqrt{96}=9.7979+$ch.

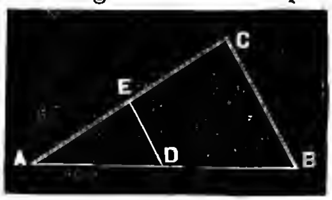


FIG. 9.

III. ∴ The base $AD=9.7979+$ch.

PROBLEMS.

1. A man travels 20 miles north, then 15 miles due east, finally 28 miles due south; what is the distance from his starting point? *Ans.* 17 mi.

2. A ladder, 50 feet long, is placed so as to reach a window 48 feet high, and on turning the ladder over to the other side of the street, it reaches a point 14 feet high. Find the breadth of the street.

3. The hypotenuse of a right-angled triangle is 55 feet and the base is ¾ of the altitude. Find the two sides.

4. The hypotenuse of a right-angled triangle is 13 feet and the sum of the sides containing the right angle is 17 feet. Find these sides.

5. In a right-angled triangle the area is half an acre, and one of the sides containing the right angle is 44 yards; find the other side in yards.

6. Find the area of a triangle whose sides are 21 feet, 20 feet, and 13 feet, respectively. Also 21 feet, 17 feet, and 10 feet.

7. In a right-angled triangle the sides containing the right angle are 30 feet and 40 feet. Find the length of a perpendicular drawn from the right angle to the hypotenuse.

8. The perimeter of a triangle is 48 feet. If one side is 10 feet and the area is 84 square feet, find the two remaining sides.

9. The area of an equilateral triangle is 30 square feet. Find the length of a side. What is the side of a square of equal area?

10. The sides of a triangle are proportional to 3, 4, and 5. If the perimeter is 84 feet, find the sides and the area.

III. TRAPEZOIDS.

Prob. XIV. To find the area of a trapezoid, having given the parallel sides and the altitude.

Formula.—$A=\frac{1}{2}(b+b')a$, where b and b' are the parallel sides and a, the altitude.

Rule.—*Multiply half the sum of the parallel sides by the altitude.*

I. What is the area of a trapezoid whose parallel sides are 15 meters and 7 meters and altitude 6 meters?

By formula, $A=\frac{1}{2}(b+b')\times a=\frac{1}{2}(15+7)\times 6=66$ **m²**.

II.
1. 7 m.$=DC$, the length of one of the parallel sides, and
2. 15 m.$=AB$, the length of the other side.
3. 22 m.$=$7 m.$+$15 m.$=$sum of the parallel sides.
4. 11 m.$=\frac{1}{2}$ of 22 m.$=$half the sum of the parallel sides.
5. 66 m².$=6\times 11=$area of the trapezoid, $ABCD$.

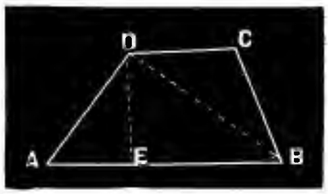


FIG. 10.

III. ∴ The area of the trapezoid is 66 m².

PROBLEMS.

1. The parallel sides of a trapezoid are 18 feet and 24 feet, and the altitude is 8 feet; find the area.

2. The parallel sides of an isosceles trapezoid are 16 feet and 20 feet, and the non-parallel sides are 10 feet each; find the area of the trapezoid.

3. The line joining the middle points of the non-parallel sides of a trapezoid is 12 feet, and the altitude is 8 feet; find the area of the trapezoid. *Ans.* 96 sq. ft.

IV. TRAPEZIUM AND IRREGULAR POLYGONS.

Prob. XV. To find the area of a trapezium or any irregular polygon.

Rule.—*Divide the figure into triangles, find the area of the triangles and take their sum.*

I. What is the area of the trapezium $ABCD$, whose diagonal AC is 84 feet, and the perpendiculars DE and BF, 56 and 22 feet, respectively?

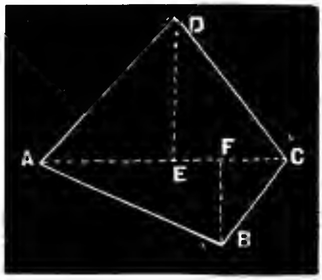


FIG. 11.

II.
1. 84 ft.$=AC=$base of the triangle ADC.
2. 56 ft.$=DE=$ altitude of ADC.
3. ∴ 2352 sq. ft.$=\frac{1}{2}(AC\times DE)$=area of the triangle ADC.
4. 84 ft.$=AC=$base of the triangle ABC.
5. 22 ft.$=BF=$ altitude of ABC.
6. ∴ 924 sq.ft.$=\frac{1}{2}(AC\times BF)$ =area of the triangle ABC.
7. 3276 sq. ft.=2352 sq. ft.+924 sq. ft.$=ADC+ABC=$ area of the trapezium $ABCD$.

III. ∴ The area of the trapezium $ABCD$ is 3276 sq. ft.

I. What is the area of the quadrilateral, $ABCD$, if AB is 30 feet, BC 17 feet, CD 25 feet, DA 30 feet, and the diagonal BD 26 feet?

By formula for the area of a triangle, $A=\sqrt{s(s-a)(s-b)(s-c)}+\sqrt{s(s-a')(s-b')(s-c')}$ where a, b, c and a', b', c', are the sides of the triangles ABD and BCD, respectively.

II.
1. 30 feet$=AB=a$,
2. 26 feet$=BD=b$, and
3. 28 feet$=DA=c$.
4. ∴ 42 feet$=\frac{1}{2}$(30 feet+26 feet+28 feet)$=\frac{1}{2}(a+b+c)=s$.
5. ∴ 336 sq. feet$=\sqrt{42\times12\times14\times16}$=the area of the triangle ABD, and
6. 204 sq. feet$=\sqrt{34\times8\times9\times17}$=the area of the triangle BDC.
7. ∴ 336 sq. feet+204 sq. ft.=540 sq. ft.=the area of $ABCD$.

III. ∴ The area of $ABCD$=540 sq. feet.

***Corollary* 1.** If a quadrilateral is such that its diagonals

AC and BD are at right angles to each other the area is given by the formula

$$area=\frac{1}{2}(AC\times BD).$$

Corollary 2. If the diagonals are inclined to each other at any angle a degrees, the area is expressed by the formula,

$$area=\frac{1}{2}(AC\times BD\times \sin a).$$

PROBLEMS.

1. In the trapezium $ABCD$, $AB=30$ in., $BC=17$ in., $CD=25$ in., $DA=28$ in., and the diagonal $BD=26$ in.; find its area. *Ans.* 540 sq. in.

2. In the quadrilateral $ABCD$, the diagonal $AC=18$ in., and the perpendicular on it from B and D are 11 inches and 9 inches respectively; find the area of the trapezoid. *Ans.* 180 sq. in.

3. In the trapezium $ABCD$, the diagonals AC and BD are perpendicular to each other and measure 16 feet and $2\frac{1}{4}$ feet respectively; find the area. *Ans.* 2 sq. yds.

4. Find the area of the trapezium $ABCD$, in which the angles ABC and CDA are right angles, and AB is 15 feet, BC is 20 feet, and CD is 7 feet. *Ans.* 234 sq. ft.

5. Find the area of the quadrilateral $ABCD$, having given that the angle ABC is 60°, ADC is a right angle, $AB=13$ chains, $BC=13$ chains, and $CD=12$ chains. *Ans.* 10.31 acres.

6. The area of a trapezium is 4 acres and the two diagonals measure 16 chains and 10 chains respectively; at what angle are the two diagonals inclined to each other? *Ans.* 30°.

Hint.—Let I be the intersection of the diagonals. Then ⊥ from B on $AC=BI$ sin. $\angle BIC$, ⊥ from D on $AC=DI$ sin. $\angle DIA(=\angle CIB.)$
∴ area of $ABCD=\frac{1}{2}AC[BI \sin. BIC+DI \sin. CIB]=\frac{1}{2}AC\times BD\times \sin. \angle BIC$.

7. Find the area of the polygon $ABCDEF$, if $AD=1675$ links, ⊥ FP from F on $AD=850$ links, ⊥ BQ from B on $AD=200$ links, ⊥ CS from C on on $AD=500$ links, ⊥ ER from E on $AD=250$ links, $AP=900$ links, $AQ=1040$ links, $AR=1200$ links, and $AS=1380$ links. *Ans.* 9.03625 acres.

8. Find the area of the field $ABCDEF$, if $AC=2900$ links, $CE=2500$ links, $EA=3600$ links, ⊥ BX from B on $AC=400$ links, ⊥ DY from D on $CE=400$ links, and ⊥ FZ from F on $AE=950$ links. *Ans.* 63 A. 3 r. 24 p.

9. Find the area of the polygon $ABCDE$, if $AB=12$ inches, $\angle ABC$ a right angle, $BC=5$ in., $CD=14$ in., $AD=15$ in., $\angle ADE$ a right angle, and $DE=8$ in. *Ans.* 1 sq. ft. 30 sq. in.

10. What is the area of a figure whose sides are 10, 12, 14, and 16 rd. in order, and the distance from starting point to opposite corner is 18 rd.? *Ans.* 1 A. 3.9— sq. rd.

11. What is the area of a figure made up of three triangles whose bases are 10, 12, and 16 rd., and whose altitudes are 9, 15, and $10\frac{1}{2}$ rd.? *Ans.* 1 A. 59 sq. rd.

V. REGULAR POLYGONS.

Prob. XVI. To find the area of a regular polygon.

Formula.—$A=\frac{1}{2}a\times p$, where p is the perimeter and a, the apothem.

Rule.—*Multiply the perimeter by half the apothem.*

The *Perimeter* of any polygon is the sum of all its sides.

The *Apothem* is the perpendicular drawn from the center to any side of the polygon.

I. What is the area of a regular heptagon whose side is 19.38 and apothem 20?

By formula, $A=\frac{1}{2}a\times p=\frac{1}{2}\times20\times(7\times19.38)=1356.6$.

II.
1. 19.38=length of one side.
2. 135.66=length of 7 sides=the perimeter.
3. 20=apothem.
4. $10=\frac{1}{2}$ of 20=half the apothem.
5. $1356.6=10\times135.66$=product of perimeter by half the apothem.

III. ∴ The area of the heptagon is 1356.6.

Prob. XVII. To find the area of a regular polygon, when the side only is given.

***Rule**.—*Multiply the square of the side of the polygon by the number standing opposite to its name in the following table of areas of regular polygons whose side is 1:*

Name.	Sides.	Multipliers.	
Triangle,	3	$\frac{1}{4}\sqrt{3}$	= .4330127.
Tetragon, or square,	4	1	= 1.0000000.
Pentagon,	5	$\frac{5}{4}\sqrt{1+\frac{2}{5}\sqrt{5}}$	= 1.7204774.
Hexagon,	6	$\frac{3}{2}\sqrt{3}$	= 2.5980762.
Heptagon,	7	$\frac{7}{4}\cot.\frac{180°}{7}$	= 3.6339124.
Octagon,	8	$2+2\sqrt{2}$	= 4.8284271.
Nonagon,	9	$\frac{9}{4}\cot. 20°$	= 6.1818242.
Decagon,	10	$\frac{5}{2}\sqrt{5+2\sqrt{5}}$	= 7.6942088.
Undecagon,	11	$\frac{11}{4}\cot.\frac{180°}{11}$	= 9.3656399.
Dodecagon,	12	$3(2+\sqrt{3})$	=11.1961524.

**Demonstration.*—Since a regular polygon can be divided into as many equal isosceles triangles as it has sides, we may find the area of one triangle and multiply this area by the number of triangles, for the whole area. Let ABC be one of these isosceles triangles, taken from a polygon of n sides, AB and BC the equal sides, and AC the base. The angle at the vertex $B=360°\div n$. $A=\frac{1}{2}(180°-360°\div n)=C$. From B let fall a perpendicular on AC at D. Then by trigonometry, $\frac{BD}{\frac{1}{2}AC}=\tan\left(90°-\frac{180°}{n}\right)$. $\therefore BD=\frac{1}{2}AC\cot\left(\frac{180°}{n}\right)$. The area of the triangle $ABC=\frac{1}{2}AC\times BD=\frac{1}{4}AC^2\cot\left(\frac{180°}{n}\right)$. ∴ The area of the polygon$=\frac{n}{4}AC^2\cot\left(\frac{180°}{n}\right)=\frac{n}{4}s^2\cot\left(\frac{180°}{n}\right)$ where s=side. By placing $s=1$, and $n=13$, 14, 15, &c., respectively, the area of polygons of 13, 14, 15, &c., side respectively, may be found.

Prob. XVIII. To find the side of an inscribed square of a triangle, having given the base and the altitude.

Formula.—$s=\frac{ab}{a+b}$, where s=side, b the base, and a the altitude.

***Rule.**—*Divide the product of the base and altitude by their sum.*

I. What is the side of an inscribed square of a triangle whose base is 14 feet and altitude 8 feet?

By formula, $s=\frac{ab}{a+b}=\frac{14\times8}{14+8}=5\frac{1}{11}$ feet.

II.
1. 8 feet=the altitude.
2. 14 feet=the base.
3. 112 sq. ft.=14×8=the product of the base and altitude.
4. 22 feet=14 ft.+8 ft.=their sum.
5. $5\frac{1}{11}$ feet=112÷22=the product divided by the sum.

III. ∴ $5\frac{1}{11}$ ft.=the side of the inscribed square.

PROBLEMS.

1. Within a given regular hexagon, drawn on a side of 10 inches, a second hexagon is inscribed by joining the middle points of the sides taken in order. Find the area of the inscribed figure. *Ans.* 194.85 sq. in.

2. Find the area of a regular pentagon on a side of 10 inches. *Ans.* 172.04 sq. in.

**Demonstration.*—Let ABC be any triangle whose base is b and altitude a. Produce AC to H, making $CH=BD$. At H, erect the perpendicular HG and make $HG=BD$. Draw AG and at C, erect the perpendicular FC, and draw FK. Then $KE=FC=EN$, and KN is the required inscribed square. For, in the similar triangles AHG and ACF, we have $AH:GH::AC:FC$, or $a+b:a::b:FC$. By inversion, and then by Division, $a:b::a-FC:FC$, or $BI:FC$. In the similar triangles ABC and KBE, $AC:KE::BD:BI$, or $BD:AC::BI:KE$. Whence $a:b::BI:KE$. $\therefore BI:KE::BI:FC$. $\therefore KE=FC$ and the figure KN has its sides equal and its angles right angles by construction. Hence, it is a square. *Q.E.D.*

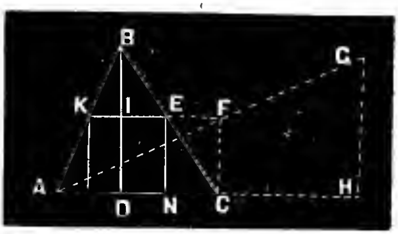


FIG. 12.

A more elegant construction, is to construct a square on the altitude, BD, and having one side adjacent to BD, lying in DH. Then join the vertex, A, of the triangle and the vertex of the square opposite the vertex, D. This line will intersect the side, BC, in E, a vertex of the required square.

In the same way, we can inscribe a rectangle similar to a given rectangle.

3. Find the area of a regular decagon on a side of 4 inches.
Ans. 123.1 sq. in.

4. Find the area of a regular heptagon inscribed in a circle, radius 6 inches. [Area of a regular n-side in terms of the radius is $\frac{n}{2}$ sin. $\frac{360°}{n} R^2$.]

5. Find the area of a regular heptagon circumscribing a circle whose radius is 12 inches. [Area of a regular n-side circumscribing a circle in terms of the radius is n tan. $\frac{180°}{n}(r)^2$].

6. A regular octagon is formed by cutting off the corners of a square whose side is 12 inches. Find the side of the octagon. [Side of octagoa $=\frac{1}{2}a(2-\sqrt{2})$, where a is the side of the square].

7. The area of a dodecagon is 300 square inches; find the radius of the circle circumscribed about it.

8. Find the area of the circular ring formed by the inscribed and circumscribed circles of a regular hexagon whose side is 20 inches. Show that for a given length of side, the area of the ring is the same whatever the number of sides of the regular polygons.

9. What is the area of a path 3 feet wide, around a hexagonal enclosure whose side is 14 feet? *Ans.* 283.17 sq. ft.

10. Find the area of the square formed by joining the middle points of the alternate sides of a regular octagon, whose side is 8 inches.
Ans. 186.51 sq. in.

11. The difference between the area of a regular octagon and a square inscribed in the same circle is 82.8 square inches. Find the radius of the circle. [Take $\sqrt{2}=1.414$]. *Ans.* 10 inches.

12. In a circle of a radius 10 inches a regular hexagon is described; in this hexagon a circle is inscribed; in this circle a regular hexagon is inscribed: and so *ad infinitum.* Find the sum of the areas of all the hexagons thus formed. *Ans.* 1039.23 sq. in.

13. In the last example, let the radius be r and the number of sides of the polygon n; find the sum of the areas of all the circles formed.
Ans. πr^2 cosec.$^2 \frac{180°}{n}$.

14. In a triangle whose base is 15 inches and altitude 10 inches a square is inscribed. Find its area.

15. A square is formed by joining the mid points of the sides of a square whose side is 10 feet; in the square thus formed, another is formed in the same way, and so on; find the side of the 7th square thus formed. *Ans.* $1\frac{1}{4}$ ft.

VI. CIRCLE.

Prob. XIX. To find the diameter of a circle, having given the height of an arc and a chord of half the arc.

Formula.—$D=k^2 \div a$, in which k=chord of half the arc and a=height.

†**Rule.**—*Divide the square of the chord of half the arc by the height of the chord.*

†*Demonstration.*— Let $AB=k$, the chord of half the arc ABC, and $BD=a$, the height of the arc ABC. Draw the diameter BE and draw the

I. What is the diameter of a circle of which the height of an arc is 5 m. and the chord of half the arc 10 m.?

By formula, $D=k^2\div a=10^2\div 5=$ 20 m.

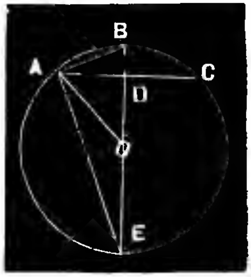


FIG. 13.

II.
1. 10 m.=AB, the length of chord of half the arc.
2. 5 m.=BD, the height of arc.
3. 100 m.²=square of chord.
4. ∴ 20 m.=100÷5=BE, the diameter of the circle.

III. ∴ The diameter of the circle is 20 meters.

Prob. XX. To find the height of an arc, having given the chord of the arc and the radius of the circle.

Formula.—$a=R-\sqrt{R^2-c^2}$, in which, R=radius and $c=\frac{1}{2}$ the chord.

***Rule.**—*From the radius, subtract the square root of the difference of the squares of the radius and half the chord.*

I. The chord of an arc is 12 feet and the radius of the circle is 10 feet. Find the height of the arc.

By formula, $a=R-\sqrt{R^2-c^2}=10-\sqrt{10^2-6^2}=2$ ft.

II.
1. 10 ft.=the radius of the circle.
2. 100 sq. ft.=square of the radius.
3. 12 ft.=the chord.
4. 6 ft.=half the chord.
5. 36 sq. ft.=square of half the chord.
6. 8 ft.=$\sqrt{100-36}$=square root of the difference of the squares of the radius and half the chord.
7. ∴ 10 ft.—8 ft.=2 ft.=height of the arc.

III. ∴ The height of the chord is 2 feet.

Prob. XXI. To find the chord of half the arc, having given the chord and height of an arc.

Formula.—$k=\sqrt{a^2+c^2}$.

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radius AO. The triangles ADB and BAE are similar, because their angles are equal. Hence, $BE:AB::AB:BD$, or $BE:k::k:a$. Whence $BE=D=k^2\div a$. *Q.E.D.*

N. B.—(1) If a and D are given, $k=\sqrt{D\times a}$; (2) if D and k are given $a=k^2\div D$.

**Demonstration.*—In Fig. 13, we have $BD=BO-DO$. But $DO=\sqrt{AO^2-DA^2}=\sqrt{[R^2-c^2]}$. ∴ $a=R-\sqrt{R^2-c^2}$. If a and R are given, (1) $2c=2\sqrt{[R^2-(R-a)^2]}=2\sqrt{(2aR-a^2)}$; if a and $2c$ are given, (2) $R=(a^2+c^2)\div 2a$.

***Rule.**—*Take the square root of the sum of the squares of the height of arc and half the chord.*

I. Given the chord=48, the height=10, find the chord of half the arc.

By formula, $k=\sqrt{a^2+c^2}=\sqrt{10^2+24^2}=\sqrt{676}=26$.

II.
1. 48=the chord.
2. 576=$\frac{1}{4}$ of 48^2=square of half the chord.
3. 10=height of chord.
4. 100=square of height of chord.
5. 676=576+100=sum of square of half of chord and height.
6. 26=$\sqrt{676}$=square root of sum of square of half of chord and height.

III. ∴ The chord of half the arc is 26.

Prob. XXII. To find the chord of half an arc, having given the chord of the arc and the radius of the circle.

Formula.—$k=\sqrt{2R^2-R\sqrt{4R^2-4c^2}}$.

†**Rule.**—*Multiply the radius by the square root of the difference of the squares of twice the radius and the chord; subtract this product from twice the square of the radius and extract the square root of the difference.*

I. Given the chord of an arc=6 and the radius of the circle =5, find the chord of half the arc.

By formula, $k=\sqrt{2R^2-R\sqrt{4R^2-4c^2}}=\sqrt{2\times5^2-5\sqrt{4\times5^2-6^2}}$ $=\sqrt{10}$.

II.
1. 5=the radius of the circle.
2. 10=twice the radius of the circle.
3. 100=square of twice the radius.
4. 6=chord of the arc.
5. 36=square of the chord.
6. 100—36=64=difference of squares of twice the radius and the chord.
7. 8=$\sqrt{64}$=square root of the above difference.
8. 40=5×8=the product of the above square root and the radius.
9. 50=2×5^2=twice the square of the radius.
10. $\sqrt{50-40}=\sqrt{10}$=chord of half the arc.

III. ∴ The chord of half the arc is $\sqrt{10}$.

**Demonstration.*—In Fig. 13, we have $AB=\sqrt{(AD^2+BD^2)}=\sqrt{[c^2+a^2]}=\sqrt{a^2+c^2}$. ∴$k=\sqrt{a^2+c^2}$. If k and $2c$ are given, (1) $a=\sqrt{k^2-c^2}$; if k and a are given, (2) $2c=2\sqrt{k^2-a^2}$.

†*Demonstration.*—From Prob. XXI., we have $k=\sqrt{a^2+c^2}$. From Prob. XX. we have $a=R-\sqrt{R^2-c^2}$. ∴ $a^2=2R^2-c^2-2R\sqrt{R^2-c^2}$. Substituting this value of a^2 in the above equation, $k=\sqrt{2R^2-R\sqrt{4R^2-4c^2}}$.

Remark.— A continued application of the formula of Problem XXII, enables us to find the ratio of the circumference of a circle to the diameter. Thus, let $R=1$ and $C_1=1$, the side of a regular inscribed hexagon.

No. of sides.	Length of Side.		Length of Perimeter.
6	$C_1=1$	$=1.00000000$	$p_1=6.00000000$
12	$C_2=\sqrt{2-\sqrt{4-C_1^2}}=(2-\sqrt{3})^{\frac{1}{2}}$	$=0.51763809$	$p_2=6.21165708$
24	$C_3=\sqrt{2-\sqrt{4-C_2^2}}=(2-(2+\sqrt{3})^{\frac{1}{2}})^{\frac{1}{2}}$	$=0.26105238$	$p_3=6.26525722$
48	$C_4=\sqrt{2-\sqrt{4-C_6^2}}=(2-(2+(2+\sqrt{3})^{\frac{1}{2}})^{\frac{1}{2}})^{\frac{1}{2}}$	$=0.13080626$	$p_4=6.27870041$
96	$C_5=\sqrt{2-\sqrt{4-C_4^2}}=(2-(2+(2+(2+\sqrt{3})^{\frac{1}{2}})^{\frac{1}{2}})^{\frac{1}{2}})^{\frac{1}{2}}$	$=0.06533817$	$p_5=6.28206396$
192	$C_6=\sqrt{2-\sqrt{4-C_5^2}}=(2-(2+(2+(2+(2+\sqrt{3})^{\frac{1}{2}})^{\frac{1}{2}})^{\frac{1}{2}})^{\frac{1}{2}})^{\frac{1}{2}}$	$=0.03272346$	$p_6=6.28290510$
384	$C_7=\sqrt{2-\sqrt{4-C_6^2}}=(2-(2+(2+(2+(2+(2+\sqrt{3})^{\frac{1}{2}})^{\frac{1}{2}})^{\frac{1}{2}})^{\frac{1}{2}})^{\frac{1}{2}})^{\frac{1}{2}}$	$=0.01636228$	$p_7=6.28311544$
768	$C_6=\sqrt{2-\sqrt{4-C_7^2}}=(2-(2+(2+(2+(2+(2+(2+\sqrt{3})^{\frac{1}{2}})^{\frac{1}{2}})^{\frac{1}{2}})^{\frac{1}{2}})^{\frac{1}{2}})^{\frac{1}{2}})^{\frac{1}{2}}$	$=0.00818121$	$p_8=6.28316941$

As the number of sides are thus continually increased, the perimeter of the polygons continually approach the circumference of the circle and can be made to differ from the circumference as little as we please, though it can never be made to exactly equal the circumference.

Hence, we may consider 6.28317 as approximately the circumference of a circle whose radius is unity.

Since $2\pi R=C$, the circumference; therefore, $\pi=\frac{1}{2}(6.28317)=3.14159$, nearly.

Students should learn to use the character, π, instead of 3.14159; for much useless labor can be saved thereby. Thus, suppose we wished to find the *mean base* of a frustum of a cone, the radii of whose upper and lower bases are 2 feet and 3 feet, respectively. By the rule, we multiply the area of the upper and lower bases together and extract the square root of the product. Now the area of the upper base is 4π sq. ft. and the area of the lower base is 9π sq. ft. Hence, the area of the mean base is $\sqrt{36\pi^2}$ sq. ft. or 6π sq. ft. By using π instead of 3.14159, we have avoided the useless labor of multiplying two numbers, consisting of five decimal places each. While this, of course, could have been avoided by simply indicating the multiplication, yet students most often perform the actual multiplication; and when the multiplication is only indicated, it is much easier and more quickly done to write one character, π, than to write the five or six characters in 3.14159.

Note.—Ahmes, an Egyptian priest who lived somewhere between the years 1700 B. C. and 1100 B. C., gives in a papyrus manuscript, forming part of the Rhind collection, the following rule for finding the area of a circle: *Cut off $\frac{1}{9}$ of the diameter and on the remainder construct a square; the area of the square thus constructed is equal to the area of the circle.*

According to this rule, if d is the diameter, $\frac{64}{81}d^2$ is the area of the circle. Since $\frac{1}{4}\pi d^2=$the area; therefore, $\frac{1}{4}\pi d^2=\frac{64}{81}d^2$, $\pi=\frac{256}{81}$, or $(\frac{16}{9})^2=3.16+$, a fair approximation for that period in the history of mathematics.

A less accurate value of π is given in I Kings, 7: 23, and II Chronicles, 4: 2, where it is stated that the circumference of a circle is 3 times its diameter.

The manuscript of Ahmes, referred to above, is called "directions for knowing all dark things," and consists of a collection of problems in arithmetic and geometry. It is believed that this manuscript is itself a copy, with emendations, of an older treatise written about 3400 B. C.

Prob. XXIII. To find the side of a circumscribed polygon, having given the radius of the circle and a side of a similar inscribed polygon.

Formula.—$K' = \frac{2KR}{\sqrt{4R^2 - K^2}}$, in which K' is the side of the circumscribed polygon and K the side of a similar inscribed polygon.

Rule.—*Divide twice the product of the side of the inscribed polygon and radius by the square root of the difference of the squares of twice the radius and the side of the inscribed polygon.*

I. When $R=1$, find one side of a regular circumscribed dodecagon.

By formula, $K' = \frac{2KR}{\sqrt{4R^2 - K^2}} = \frac{2K}{\sqrt{4 - K^2}}$. The formula does not lead to a direct result, since K is not given. But by the formula of Prob. XXI., if k is replaced by K we have $K = \sqrt{2 - \sqrt{4-1}}$ for $2c=1$, since it is the side of a regular inscribed hexagon, and $K = \sqrt{2 - \sqrt{3}}$, since $2c$ is a side of a regular inscribed dodecagon.

$$\therefore K' = \frac{2K}{\sqrt{4-K^2}} = \frac{2\sqrt{2-\sqrt{3}}}{\sqrt{4-2+\sqrt{3}}} = 2\frac{\sqrt{2-\sqrt{3}}}{\sqrt{2+\sqrt{3}}} = .535898.$$

PROBLEMS.

1. The driving-wheel of a locomotive engine 6 feet 3 inches in diameter, makes 110 revolutions a minute; find the rate at which it is traveling. *Ans.* 24.54 miles per hour.

2. If the driving-wheel of a bicycle makes 560 revolutions in traveling a mile, what is its radius? [Take $\pi = 3\frac{1}{7}$]. *Ans.* 1½ feet.

3. Find the area of a walk 7 feet wide, surrounding a circular pond 252 in diameter. [Take $\pi = 3\frac{1}{7}$]. *Ans.* 5390 sq. ft.

4. A wire equal to the radius of a circle is bent so as to fit the circumference. How many degrees in the angle formed by joining its ends with the center of the circle? [Take $\pi = 3.14159265$]. *Ans.* 57.2957795°.

Definition.—The angle subtended at the center of a circle by an arc equal in length to the radius is called a *radian.*

5. A wire is bent into the form of a circle whose radius is 30 inches. If the same wire be bent into the form of a square, what would be the length of its side?

6. A circle and a square have the same perimeter. What is the difference between their areas?

7. Two tangents drawn from an external point to a circle are 21 inches long and make angles with each other of 90°. Find the area of the circle.

8. A bicycle driving-wheel is 28 inches in diameter, the sprocket-wheel has 17 sprockets, and the rear sprocket-wheel 7 sprockets; what is the gear of the wheel?

Hint.—One revolution of the sprocket-wheel makes $17 \div 7 = \frac{17}{7}$ revolutions of the rear sprocket-wheel, or $\frac{17}{7}$ revolutions of the driving wheel, the rear sprocket-wheel and the driving-wheel being rigidly connected. $\therefore \pi\times28\times\frac{17}{7}=\pi\times68$ inches, the distance traveled in one revolution of the sprocket-wheel. 68 inches is the gear of the wheel. Gear $=(n\div m)D$, where n is the number of sprockets in the sprocket-wheel, m the number of sprockets in the rear sprocket-wheel, and D the diameter of the driving-wheel in inches.

9. (*a*) What is the gear of a bicycle whose driving-wheel is 30 inches in diameter, whose sprocket-wheel has 19 sprockets, and whose rear sprocket-wheel has 6 sprockets? (*b*) How many revolutions of the sprocket-wheel will be required to travel a mile? *Ans.* (*a*) 95 inches.

10. What is the distance from the center of a chord 70 inches long in a circle whose radius is 37 inches? *Ans.* 12 inches.

11. In a circle whose radius is 9 inches, the chord of half an arc is 12 inches; find the chord of the whole arc. *Ans.* 17.89 inches.

12. The length of an arc of a circle is 143 inches and its central angle is 9° 6′; find the radius of the circle. *Ans.* 900 inches.

13. In a circle of a radius of 37 inches, find the length of the minor arc whose chord is 24 inches. *Ans.* 24.44 inches.

14. The radius of a circle is 21 inches; find the length of an arc which subtends an angle of 60° at the center.

15. The radius of a circle is 9 feet 4 inches; what angle is subtended at the center by an arc of 28 inches?

16. The chord of an arc is 48 inches and its height is 7 inches; find the length of the arc. [Arc$=\frac{1}{3}(8b-a)$, where b is the chord of half the arc and a is the chord of the whole arc.] *Ans.* 50⅔ inches.

17. In a circle whose diameter is 72 inches, find the length of the arc whose height is 8 inches.

18. Find the area of a sector of a circle whose radius is 21 inches and the angle between the radii 40°.

19. Find the area of the sector of a circle having given the arc 32 inches and the radius 17 inches. *Ans.* 272 sq. in.

20. Angle of a sector is 36° and its area is 385 square feet; find the length of its arc. *Ans.* 22 feet.

21. Find the area of a segment cut off by a chord whose length is 14 inches from a circle of a radius of 25 inches. *Ans.* 9.37 inches.

22.* A regular pentagon is inscribed in a circle of a radius 10 inches; find the area of a minor segment cut off from the circle by one of its sides. *Ans.* 15.27 sq. in.

23. Find the area of a segment whose chord is 30 inches and height is 8 inches. *Ans.* 168.16 sq. in.

24. Find the area of a circle inscribed in a sector whose angle is 120° and whose radius is 10 inches.

25. A line AB is 20 inches long, and C is its middle point. On AB, AC, and CB semicircles are described. Find the area of the circle inscribed in the space inclosed by the three semicircles. *Ans.* $r=3\frac{1}{3}$ inches.

26. Two equal circles, each of a radius 9 inches, touch each other externally, and a common tangent (direct) is drawn to them; find the area of the space inclosed between the circles and the tangent. *Ans.* 7.53 sq. inches.

27. Three circles of radius 3 feet are placed so that they touch each other; find the area of the curvilinear space inclosed by them. *Ans.* 207 sq. in.

28. From the angular points of a regular hexagon, whose side is 10 inches, six equal circles, radii 5 inches, are drawn; find the area of the figure inclosed between the circles. *Ans.* $50(3\sqrt{3}-\pi)$ sq. in.

29. Two equal circles of radius 5 inches are described so that the center of each is on the circumference of the other; find the area of the curvilinear figure intercepted between the two circumferences.
Ans. 30.71 sq. in.

30. Two equal circles of radius 5 inches intersect so that their common chord is equal to their radius; find the area of the curvilinear figure intercepted between the two circumferences. *Ans.* 4.53 sq. in.

31. Three circles, radii 10, 12, and 16 inches respectively, touch each other; find the radius of a circle touching the three circles. [See Prob. CLXVI.]

32. Find the area of the largest semicircle that can be inscribed in a trapezium $ABCD$ whose sides are $AB=AD=30$ feet, and $CB=CD=40$ feet.

33. Find the area of the equilateral triangle formed by the diameters of three equal semicircles drawn with an equilateral triangle, side=60 feet, in such a way that each semicircle is tangent to two sides of the triangle.

34. Three equal circles are inscribed in an equilateral triangel, side=30 feet, in such a way that each circle is tangent to the two others and tangent to two sides of the triangle. Find the area between the circles and the sides of the triangle.

VII. RECTIFICATION OF PLANE CURVES AND QUADRATURES OF PLANE SURFACES.

1. ***To Rectify a Curve*** is to find its length. The term arises from the conception that a right line is to be found which has the same length

2. ***The Quadrature*** of a surface is finding its area. The term arises from the conception that we find a square whose area is equal to the area of the required surface.

The formula for the rectification of plane curves is

$$s=\int\sqrt{dx^2+dy^2}=\int\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}\,dx,$$ when the curve is referred to rectangular co-ordinates.

$$s=\int\sqrt{\left\{1+r^2\left(\frac{d\theta}{dr}\right)^2\right\}}\,dr, \text{ or } s=\int\sqrt{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}}\,d\theta,$$ when the curve is referred to polar co-ordinates.

$$s=\int\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2+\left(\frac{dz}{dx}\right)^2\right\}}\,dx$$
$$s=\int\sqrt{\left\{1+\left(\frac{dx}{dy}\right)^2+\left(\frac{dz}{dy}\right)^2\right\}}\,dy$$
$$s=\int\sqrt{\left\{1+\left(\frac{dx}{dz}\right)^2+\left(\frac{dy}{dz}\right)^2\right\}}\,dz$$
are formulæ for the rectification of curves of double curvature, when referred to rectangular co-ordinates.

$$s=\int\sqrt{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2+r^2\sin^2\theta\left(\frac{d\varphi}{d\theta}\right)^2\right\}}\,d\theta$$

$$s=\int\sqrt{\left\{1+r^2\left(\frac{d\theta}{dr}\right)^2+r^2\sin^2\theta\left(\frac{d\varphi}{dr}\right)^2\right\}}\,dr$$

$$s=\int\sqrt{\left\{r^2\sin^2\theta+\left(\frac{dr}{d\varphi}\right)^2+r^2\left(\frac{d\theta}{d\varphi}\right)^2\right\}}\,d\varphi$$

are formulæ for the rectification of curves of double curvature, referred to polar co-ordinates.

$A=\int ydx$ or $\int xdy$ is the formula for the quadrature of any plane surface referred to rectangular co-ordinates.

$A=\int\frac{1}{2}r^2d\theta$ is the formula for the quadrature of plane surfaces, referred to polar co-ordinates.

3. ***A Surface of Revolution*** is the surface generated by a line (right or curved) revolving around a fixed right line as an axis, so that sections of the volume generated, made by a plane perpendicular to the axis are circles.

$S=2\pi\int y\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}\,dx$ is the formula for a surface of revolution, referred to rectangular co-ordinates.

$S=2\pi\int yds=2\pi\int r\sin\theta\sqrt{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}}\,d\theta$ is the formula for a surface of revolution, referred to polar co-ordinates.

$V=\pi\int y^2dx$ or x^2dy is the formula for the volume of a solid of revolution referred to rectangular co-ordinates.

$V=\int\int\int dxdydz$ and $V=\int\int zdxdy$ are formulæ for the cubature of solids, requiring triple and double integration.

$V=\int\int zrd\theta dr$ and $V=\int\int\int r^2\sin\theta d\varphi d\theta dr$ are the formulæ for cubature of solids referred to polar co-ordinates. From the equation to the surface of the solid, z must be expressed as a function of r and θ.

$x^2+y^2=R^2$ is the rectangular equation of a circle referred to the center.

$y^2=2Rx-x^2$ is the rectangular equation of a circle referred to the left hand vertex as origin of co-ordinates.

$r=2R\cos.\theta$ is the equation of the circle referred to polar co-ordinates.

Prob. XXIV. To find the circumference of a circle, the radius being given.

Formula.—$C.=4\int_0^R\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}dx=4\int_0^R\left(\frac{x^2+y^2}{y^2}\right)^{\frac{1}{2}}dx=4\int_0^R\frac{-Rdx}{(R^2-x^2)^{\frac{1}{2}}}=4R(1+\frac{1}{2.3}+\frac{1.3}{2.4.5}+\frac{1.3.5}{2.4.6.7}+\frac{1.3.5.7}{2.4.6.8.9}+\&c)=4R\times1.570796+=3.141592\times2R=2\pi R$, in which $\pi=3.141592+$. Since the diameter is twice the radius, we have $2\pi R=\pi D$, in which D is the diameter. $\therefore$ $C.=2\pi R=\pi D$. $\therefore$ (1) $R=\frac{C}{2\pi}$, (2) $D=\frac{C}{\pi}$, where C is the circumference.

Rule.—*Multiply twice the radius or the diameter by 3.141592.*

I. What is the circumference of a circle whose radius is 17 rods?

By formula, $C=2\pi R=3.141592\times34$ rods $=106.814128$ rods.

II. 1. 17 rods=the radius.
2. 34 rods=2×17 rods=the diameter.
3. 106.814128 rods=3.141592×34 rods=the circumference.

III. ∴ The circumference is 106.814128 rods.

Note.—The ratio of the circumference to the diameter can not be exactly ascertained. An untold amount of mental energy has been expended upon this problem; but all attempts to find an exact ratio have ended in utter failure. Many persons not noted along any other line, claimed to have found this *clavem impossibilitibus* by which they have unlocked all the difficulties that have encumbered the quadrature of the circle for more than two thousand years. The Quadrature of the Circle is to find a square whose area shall be exactly equal to that of the circle. This can not be done, since the ratio of the circumference to the diameter can not be exactly ascertained. Persons claiming to have held communion with the "gods" and extorted from them the exact ratio are ranked by mathematicians in the same class with the inventors of Perpetual Motion and the discoverers of the Elixir of Life, Alkahest, the Fountain of Perpetual Youth, and the Philosopher's Stone. Lambert, an Alsacian mathematician, proved, in 1761, that this ratio is incommensurable. In 1881, Lindemann, a German mathematician, demonstrated that this ratio is transcendental, and that the quadrature of the circle by means of the ruler and compass only, or by means of any algebraic curve, is *impossible*. Its value has been computed to several hundred decimal places. Archimedes, in 287 B. C., found it to be between $3\frac{10}{71}$ and $3\frac{1}{7}$; Metius, in 1640, gave a nearer approximation in the fraction $\frac{355}{113}$; and, in 1873, Mr. W. Shank presented to the Royal Society of London a computation extending the decimal to 707 places. The following is its value to 600 decimal places:

3. 141, 592, 653, 589, 793,238,462,643,383, 279, 502,884, 197, 169,
399,375,105,820,974,944,592,307,816,406,286,208,998,628,034,825,
342,117,067,982,148,086,513,282,306,647,093,844,609,550,582,231,
725,359,408,128,481,117,450,284,102,701,938,521,105,559,644,622,
948,954,930,381,964,428,810,975,665,933,446,128,475,648,233,786,

783,165,271,201,909,145,648,566,923,460,348,610,454,326,648,213, 393,607,260,249,141,273,724,587,006,606,315,588,174,881,520,920, 962,829,254,091,715,364,367,892,590,360,011,330,530,548,820,466, 521,384,146,951,941,511,609,433,057,270,365,759,591,953,092,186, 117,381,932,611,793,105,118,548,074,462,379,834,749,567,351,885, 752,724,891,227,938,183,011,949,129,833.673,362,441,936.643.086, 021,395,016,092,448,077,230,943,628,553,096,620,275,569,397,986, 950,222,474,996,206,074,970,304,123,669+.

$\frac{1}{4}\pi=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\&c.$

$\frac{1}{6}\pi^2=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\&c.$* Bernoulli's Formula.

$\frac{1}{2}\pi=\left[\frac{2.4.6.8.10....\&c.}{3.5.7.9.11....\&c.}\right]^2$ .... Wallis's Formula, 1655.

$$\frac{1}{2}\pi=1+\cfrac{1}{1+\cfrac{1.2}{1+\cfrac{2.3}{1+\cfrac{3.4}{1+\cfrac{1}{1+\cfrac{4.5}{1+}}}}}}$$ .... Sylvester's Formula, 1869.

$$\frac{4}{\pi}=1+\cfrac{1^2}{2+\cfrac{3^2}{2+\cfrac{5^2}{2+\ldots}}}$$ ...... Buckner's Formula.

Note.—The Greek letter, π is the initial letter of the word περιφέρεια =*peripheria*, meaning periphery or circumference. It was first used to represent the ratio of the circumference to the diameter by William Jones in his *Synopsis Palmariorum Matheseos*, 1706, and came into general use through the influence of Euler.

Prob. XXV. To find the length of any arc of a circle, having given the chord of the arc and the height of the arc, i. e., the versed sine of half the arc.

(*a*). ***Formula.***—$s=\int\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int\left(\frac{x^2+y^2}{y^2}\right)dx=-\int\frac{Rdx}{(R^2-x^2)^{\frac{1}{2}}}=R\sin^{-1}\frac{x}{R}=R\left[\frac{x}{R}+\frac{x^3}{2.3.R^3}+\frac{1.3x^5}{2.4.5.R^5}+\frac{1.3.5x^7}{2.4.6.7.R^7.}+\&c.\right],=\frac{1}{2a}(a^2+c^2)\left[\frac{c^2-a^2}{a^2+c^2}+\frac{1}{6}\frac{(c^2-a^2)^3}{(a^2+c^2)^3}+\frac{3}{40}\frac{(c^2-a^2)^5}{(a^2+c^2)^5}+\frac{5}{112}\left(\frac{c^2-a^2}{a^2+c^2}\right)^7+\&c.\right]$, in which a=the altitude of the arc and c=half the chord of the arc.

Note.*—This series was discovered by Bernoulli, but he acknowledged his inability to sum it. Euler found the result to be $\frac{1}{6}\pi^2$. For an interesting discussion of the various formulæ for π, see *Squaring the Circle*, *Britannica Encyclopedia* (*b*.) **Formula.—s=arc$=\frac{1}{3}(8b-a)$*, where a is the chord of the whole arc and b the chord of half the arc.

Rule from (*b*). *From eight times the chord of half the arc subtract the chord of the whole arc; one-third of the remainder will be the length of the arc, approximately.*

I. Find the length of the arc whose chord is 517638 feet and whose half chord is 261053.6 feet.

By formula (*b*), $s=\frac{1}{3}(8b-a)=\frac{1}{3}$ (8×261053.6—517638)= 52359.88 feet.

II.
1. 261053.6 feet=length of chord of half arc.
2. 2088428.8 feet=8×261053.6 feet=eight times the length of chord of half arc.
3. 517638 feet=length of chord of whole arc.
4. 1570790.8 feet=2088428.8 feet—517638 feet=difference between eight times chord of half arc and chord of whole arc.
5. 52359.69 feet=$\frac{1}{3}$ of 1570790.8 feet=length of arc, nearly.

III. ∴ The length of the arc is 52359.69 feet.

Note.—This important approximation is due to Huygens, (he wrote his name Hugens. It is also sometimes spelled Huyghens), a Danish mathematician, born at the Hague, April 14, 1629, and died in the same town in 1695. For a brief biography of this noted mathematician, see Ball's *A Short History of Mathematics*, pp. 302–306.

The following is Newton's demonstration:

Let R be the radius of the circle, L the length of the arc, A the chord of the arc, and B the chord of half the arc.

Then $\frac{A}{R}=2\sin.\frac{L}{2R}$, $\frac{B}{R}=2\sin.\frac{L}{4R}$.

Since, $\sin. x=\frac{x}{1}-\frac{x^3}{3!}+\frac{x^5}{5!}-$etc. (See Bowser's *Treatise on Trigonometry*, or any other good work on the subject), we have

$$\frac{A}{R}=2\left(\frac{\frac{L}{2R}}{1}-\frac{\left(\frac{L}{2R}\right)^3}{3!}+\frac{\left(\frac{L}{2R}\right)^5}{5!}-\text{etc.}\right) \text{ and}$$

$$\frac{B}{R}=2\left(\frac{\frac{L}{4R}}{1}-\frac{\left(\frac{L}{4R}\right)^3}{3!}+\frac{\left(\frac{L}{4R}\right)^5}{5!}-\text{etc.}\right).$$

$$\therefore A=L-\frac{L^3}{4!R^2}+\frac{L^5}{2^4 5!R^4}-\text{etc., and}$$

$$8B=4L-\frac{L^3}{4!R^2}+\frac{L^5}{2^6 5!R^4}-\text{etc.}$$

$$\therefore \tfrac{1}{3}(8B-A)=L\left(1-\frac{L^4}{7680R^4}\right),\text{ nearly,}=L\text{, nearly.}$$

In the problem proposed, the radius is 100000 feet and the arc is 30°. Using π=3.1415926, L=52359.88 feet. ∴ The result by the formula lacks only about 2 inches of being the same.

(*c.*) ***Formula.***—$s=\text{arc}=2\sqrt{(a^2+c^2)}\times\left\{1+\frac{10a^2}{60c^2+33a^2}\right\}$.

This formula is a very close approximation to the true length of the arc when *a* and *c* are small. The first formula may be extended to any desired degree of accuracy.

Rule from (*c*).—*Divide 10 times the square of the height of the arc by 15 times the square of the chord and 33 times the height of the chord; multiply this quotient increased by 1, by 2 times the square root of the sum of the squares of the height and half the chord.*

I. The chord of an arc is 25, and versed-sine 15, required the length of the arc.

By formula (*a*), $arc=\frac{a^2+c^2}{2a}\left[\frac{c^2-a^2}{a^2+c^2}+\frac{1}{6}\left\{\frac{c^2-a^2}{a^2+c^2}\right\}^3+\frac{3}{40}\left\{\frac{c^2-a^2}{a^2+c^2}\right\}^5+\frac{5}{112}\left\{\frac{c^2-a^2}{a^2+c^2}\right\}^7+\&c.\right]=\frac{15^2+25^2}{2\times15}\left[\frac{25^2-15^2}{15^2+25^2}+\frac{1}{6}\times\left\{\frac{25^2-15^2}{15^2+25^2}\right\}^3+\frac{3}{40}\left\{\frac{25^2-15^2}{15^2+25^2}\right\}^5+\frac{5}{112}\left\{\frac{25^2-15^2}{15^2+25^2}\right\}^7+\&c.\right]=53.58$ +ft.

II.
1. 25 ft.=length of the chord.
2. 15 ft.=height of the arc, or the versed-sine.
3. 2250 sq. ft.=10 times 15^2=10 times the square of the height of the arc.
4. 9375 sq. ft.=15 times 25^2=15 times the square of the chord.
5. 7425 sq. ft.=33 times 15^2=33 times the square of the height of arc.
6. 17800 sq. ft.=7425 sq. ft.+9375 sq. ft.
7. $\frac{45}{356}=2250\div17800=10$ times $15^2\div(15$ times 25^2+33 times $15^2)$.
8. $1+\frac{45}{356}=\frac{401}{356}=1+10$ times $15^2\div(15$ times 25^2+33 times $15^2)$.
9. $381\frac{1}{4}$ sq. ft.$=15^2+(12\frac{1}{2})^2$.
10. 53.58 ft.$=\frac{401}{356}\times\sqrt{15^2+(12\frac{1}{2})^2}=\frac{401}{356}\times\frac{5}{2}\sqrt{61}$=length of arc, nearly.

III. ∴ 53.58 ft.=length of the arc.

Prob. XXVI. To find the area of a circle having given the radius, diameter, or circumference.

Formula.—$A=4\int y\,dx,=4\int_0^R(R^2-x^2)^{\frac{1}{2}}dx=4\left[\frac{1}{2}\times(R^2-x^2)+\frac{1}{2}R^2\sin^{-1}\frac{x}{R}\right]_0^R=2R^2\left[\frac{x}{R}+\frac{x^3}{2.3R^3}+\frac{1.3.x^5}{2.4.5.R^5}+\frac{1.3.5.\,x^7}{2.4.6.7R^7}+\&c=\frac{1}{2}\pi\right]=\pi R^2=\frac{1}{4}\pi D^2=\frac{C^2}{4\pi}=\frac{1}{2}R\times C$, when the radius and circumference are given. $\therefore$ (1) $R=\sqrt{A\div\pi}$, (2) $D=\sqrt{4A\div\pi}=2R=2\sqrt{A\div\pi}$, and (3) $C=\sqrt{4\pi A}=2\sqrt{\pi A}$.

Rule I.—*The area of a circle equals the square of the radius multiplied by 3.141592; or (2) the square of the diameter multiplied by .785398; or (3) the square of the circumference multiplied by .07958; or (4) the circumference mnltiplied by $\frac{1}{4}$ of the diameter; or (5) the circumference multiplied by $\frac{1}{2}$ of the radius.*

Rule II.—*Having given the area. (1) To find the radius: Divide the area by 3.141592, and extract the square root of the quotient. (2) To find the diameter: Divide the area by 3.141592 and multiply the square root of the quotient by 2. (3) To find the circumference: Multiply the area by 3.141592 and multiply the square root of the product by 2.*

I. What is the area of a circle whose radius is 7 feet?

By formula, $A=\pi R^2=3.141592\times7^2=153.93804+$ sq. ft.

II. 1. 7 ft.=the radius.
2. 49 sq. ft.$=7^2$=square of the radius.
3. 153.93804 sq. ft.$=3.141592\times49$ sq. ft.=area of the circle.

III. $\therefore$ 153.93804 sq. ft.=area of the circle.

I. What is the area of a circle whose diameter is 4 rods?

By formula, $A=\frac{1}{4}\pi D^2=\frac{1}{4}\times3.141592\times4^2=12.566368$ sq. ft.

II. 1. 4 ft.=the diameter.
2. 16 sq. ft.=square of the diameter.
3. 12.566368 sq. ft.$=\frac{1}{4}\times3.141592\times4^2=.785398\times16$ sq. ft. =area of the circle.

III. $\therefore$ 12.566368 sq. ft.=area of the circle.

I. What is the area of a circle whose circumference is 5 meters?

By formula, $A=\frac{C^2}{4\pi}=\frac{25^2}{4\pi}=1.989\ m^2$.

II. 1. 5 m.=the circumference.
2. 25 m.^{2}=the square of the circumference.
3. 1.989 m.^{2}=.07958$\times$25 m.^{2}=the area of the circle.

III. $\therefore$ 1.989 m.^{2}=the area of the circle.

Remark.—We might have found the radius by formula (1) under Prob. XXIV and then applied the first of Rule I. above. We might have found the radius by formula (1) of Prob. XXIV and then applied (5) of Rule I. above.

I. What is the circumference of a circle whose area is 10 A.?

By formula (3), $C=2\sqrt{\pi A}=2\sqrt{3.141592\times1600}=80\sqrt{\pi}80\times$ $1.7724539=141.796312$ rods.

II.
1. 10 A.=1600 sq. rds.=the area of the circle.
2. $1600\div\pi$=the square of the radius.
3. $\therefore 40\sqrt{\frac{1}{\pi}}=\frac{40}{\pi}\sqrt{\pi}$=the radius.
4. $\frac{80}{\pi}\sqrt{\pi}=\pi 2$ times $\frac{40}{\pi}\sqrt{\pi}$=the diameter.
5. $80\sqrt{\pi}=\pi\times\frac{40}{\pi}\sqrt{\pi}=141.796312$ rods=the circumference.

III. $\therefore$ 141.796312 rods=the circumference of the circle.

I. With what length of rope must a horse be tied to a stake so that he can graze over one acre of grass and no more?

By formula (1), $R=\sqrt{A\div\pi}=\sqrt{160\div\pi}=4\sqrt{\frac{10}{\pi}}=7.1364+$ rd.

II.
1. 1 A.=160 sq. rd.=area of the circle over which the horse can graze.
2. $160\div\pi$=square of the radius.
3. $\sqrt{160\div\pi}=4\sqrt{10\div\pi}=7.1364$ rd.=radius or length of rope.

III. $\therefore$ 7.1364 rd.=length of the rope.

Prob. XXVII. To find the area of a sector, or that part of a circle which is bounded by any two radii and their included arc, having given the chord of the arc and the height of the arc.

$$*\textit{Formula.}—A=\int ydx=2\int(R^2-x^2)^{\frac{1}{2}}dx=x(R^2-x^2)^{\frac{1}{2}}+R^2\sin^{-1}\frac{x}{R}=\frac{c^2-a^2}{2a}\left\{\left(\frac{a^2+c^2}{2a}\right)^2-\left(\frac{c^2-a^2}{2a}\right)^2\right\}^{\frac{1}{2}}+\left(\frac{a^2+c^2}{2a}\right)^2\sin^{-1}\frac{c^2-a^2}{c^2+a^2}=c\left(\frac{c^2-a^2}{2a}\right)+\left(\frac{a^2+c^2}{2a}\right)^2\sin^{-1}\frac{c^2-a^2}{c^2+a^2}=c\left(\frac{c^2-a^2}{2a}\right)$$

$$+\left(\frac{a^2+c^2}{2a}\right)^2\left\{\frac{c^2-a^2}{c^2+a^2}+\frac{1}{1.2.3}\left(\frac{c^2-a^2}{c^2+a^2}\right)^3+\frac{1}{1.2.3.4.5}\left(\frac{c^2-a^2}{c^2+a^2}\right)^5+\frac{1}{1.2.3.4.5.6.7}\left(\frac{c^2-a^2}{c^2+a^2}\right)^7+\&c.\right\}$$

in which c is half the chord of the arc and a the height of arc.

Demonstration.—Let $AB=x$, $BD=y$, and $R=AD$=the radius of the circle. Then $x^2+y^2=R^2$, the equation of the circle referred to the center. Now $A=2\int ydx$; but $y=(R^2-x^2)^{\frac{1}{2}}$, from the equation of the circle.

$\therefore A=2\int(R^2-x^2)^{\frac{1}{2}}dx=x(R^2-x^2)^{\frac{1}{2}}+R^2\sin^{-1}\frac{x}{R}$. But $x=R-a$ and $y=c$.

Hence $A=(R-a)[R^2-(R-a)^2]+R^2\sin^{-1}\frac{R-a}{R}$. But, from (2) Prob. XX, R

Rule.—(1) *Find the length of the arc by Problem XXV, and then multiply the arc by half the radius which may be found by Problem XX, in which c and a are known and R is the unknown quantity.*

(2) *If the arc is given in degrees, take such a part of the whole area of the circle as the number of degrees in the arc is of 360°.*

I. Find the area of the sector, the chord of whose arc is 40 feet, and the versed-sine of half the arc 15 feet.

By formula, $A=c\left(\frac{c^2-a^2}{2a}\right)+\left(\frac{a^2+c^2}{2a}\right)^2\left\{\frac{c^2-a^2}{c^2+a^2}+\frac{1}{1.2.3}\left(\frac{c^2-a^2}{c^2+a^2}\right)^3+\&c.\right\}=20\left(\frac{20^2-15^2}{30}\right)+\left(\frac{15^2+20^2}{30}\right)^2\left\{\frac{20^2-15^2}{20^2+15^2}+\frac{1}{1.2.3}\left(\frac{20^2-15^2}{20^2+15^2}\right)^3+\frac{1}{1.2.3.4.5}\left(\frac{20^2-15^2}{20^2+15^2}\right)^5+\&c.\right.=558.125$ sq. ft.

II.
1. 53.58 ft. $=2(15^2+20^2)^{1/2}\times\left(1+\frac{10\times15^2}{60\times20^2+33\times15^2}\right)=$ length of the arc, by (*b*), Prob. XXV.
2. $20\frac{5}{6}$ ft. $=\frac{a^2+c^2}{2a}=$ radius of the circle, by solving the formula of Prob. XX with respect to R.
3. ∴ 558.125 sq. ft. $=\frac{1}{2}(20\frac{5}{6}\times53.58)=$ area of the sector.

III. ∴ 558.125 sq. ft. = the area of the sector.

$=\frac{a^2+c^2}{2a}$. Hence, $A=\left(\frac{c^2-a^2}{2a}\right)c+\left(\frac{c^2-a^2}{2a}\right)^2\sin^{-1}\left(\frac{c^2-a^2}{c^2+a^2}\right)$ But, fron Trigonometry, we have $\sin^{-1}\theta=\theta-\frac{1}{1.2.3}\theta^3+\frac{1}{1.2.3.4.5}\theta^5-\&c.$ Hence, $A=\left(\frac{c^2-a^2}{2a}\right)c+\left(\frac{c^2-a^2}{2a}\right)^2\left\{\frac{c^2-a^2}{c^2+a^2}-\frac{1}{1.2.3}\left(\frac{c^2-a^2}{c^2+a^2}\right)^3+\frac{1}{1.2.3.4.5}\left(\frac{c^2-a^2}{c^2+a^2}\right)^5-\&c.\right\}$

In this formula, $\left(\frac{c^2-a^2}{2a}\right)c$ is the area of the triangle DEA. For $x=R-a=\frac{a^2+c^2}{2a}-a=\frac{c^2-a^2}{2a}=$ the altitude and c is half the base of the triangle. ∴ $\left(\frac{c^2-a^2}{2a}\right)c=$ the area of the triangle DEA. Therefore, if we subtract the area of the triangle DEA from the area of the sector, we shall have the area of the segment DEC. Hence, the area of the segment DEC is $\left(\frac{c^2-a^2}{2a}\right)^2\left\{\left(\frac{c^2-a^2}{c^2+a^2}\right)-\frac{1}{1.2.3}\left(\frac{c^2-a^2}{c^2+a^2}\right)^3+\frac{1}{1.2.3.4.5}\left(\frac{c^2-a^2}{c^2+a^2}\right)^5-\&c.\right\}$. This result may be carried to any desired degree of accuracy.

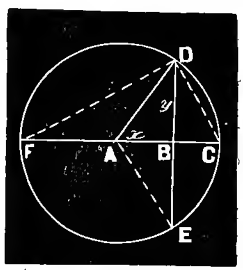

FIG 14.

I. What is the area of a sector whose arc is 40° and the radius of the circle 9 feet?

II.
1. 9 ft.=radius of the circle.
2. $\pi R^2=\pi 9^2$=area of the whole circle.
3. 40°=length of the arc.
4. 40°=$\frac{1}{9}$ of 360°.
5. $\pi 9=\frac{1}{9}$ of $\pi 9^2$=28.274328 sq. ft.=area of the sector.

III. ∴ The area of the sector is 28.274328 sq. ft.

Prob. XXVIII. To find the area of the segment of a circle, having given the chord of the arc and the height of the segment, i. e., the versed-sine of half the arc.

Formula.—(a) $A=\left(\frac{c^2-a^2}{2a}\right)^2\left\{\left(\frac{c^2-a^2}{c^2+a^2}\right)-\frac{1}{1.2.3}\left(\frac{c^2-a^2}{c^2+a^2}\right)^3+\frac{1}{1.2.3.4.5}\left(\frac{c^2-a^2}{c^2+a^2}\right)^5-\text{\&c.}\right\}$

(b) $A=\frac{a^3}{2(2c)}+\frac{2}{3}(2ca)=\frac{a^3}{4c}+\frac{4ca}{3}$.

***Rule.**—*Divide the cube of the height by twice the base and increase the quotient by two-thirds of the product of the height and base.*

I. What is the area of a segment whose base is 2 feet and altitude 1 foot?

By formula (b), $A=\frac{a^3}{2(2c)}+\frac{2}{3}(2ca)=\frac{1^3}{2\times2}+\frac{2}{3}(2\times1)=1\frac{7}{12}$sq.ft.

**Demonstration.*— In the last figure, let $BC=a$=altitude of the segment and $DE=2c$=the base of the segment. Then $BD^2=BC\times BF=a(2R-a)=c^2$. Whence $R=\frac{c^2+a^2}{2a}$, and $AD=R-a=\frac{c^2+a^2}{2a}-a=\frac{c^2-a^2}{2a}$. $DC=k=\sqrt{c^2+a^2}$. By Trigonometry, $\frac{BD}{AD}=\sin\angle DAC$, or $\frac{c}{R}=\sin\angle DAC$. Now $2\pi R=360°$. ∴ $R=\frac{180°}{\pi}$. Therefore, $R:arcDC::\frac{180°}{\pi}:?=\frac{arcDC}{R}\times\frac{180°}{\pi}$. Let $s=arc$ DCE. Then the $\angle DAC=\frac{\frac{1}{2}s}{R}=\frac{s}{2R}$. ∴ $\frac{c}{R}=\sin\frac{s}{2R}$. In like manner, from the right angled triangle FDC, $\frac{DC}{FC}=\sin\angle CFD$, or since the $\angle CFD$=the $\frac{1}{2}\angle CAD$, $\frac{k}{2R}=\sin\frac{s}{4R}$. Now since the sine of any angle $\theta=\theta-\frac{1}{1.2.3}\theta^3+\frac{1}{1.2.3.5}\theta^5-\frac{1}{1.2.3.5.7}\theta^7+\text{\&c.}$, the above equation becomes

$\frac{c}{R}=\frac{s}{2R}-\frac{1}{1.2.3}\left(\frac{s}{2R}\right)^3+\frac{1}{1.2.3.5}\left(\frac{s}{2R}\right)^5-\text{\&c.}$.....(1), and

$\frac{k}{2R}=\frac{s}{4R}-\frac{1}{1.2.3}\left(\frac{s}{4R}\right)^3+\frac{1}{1.2.3.5}\left(\frac{s}{4R}\right)^5-\text{\&c.}$.....(2). Multiplying equation (2) by 8 and subtract equation (1) in order to eliminate the term containing s^3, we have approximately, $\frac{4k-c}{s}=\frac{3s}{2R}-\frac{3}{4}\left(\frac{1}{1.2.3.5}\right)\left(\frac{s}{2R}\right)^5+\text{\&c.}$ Omitting the negative quantity, since it is very small in comparison with s and because it is still more diminished by a succeeding positive quantity, we have

II.
1. 1 ft.=altitude of the segment.
2. 2 ft.=base of the segment.
3. 4 ft.=twice the base of the segment.
4. 1 cu. ft.=cube of the height of the segment.
5. $\frac{1}{4}$ sq. ft.=1÷4=quotient of the cube of the height and twice the base.
6. 2 sq. ft.=2×1=product of the height and base.
7. $1\frac{1}{3}$ sq. ft.=$\frac{2}{3}$ of the product of the height and base.
8. ∴ $1\frac{1}{3}$ sq. ft.+$\frac{1}{4}$ sq. ft.=$1\frac{7}{12}$sq. ft.=area of the segment.

III. ∴ The area of the segment is $1\frac{7}{12}$ sq. ft.

Prob. XXIX. To find the area of a circular zone, or the space included between any two parallel chords and their intercepted arcs.

Formula.—(a) $A=\left(\frac{c^2-a^2}{2a}\right)^2\left\{\frac{c^2-a^2}{c^2+a^2}-\frac{1}{1.2.3}\left(\frac{c^2-a^2}{c^2+a^2}\right)^2+\frac{1}{1.2.3.4.5}\left(\frac{c^2-a^2}{c^2+a^2}\right)^5-\&c.\right\}-\left(\frac{c'^2-a'^2}{2a'}\right)^2\left\{\frac{c'^2-a'^2}{c'^2+a'^2}-\frac{1}{1.2.3}\left(\frac{c'^2-a'^2}{c'^2+a'^2}\right)^3+\frac{1}{1.2.3.4.5}\left(\frac{c'^2-a'^2}{c'^2+a'^2}\right)^5-\&c.\right\}$.

(b) $A=\frac{a^3}{2(2c)}+\frac{2}{3}(2ca)-\left[\frac{a'^3}{2(2c')}+\frac{2}{3}(2c'a')\right]$

Rule.—*Find the area of each segment by Prob. XXVIII., and take the difference between them, if both chords are on the same side of the center; if on opposite sides of the center, subtract the sum of the areas of the segments from the whole area of the circle.*

I. What is the area of a zone, one side of which is 96, and the other 60, and the distance between them 25?

Let $AB=60=2c'$, $CD=96=2c$, and $HK=25=h$. Then $AH=30=c'$ and $CK=48=c$. Let $OA=R$. Then $OH=\sqrt{R^2-c'^2}$, $OK=\sqrt{R^2-c^2}$. But $OH=OK+HK$; ∴ $\sqrt{R^2-c'^2}=\sqrt{R^2-c^2}+h$. Whence $R=\frac{1}{2h}\sqrt{4c^2h^2+(c^2-h^2-c'^2)^2}$.

∴ $LH=a'=R-OH=R-\sqrt{(R^2-c^2)}=\frac{1}{2h}\sqrt{4c^2-h^2+(c^2=h^2-c'^2)^2}-\frac{1}{2h}\sqrt{4h^2(c^2-c'^2)}$

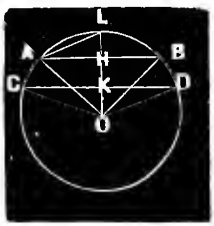

FIG. 15.

$S=\frac{8k-2c}{3}=\frac{8\sqrt{c^2+a^2}-2a}{3}=\frac{2}{3}(4\sqrt{c^2+a^2}-a)$. This is the approximate length of an arc in terms of its height and base. Now the area of the segment $DCE=\frac{1}{2}AC\times arc\ DCE$—area of the triangle $DEA=\frac{1}{2}R\times S-\frac{1}{2}AB\times DE=\frac{1}{2}\left(\frac{c^2+a^2}{2a}\right)^2\times\frac{2}{3}(4\sqrt{c^2+a^2}-a)-\frac{1}{2}\left(\frac{c^2-a^2}{2a}\right)c=\frac{1}{6a}[4(c^2+a^2)^{\frac{3}{2}}-4c^3+2ca^2]=\frac{1}{6a}[\sqrt{(16c^6+48c^4a^2+48c^2a^4+16a^6)}-4c^3+2ca^2]=\frac{1}{6a}\left[\left(4a^3+6c^2a+\frac{3a^4}{2c}\right)\text{nearly}-4c^3+2ca^2\right]=\frac{1}{6a}\left(8ca+\frac{3a^4}{2c}\right)=\frac{a^3}{2(2c)}+\frac{2}{3}(2ca)$. Q. E. D.

$+(c^2-h^2-c'^2)^2$. In like manner, $LK=a=R-\sqrt{R^2-c^2},=\frac{1}{2h}$ $\sqrt{4c^2h^2+(c^2-h^2-c'^2)^2}-\frac{1}{2h}(c^2h^2-c'^2).^2$

$\therefore$ By formula (*b*), $A=\frac{a^3}{2(2c)}+\frac{2}{3}(2ca)-\left[\frac{a'^3}{2(2c')}+\frac{2}{3}(2c'a')\right],=$ $\left\{\frac{1}{2h}\left[\sqrt{4c^2h^2+(c^2-h^2-c'^2)^2}-(c^2-h^2c'^2)^2\right]\right\}^3\div2.2c+\frac{2}{3}\times2c\times\frac{1}{2h}$ $\left[\sqrt{4c^2h^2+(c^2-h^2-c'^2)^2}-(c^2-h^2-c'^2)^2\right]-\left[\left\{\frac{1}{2h}\sqrt{4c^2h^2+(c^2-h^2-c'^2)^2}-\sqrt{4h^2(c^2-c'^2)+(c^2-h^2-c'^2)^2}\right]\right\}^3\div2(2c)+\frac{2}{3}\times2c'$ $\times\frac{1}{2h}[\sqrt{4h^2c^2+(c^2-h^2-c'^2)^2}-\sqrt{4h^2(c^2-c'^2)+(c^2-h^2-c'^2)^2}]\Big\}$ $=2547-408\frac{1}{3}=2138\frac{2}{3}$.

II.
1. $50=R=\frac{1}{2h}\sqrt{4c^2h^2+(c^2-h^2-c'^2}=\frac{1}{50}\sqrt{4\times48^2\times25^2+(48^2-25^2-30^2)}$=radius of the circle.
2. $OK=\sqrt{R^2-c^2}=\sqrt{50^2-48^2}=14$.
3. $\therefore$ $LK=a=50-14=36$=altitude of segment *CLD*.
4. $OH=\sqrt{R^2-c'^2}=\sqrt{50^2-30^2}=40$.
5. $\therefore$ $LH=a'=50-40=10$=altitude of the segment *ALB*.
6. $\therefore$ $\frac{36^3}{2\times96}+\frac{2}{3}(96\times36)=2547$=area of segment *CDBLA*.
7. $\frac{10^3}{2\times60}+\frac{2}{3}(60\times10)=408\frac{1}{3}$=area of the segment *ABL*.
8. $\therefore$ $2547-408\frac{1}{3}=2138\frac{2}{3}$=area of the zone *CDBA*.

III. $\therefore$ $2138\frac{2}{3}$=area of the zone *ABDC*.

Note.—**This result is only approximately correct. The radius of the circle may be found by the following rule:**

Subtract half the difference between the two half-chords from the greater half-chord, multiply the remainder by said difference, divide the product by the width of the zone, and add the quotient to half the width. To the square of this sum add the square of the less half chord, and take the square root of the sum.

This rule is derived from the formula in the above solution, in which

$$R=\sqrt{\frac{h^2c^2+(c^2-h^2-c'^2)^2}{4h^2}}=\sqrt{\left[\frac{(c^2+h^2-c'^2)^2+4h^2c'^2}{4h^2}\right]},$$

Prob. XXX. To find the area of a circular ring, or the space included between the circumference of two concentric circles.

Formula.—(*a.*) $A=\pi\ (R^2-r^2)$, in which R and r are the radii of the circles.

(*b.*) *$A=\frac{1}{4}\pi c^2$, in which c is a chord of the larger circle tangent to the smaller circle.

I. Required the area of a ring the radii of whose bounding circles are 9 and 7 respectively.

By formula (*a*), $A=\pi(R^2-r^2)=\pi(9^2-7^2)=32\pi=$ 100.530944.

II.
1. $9=R=$radius of the larger circle, and
2. $7=r=$radius of the smaller circle.
3. $\pi 9^2=\pi R^2=$area of larger circle, and
4. $\pi 7^2=\pi r^2=$area of smaller circle.
5. $\therefore \pi 9^2-\pi 7^2=\pi(9^2-7^2)=32\pi=$ 100 530944=area of the ring.

III. $\therefore$ 100.530944=the area of the ring.

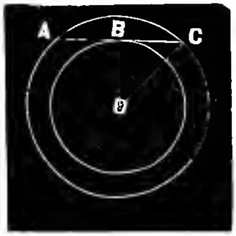

FIG. 16.

****Demonstration.***—Let ABC be the chord of the large circle, which is tangent to the smaller circle, and let $ABC=c$. Then $BC=\frac{1}{2}c=\sqrt{(OC^2-OB^2)}=\sqrt{(R^2-r^2)}$. $\therefore \frac{1}{4}c^2=R^2-r^2$ and $\frac{1}{4}\pi c^2=\pi(R^2-r^2)$. But $\pi(R^2-r^2)$ is the difference of the areas of the two circles or the area of the ring. $\therefore \frac{1}{4}\pi c^2=$the area of the ring. *Q. E. D.*

Prob. XXXI. To find the areas of circular lunes, or the spaces between the intersecting arcs of two eccentric circles.

$$\textbf{\textit{Formula.}}\text{—}A=\frac{a'^3}{2(2c)}+\tfrac{2}{3}(2ca)-\left[\frac{a'^3}{2(2c)}+\tfrac{2}{3}(2ca')\right].$$

Rule.—*Find the area of the two segments of which the lunes are formed, and their difference will be the area required.*

I. The chord AB is 20, and the height DC is 10, and DE 2; find the area of the lune $AEBC$.

$$=\sqrt{\left[\left(\frac{c^2-c'^2}{2h}+\tfrac{1}{2}h^2\right)+c'^2\right]}=\sqrt{\left\{\left[\frac{\left(\frac{c-c'}{2}\right)(c-c')}{h}+\tfrac{1}{2}h\right]^2+c'^2.\right\}}.$$

If now we find the altitudes of the two segments and then find the length of the arcs of the segments by formula (*b*), Prob. XXV, and then find the area of the sectors by multiplying the length of the arcs by half the radius, from the areas of the sectors subtract the triangles formed by the radii of the circles and the chord of the arcs, we shall then have the area of the two segments. Taking their difference, we shall have for the area of the zone 2136.75, which is a nearer approximation to the true area.

By formula, $A=\frac{a}{2(2c)}+\frac{2}{3}(2ca)-\left[\frac{a'^3}{2(2c)}+\frac{2}{3}(2ca')\right]=\frac{10^3}{2\times20}$ $+\frac{2}{3}(20\times10)-\left[\frac{2^3}{2\times20}+\frac{2}{3}(20\times2)\right]=131\frac{7}{15}$.

II.
1. AB=chord=20.
2. DE = height of segment $AEBD$=2. [$ACBD$=10
3. DC = height of segment
4. $\frac{10^3}{2\times20}+\frac{2}{3}(20\times10)=158\frac{1}{3}$= area of the segment $ACBD$.
5. $\frac{2^3}{2\times20}+\frac{2}{3}(20\times2)=26\frac{13}{15}$= area of the segment $AEBD$.

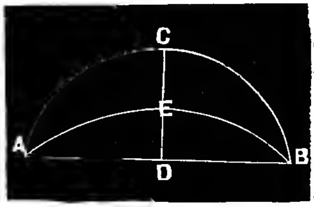

FIG. 17.

III. ∴ $158\frac{1}{3}-26\frac{13}{15}=131\frac{7}{15}$=area of the lune $ACBE$.

VIII. CONIC SECTIONS.

DEFINITIONS.

1. ***The Conic Sections*** are such plane figures as are formed by the cutting of a cone.

2. If a cone be cut through the vertex, by a plane which also cuts the base, the sections will be a *triangle*.

3. If a right cone be cut in two parts, by a plane parallel to the base, the section will be a *circle*.

4. If a cone be cut by a plane which passes through its two slant sides in an oblique direction, the section will be an *ellipse*.

5. ***The Transverse Axis*** of an ellipse is its longest diameter.

6. ***The Conjugate Axis*** of an ellipse is its shortest diameter.

7. ***An Ordinate*** is a right line drawn from any point of the curve, perpendicular to either of the axes.

8. ***An Abscissa*** is that part of the diameter which is contained between the vertex and the *ordinate*.

9. ***A Parabola*** is a section formed by passing a plane through a cone parallel to any of its slant sides.

10. ***The Axis*** of a parabola is a right line drawn from the vertex, so as to divide the figure into two equal parts.

11. ***The Ordinate*** is a right line drawn from any point in the curve perpendicular to the axis.

12. ***The Abscissa*** is that part of the axis which is contained between the vertex and the ordinate.

13. ***An Hyperbola*** is a section formed by passing a plane through a cone in a direction to make an angle at the base greater than that made by the slant height. It will thus pass through the symmetrical opposite cone.

14. ***The Transverse Diameter*** of an hyperbola, is that part of the axis intercepted between the two opposite cones.

15. ***The Conjugate Diameter*** is a line drawn through the center perpendicular to the transverse diameter

16. ***An Ordinate*** is a line drawn from any point in the curve perpendicular to the axis.

17. ***The Abscissa*** is the part of the axis intercepted between that ordinate and the vertex.

1. ELLIPSE.

$a^2y^2+b^2x^2=a^2b^2$ is the equation of an ellipse referred to the center.

$y^2=\frac{b^2}{a^2}(2ax-x^2)$ is the equation of the ellipse referred to left hand vertex.

In these equations, a is the semi-transverse diameter and b the semi-conjugate diameter; y is any ordinate and x is the corresponding abscissa. When any three of these quantities are given the fourth may be found by solving either of the above equations with reference to the required quantity.

$\rho=\frac{a(1-e^2)}{1-e\cos\theta}$ is the polar equation referred to the centre, and

$\rho=\frac{a(1-e^2)}{1+e\cos\theta}$ is the polar equation referred to the left hand vertex.

Prob. XXXII. To find the circumference of an ellipse, the transverse and conjugate diameters being known.

Formula.—cir. $=C=4\int\sqrt{dy^2+dx^2}=4\int\sqrt{\frac{x^2}{y^2}(1-e^2)^2 dx^2+dx^2}=4\int\frac{\sqrt{y^2+x^2(1-e^2)^2}}{y}dx=4\int\sqrt{\frac{a^2-e^2x^2}{a^2-x^2}}dx=$

$$4\int_0^a\frac{x}{\sqrt{a^2-x^2}}\sqrt{1-\frac{e^2x^2}{a^2}}dx=4\Big(a\ \sin^{-1}\frac{x}{a}-\frac{e^2}{2a}\Big\{\frac{a^2}{2}\sin^{-1}\frac{x}{a}-\frac{x}{2}\sqrt{a^2-x^2}\Big\}-\frac{e^4}{2.4a^3}\Big[\frac{3a^2}{2}\Big\{\frac{a^2}{2}\sin^{-1}\frac{x}{a}-\frac{x}{2}\sqrt{a^2-x^2}-\frac{x^3}{4}$$

$$\sqrt{a^2-x^2}\Big]-\frac{3e^6}{2.4.6a^5}\left\{\frac{5a^2}{6}\Big[\frac{3a^2}{2}\left(\frac{a^2}{2}\sin^{-1}\frac{x}{a}-\frac{x}{2}\sqrt{a^2-x^2}\right)-\frac{x^3}{4}\sqrt{a^2-x^2}\Big]-\frac{x^5}{6}\sqrt{a^2-x^2}\right\}-\&c.\Big)_o^a=4\Big(\frac{\pi a}{2}-\frac{e^2}{2a}\left(\frac{a}{2}\cdot\frac{\pi a}{2}\right)-\frac{e^4}{2.4a^3}\Big[\frac{3a^2}{4}\left(\frac{a}{2}\cdot\frac{\pi a}{2}\right)\Big]-\frac{3e^6}{2.4.6a^5}\left\{\frac{5a^2}{6}\Big[\frac{3a^2}{4}\left(\frac{a}{2}\cdot\frac{\pi a}{2}\right]\right\}\right)-\&c.\Big)=$$

$$2\pi a\left\{1-\frac{e^2}{2.2}-\frac{3e^4}{2.2.4.4}-\frac{3.3.5.e^6}{2.2.4.4.6.6}-\&c,\right\}\text{ in which } e=\sqrt{\frac{a^2-b^2}{a^2}}$$

Rule.—*Multiply the square root of half the sum of the squares of the two diameters by 3.141592, and the product will be the circumference, nearly.*

I. What is the circumference of an ellipse whose axes are 24 and 18 feet respectively?

By formula, $Cir.=C=2\pi\times12\left\{1-\frac{1}{2.2}\left(1-\frac{9^2}{12^2}\right)^2-\frac{3}{2.2.4.4}\left(1-\frac{9^2}{12^2}\right)^4-\&c.\right.=2\pi\times12\times.87947.=66.31056$ ft., nearly.

II
1. 576 sq. ft.$=24^2$=square of the transverse diameter.
2. 324 sq. ft.$=18^2$=square of the conjugate diameter.
3. 900 sq. ft.=sum of the squares of the diameters.
4. 450 sq. ft.=half the sum of the squares of the diameters.
5. $15\sqrt{2}$ft.$=\sqrt{450}$=square root of half the sum of the squares of the diameters.
6. $\pi15\sqrt{2}$ ft.= 66.6434 ft., nearly, =the circumference of the ellipse.

III. ∴ The circumference of the ellipse is 66.6434 ft. nearly, by the rule.

Prob. XXXIII. To find the length of any arc of an ellipse, having given the ordinate, abscissa, and either of the diameters.

Formula.—$s=2[\frac{1}{2}\pi a\{1-(\frac{1}{2})^2\frac{e^2}{1}-(\frac{1}{2}\cdot\frac{3}{4})^2\frac{e^4}{3}-(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6})^2\frac{e^6}{5}-\&c\}]-a\sin^{-1}\frac{x}{a}-\frac{e^2}{2a}\left\{\frac{a^2}{2}\sin^{-1}\frac{x}{a}-\frac{x}{2}\sqrt{a^2-x^2}\right\}-\frac{e^4}{2.4a^3}\Big[\frac{3a^2}{4}\left\{\frac{a^2}{2}\sin^{-1}\frac{x}{a}-\frac{x}{2}\sqrt{a^2-x^2}-\frac{x^3}{4}\sqrt{a^2-x^2}\Big]\right\}-\&c.$, in which x is the abscissa; a the semi-transverse diameter; and $e=\sqrt{\frac{a^2-b^2}{a^2}}$=the eccentricity of the ellipse.

Rule.—*Find the length of the quadrant CB by Prob. XXXI and CF by substituting the value of x in the above series. Twice the difference between these arcs will give the length of the arc FBG.*

I. What is the length of the arc FBG, if $OE=x=9, EF=y=8$, and $OC=b=10$?

Since $a^2y^2+b^2x^2=a^2b^2$, we find, by substituting the values of x, y, and b, $a=15$. Then by the formula,

$$FBG=s=2\left[\tfrac{1}{2}\pi a\left\{1-(\tfrac{1}{2})^2\frac{e^2}{1}-\left(\frac{1}{2}.\frac{3}{4}\right)^2\frac{e^4}{3}-\left(\frac{1}{2}.\frac{3}{4}.\frac{5}{6}\right)^2\frac{e^6}{5}-\&c.\right\}\right.$$

$$-a\sin^{-1}\frac{x}{a}-\frac{e^2}{2a}\left(\frac{a^2}{2}\sin^{-1}\frac{x}{a}-\frac{x}{2}\sqrt{a^2-x^2}\right)$$

$$-\frac{e^4}{2.4a^3}\left[\frac{3a^2}{4}\left(\frac{a^2}{2}\sin^{-1}\frac{x}{a}-\frac{x}{2}\sqrt{a^2-x^2}-\right.\right.$$

$$\left.\left.\left.\frac{x^3}{4}\sqrt{a^2-x^2}\right)\right]-\&c.\right]=\pi 15\left\{1-(\tfrac{1}{2})^2\frac{e^2}{1}\right.$$

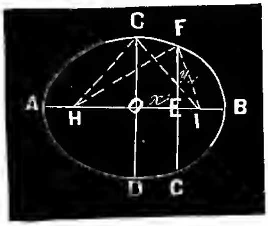

FIG. 18.

$$\left.-(\tfrac{1}{2}.\tfrac{3}{4})^2\frac{e^4}{3}-(\tfrac{1}{2}.\tfrac{3}{4}.\tfrac{5}{6})^2\frac{e^6}{5}-\&c.\right\}-2\left\{15\sin^{-1}\frac{9}{15}\right.$$

$$-\frac{e^2}{2.15}\left(\frac{15^2}{2}\sin^{-1}\frac{9}{15}-\frac{9}{2}\sqrt{15^2-9^2}\right)-\frac{e^4}{2.4.15^3}\left[\frac{3.15^2}{4}\left(\frac{15^2}{2}\sin^{-1}\frac{9}{15}-\right.\right.$$

$$\left.\left.\frac{9}{2}\sqrt{15^2-9^2}-\frac{9^3}{4}\sqrt{15^2-9^2}\right]-\&c.\right\}=\pi 15\times.815-2\left\{\frac{37}{12}\pi-\frac{e^2}{30}\right.$$

$$\left.\left(\frac{185}{8}\pi-72\right)-\frac{e^4}{8.15^3}\left[\frac{3.15^2}{4}\left(\frac{185}{8}\pi-72-3^7\right]-\&c.\right\}=38.406-.$$

Prob. XXXIV. To find the area of an ellipse, the transverse and conjugate diameters being given.

Formula.—$A=4\int ydx=4\frac{b}{a}\int_0^a(\sqrt{a^2-x^2})dx=\pi ab$, in which a and b are the semi-transverse, and semi-conjugate diameters.

Rule.—*Multiply the product of the semi-diameters by $\pi=3.141592$, or multiply the product of the diameters by $\frac{1}{4}\pi=.785398$.*

I. What is the area of an ellipse whose traverse diameter is 70 feet and conjugate diameter 50 feet?

By formula, $A=\pi ab=\pi 35\times 25=2748.893$ sq. ft.

II. 1. 35 ft.$=\frac{1}{2}$ of 70 ft.=length of the semi-transverse diameter.
  2. 25 ft.$=\frac{1}{2}$ of 50 ft.=length of the semi-conjugate diameter.
  3. ∴ 2748.893 sq. ft.$=\pi\times 35\times 25$=the area of the ellipse.

III. ∴ The area of the ellipse is 2748.893 sq. ft.

NOTE.—$\pi ab=\sqrt{\pi a^2.\pi b^2}$. ∴ The area of an ellipse is a mean proportional between the circumscribed and inscribed circles.

Prob. XXXV. To find the area of an elliptic segment, having given the base of the segment, its height, and either diameter of the ellipse, the base being parallel to either diameter.

Formulae.—(a) $A=\int ydx$, or $\int xdy=\frac{b}{a}\int x(a^2-x^2)^{\frac{1}{2}}dx=$ $ab[1-\frac{1}{2}\left(\frac{x}{a}\right)^2-2\left(\frac{x}{2a}\right)^4-4\left(\frac{x}{2a}\right)^6-2.5\left(\frac{x}{2a}\right)^8-2^2.7\left(\frac{x}{2a}\right)^{10}-2^2.3.7\left(\frac{x}{2a}\right)^{12}-\&c.]$, or $\frac{a}{b}\int(b^2-y^2)^{\frac{1}{2}}dy=ab[1-\frac{1}{2}\left(\frac{y}{b}\right)^2-2\left(\frac{y}{2b}\right)^4-4\left(\frac{y}{2b}\right)^6-2.5\left(\frac{y}{2b}\right)^8-2^2.7\left(\frac{y}{2b}\right)^{10}-2^2.3.7\left(\frac{y}{2b}\right)^{12}-\&c.]$

(b) $A=\int ydx=\frac{b}{a}\left[x(a^2-x^2)^{\frac{1}{2}}+a^2\sin^{-1}\frac{x}{a}\right]$.

The former formula of (a) gives the area of a segment whose base is parallel to the conjugate diameter and the latter the area of a segment whose base is parallel to the transverse diameter.

Rule.—*Find the area of the corresponding segment of the circle described upon the same axis to which the base of the segment is perpendicular. Then this axis is to the other axis as the area of the circular segment is to the area of the elleptic segment.*

2. PARABOLA.

$y^2=2px$ is the rectangular equation of the parabola.

$\rho=\frac{p}{1-\cos\theta}$ is the polar equation of the parabola.

In the rectangular equation, $HG=y$, the ordinate; $AG=x$, the abscissa; $AF=AE=\frac{1}{2}p$. If any two of these are given the remaining one may be found from the equation. p is a constant quantity.

Prob. XXXVI. To find the length of any arc of a parabola cut off by a double ordinate.

Formula.—$s=2\int\sqrt{dy^2+dx^2}=\frac{2}{p}\int(p^2+y^2)^{\frac{1}{2}}dy=$ $\frac{y}{p}\sqrt{p^2+y^2}+p\log[y+\sqrt{p^2+y^2}]+C=\frac{y}{p}\sqrt{p^2+y^2}+$ $p\log\left[\frac{y+\sqrt{p^2+y^2}}{p}\right]$, or $\frac{2}{p}\int(p^2+y^2)^{\frac{1}{2}}dy=2\,(y+\frac{1}{2}\cdot\frac{1}{3}\frac{y^3}{p^2}-\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{5}\frac{y^5}{p^4}+\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{7}\frac{y^7}{p^6}-\&c)$.

Rule.— *When the abscissa is less than half the ordinate: To the square of the ordinate add $\frac{4}{3}$ of the square of the abscissa and twice the square root of the sum will be the length of the arc.*

I. What is the length of the arc *KAH*, if *AG* is 2 and *GH* 6?

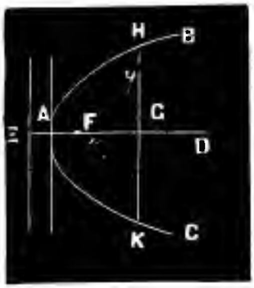

FIG. 19.

By formula, $s=\frac{y}{p}\sqrt{p^2+y^2}+p\log\left[\frac{y+\sqrt{p^2+y^2}}{p}\right]=\frac{6}{p}\sqrt{p^2+6^2}+p\log\left[\frac{6+\sqrt{p^2+6^2}}{p}\right]$. Since $y^2=2px$, we have $p=\frac{y^2}{2x}=\frac{36}{4}=9$. $\therefore s=\frac{2}{3}(3\sqrt{13})+9\log\left[\frac{6+3\sqrt{13}}{9}\right]=2\sqrt{13}+9\log\frac{1}{3}(2+\sqrt{13})$, or by series, $s=2(y+\frac{1}{2}\cdot\frac{1}{3}\frac{y^3}{p^2}-\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{5}\frac{y^5}{p^4}+\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{7}\frac{y^7}{p^6}-\&c.]=12.7105$ =length of the arc, nearly.

II.
1. $2=AG=$the abscissa.
2. $6=GH=$the ordinate.
3. $36=$the square of the ordinate.
4. $\frac{16}{3}=\frac{4}{3}$ of $2^2=\frac{4}{3}$ of the square of the abscissa.
5. $2\sqrt{(\frac{16}{3}+36)}=12.858=$the length of the arc, nearly.

III. . . 12.858=length of the arc, nearly.

Prob. XXXVII. To find the area of a parabola, the base and height being given.

Formula.—$A=2\int ydx=2\int\frac{1}{p}y^2dy=\frac{2y^3}{3p}=\frac{4}{3}xy=\frac{2}{3}(x.\ 2y)$, i. e., the area of parabola *HKA* is $\frac{2}{3}$ of the circumscribed rectangle.

Rule.—*Multiply the base by the height and $\frac{2}{3}$ of the product will be the area.*

I. What is the area of a parabola whose double ordinate is 24m. and altitude 16m.?

By formula, $A=\frac{2}{3}(x.\ 2y)=\frac{2}{3}(16\times24)=256\text{m}^2$.

II.
1. 24m.$=HK$(in last figure)=the double ordinate, or base of the parabola.
2. 16m.$=AG=$the altitude of the parabola.
3. $\therefore 384\text{m}^2=16\times24=$the area of the rectangle circumscribed about the parabola.
4. $\frac{2}{3}$ of $384\text{m}^2=256\text{m}^2=$the area of the parabola.

III. $\therefore$ The area of the parabola is 256m^2.

Prob. XXXVIII. To find the area of a parabolic frustum having given the double ordinates of its ends and the distance between them.

Formula.—$A=\frac{2}{3}a\times\frac{B^3-b^3}{B^2-b^2}$, in which a is the distance between the double ordinates, B the greater and b the lesser double ordinate.

Rule.—*Divide the difference of the cubes of the two ends by the difference of their squares and multiply the quotient by $\frac{2}{3}$ of the altitude.*

I. What is the area of a parabolic frustum whose greater base is 10 feet, lesser base 6 feet, and the altitude 4 feet?

By formula, $A=\frac{2}{3}a\times\frac{B^3-b^3}{B^2-b^2}=\frac{2}{3}\times4\times\frac{10^3-6^3}{10^2-6^2}=32\frac{2}{3}$ sq. ft.

II.
1. 10 ft.=the greater base,
2. 6 ft.=the lesser base, and
3. 4 ft.=the altitude.
4. 784 cu. ft.$=10^3-6^3$=the difference of the cubes of the two bases.
5. 64 sq. ft.$=10^2-6^2$=the difference of the squares of the two bases.
6. $12\frac{1}{4}$ ft.$=784\div64$=the quotient of the difference of the cubes by the difference of the squares.
7. $\therefore \frac{2}{3}\times(4\times12\frac{1}{4})=32\frac{2}{3}$ sq. ft.=the area of the frustum.

III. $\therefore$ The area of the frustum is $32\frac{2}{3}$ sq. ft.

3. HYPERBOLA.

1. $a^2y^2-b^2x^2=-a^2b^2$ is the equation of the hyperbola referred to its axes in terms of its semi-axes.

2. $y^2=-\frac{b^2}{a^2}(2ax-x^2)$ is the equation of a hyperbola referred to its transverse axis and a tangent at the left hand vertex.

3. $\rho=\frac{a(1-e^2)}{1-e\cos\theta}$ is the polar equation of the hyperbola.

Having given any three of the four quantites a, b, x, y, the other may be found by solving the rectangular equation with reference to the required quantity.

Prob. XXXIX. To find the length of any arc of an hyperbola, beginning at the vertex.

Formula.—$s=\sqrt{dy^2+dx^2}=\int\left(\frac{(a^2+b^2)y^2+b^4}{b^2(y^2+b^2)}\right)dx=$ $y\left(1+\frac{1}{1.2.3}\frac{a^2x^2}{b^4}-\frac{1.1.3}{1.2.3.4.5}\frac{a^4+4a^2b^2}{b^8}y^4+\frac{1.1.3.3.5}{1.2.3.4.5.6.7}\frac{a^6+4a^4b^2+8a^2b^4}{b^{12}}y^6-\&c.\right)$

Rule.—*1. Find the parameter by dividing the square of the conjugate diameter by the transverse diameter.*

2. To 19 times the transverse, add 21 times the parameter of the axis, and multiply the sum by the quotient of the abscissa divided by the transverse.

3. To 9 times the transverse, add 21 times the parameter, and multiply the sum by the quotient of the abscissa divided by the transverse.

4. To each of the products thus found, add 15 times the parameter, and divide the former by the latter; then this quotient being multiplied by the ordinate will give the length, nearly.

(*Bonnycastle's Rule.*)

Note.—*A parameter* is a double ordinate passing through the focus.

I. In the the hyperbola DAC, the transverse diameter GA $=80$, the conjugate $HI=60$, the ordinate $BC=10$, and the abscissa $AB=2.1637$; what is the length of the arc DAC?

By formula, $DAC=$

$$s=2x\left(1+\frac{1}{1.2.3}\frac{a^2x^2}{b^4}-\frac{1.1.3}{1.2.3.4.5}\frac{a^4+4a^2b^2}{b^8}x^4+\frac{1.1.3.3.5}{1.2.3.4.5.6.7}\frac{a^6+4a^4b^2+8x^2b^4}{b^{12}}\right)=$$

20.658.

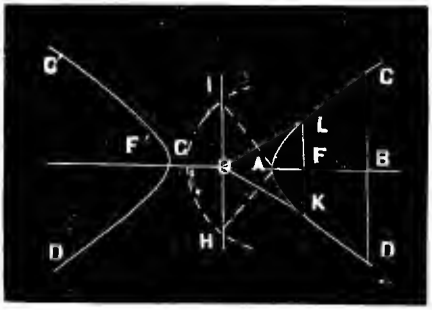

FIG. 20.

II.

1. $45=2OI^2\div OA=2b^2\div a=$the parameter LK which, in the figure, should be drawn to the right of DC, to be consistent with the nature of the problem.
2. $1520=19\times 80=19$ times the transverse diameter.
3. $945=21\times 45=21$ times the parameter.
4. $2465=$sum of these two products.
5. $.02704=2.1637\div 80=$quotient of abscissa and transverse diameter.
6. $2465\times .02704=66.6536=$sum of the *products* multiplied by the said quotient. Also,
7. $[(80\times 9)+(45\times 21)]\times\frac{2.1637}{80}=(720+945)\times .02704=$ 45.0216. Whence
8. $(15\times 45+66.6536)\div(15\times 45+45.0216)=741.6536\div$ $720.0216=1.03004$.
9. $\therefore 1.03004\times 10=10.3004=$length of the arc AC, nearly.

III. ∴ The length of the arc is 10.3004.

Prob. XL. To find the area of an hyperbola, the transverse and conjugate axes and abscissa being given.

Formulae.—(*a*) $A=2\int ydx=2\frac{b}{a}\int_a^{x'}(x^2-a^2)^{\frac{1}{2}}dx=\frac{b}{a}x'\sqrt{x'^2-a^2}-ab\log\left[\frac{x'+\sqrt{x'^2-a^2}}{a}\right]=x'y'-ab\log\left[\frac{x'+\sqrt{x'^2-a^2}}{a}\right]$; or, (*b*) $A=4xy\left\{\frac{1}{3}-\frac{1}{1.3.5}\frac{x^2}{x^2+x^2}-\frac{1}{3.5.7.}\left(\frac{x^2}{a^2+x^2}\right)^2-\frac{1}{5.7.9}\left(\frac{x^2}{a^2+x^2}\right)^3-\&c.\right\}$

Rule.—*1. To the product of the transverse diameter and abscissa, add $\frac{5}{7}$ of the square of the abscissa, and multiply the square root of the sum by 21.*

2. Add 4 times the square root of the product of the transverse diameter and abscissa, to the product last found and divide the sum by 75.

3. Divide 4 times the product of the conjugate diameter and abscissa by the transverse diameter, and this last quotent multiplied by the former will give the area required, nearly.—Bonnycastle's Rule.

I. If, in the hyperbola *DAC*, the transverse axis *AG* is 30, the conjugate diameter *HI*, 18 and the abscissa *AB*, 10; what is the area of the hyperbola *DAC*?

By formula (*a*), $A=x'y'-ab\log_e\left[\frac{x'+\sqrt{x'^2-a^2}}{a}\right]=25y'-15\times9\log_e\left[\frac{25+\sqrt{26^2-15^2}}{15}\right]=300-135\log_e\left[\frac{25+\sqrt{400}}{15}\right]=300-135\log_e 3=300-135\times1.09861228=151.687343$, y' being found from the equation $a^2y'^2-b^2x'^2=-a^2b^2$. in which $a=15$, $b=9$ and $x'=15+10=25$.

II.
1. *1.* $21\sqrt{30\times10+\frac{5}{7}\times10^2}=21\sqrt{300+500\div7}=21\sqrt{371.42857}=21\times19.272=404.712$, by the first part of the rule.
2. *2.* $(4\sqrt{30\times10}+404.712)\div75=(4\times17.3205+404.712)\div75=6.3199$, by the second part of the rule.
3. *3.* $\therefore\frac{18\times10\times4}{30}\times6.3199=151.6776$, by the third part of the rule, =the area of the hyperbola, nearly.

III. ∴ 151.6776=the area of the hyperbola.

Prob. XLI. To find the area of a zone of an hyperbola.

$$\textit{Formula.}—A=2\frac{b}{a}\int_{x_1}^{x_2}(x^2-a^2)^{\frac{1}{2}}dx$$

$$=\frac{b}{a}x_2\sqrt{x_2^2-a^2}-ab\log_e\left[\frac{x_2+\sqrt{x_2^2-a^2}}{a}\right]-\frac{b}{a}x_1\sqrt{x_1^2-a^2}+$$

$$ab\log_e\left[\frac{x_1+\sqrt{x_1^2-a^2}}{a}\right]=x_2y_2-x_1y_1-ab\log_e\left[\frac{x_2+\sqrt{x_2^2-a^2}}{a}\right]+$$

$$ab\log_e\left[\frac{x_1+\sqrt{x_1^2-a^2}}{a^2}\right]=x_2y_2-x_1y_1-ab\log_e\left[\frac{x_2+\sqrt{x_2^2-a^2}}{x_1+\sqrt{x_1^2-a^2}}\right],$$

in which (x_2, y_2), and (x_1, y_1) are the co-ordinates of the points C and L respectively.

I. What is the area of a zone of an hyperbola whose transverse diameter is $2a=10$ feet, conjugate diameter $2b=6$ feet, the lesser double ordinate of the zone being 8 feet and the greater 12 feet?

By formula, $A=x_2y_2-x_1y_1-ab\log_e\left\{\frac{x_2+\sqrt{(x_2^2-a^2)}}{x_1+\sqrt{(x_1^2-a^2)}}\right\}$

$=6x_2-4x_1-15\log_e\left(\frac{bx_2+ay_2}{bx_1+ay_1}\right)=6x_2-4x_1-15\log_e\left(\frac{3x_2+30}{3x_1+20}\right)$,

But from the equation, when $y=y_2=6$, $x=x_2=10\sqrt{6}$ and when $y=y_1=4$, $x=x_1=13\frac{1}{3}$. Substituting these values of x_2 and y_2,

we have $A=50\sqrt{6}-66\frac{2}{3}-30\log_e\left(\frac{30\cdot\sqrt{6}+30}{50+20}\right)$

$=\left\{50\sqrt{6}-66\frac{2}{3}-30\log_e[\frac{3}{7}(\sqrt{6}+1)]\right\}$ sq. ft.

Prob. XLII. To find the area of a sector of an hyperbola, KALO.

$$\textit{Formula.}—A=ab\log_e\left(\frac{x}{a}+\frac{y}{b}\right).$$

Rule.—*Find the area of the segment AKL by Prob. XL., and subtract it from the area of the triangle KOL.*

I. What is the area of the sector OAL (Fig. 20) if $OA=a=5$, $OI=b=3$, and $LF=y=4$?

By formula, $A=\frac{1}{2}ab\log_e\left(\frac{x}{a}+\frac{y}{b}\right)=7\frac{1}{2}\log_e(\frac{1}{5}x+\frac{4}{3})=7\frac{1}{2}$ $\log_e[\frac{1}{15}(3x+20)]$. But when $y=4$, $x=13\frac{1}{3}$. Hence, $A=7\frac{1}{2}\log_e(\frac{14}{3})$.

IX. HIGHER PLANE CURVES.

1. ***Higher Plane Curves*** are loci whose equations are above the second degree, or which involve transcendental functions, *i. e.*, a function whose degree is infinite.

1. THE CISSOID OF DIOCLES.

1. ***The Cissoid of Diocles*** is the curve generated by the vertex of a parabola rolling on an equal parabola.

2. If pairs of equal ordinates be drawn to the diameter of a circle, and through one extremity of this diameter and the point in the circumference through which one of the ordinates is let fall, a line be drawn, the locus of the intersection of this line and the equal ordinate, or that ordinate produced is the *Cissoid of Diocles*.

3. $y^2=\dfrac{x^3}{2a-x}$ is the equation of the cissoid referred to rectangular axes.

$\rho=2a\sin\theta\tan\theta$ is the polar equation of the curve.

Prob. XLIII. To find the length of an arc OAP of the cissoid.

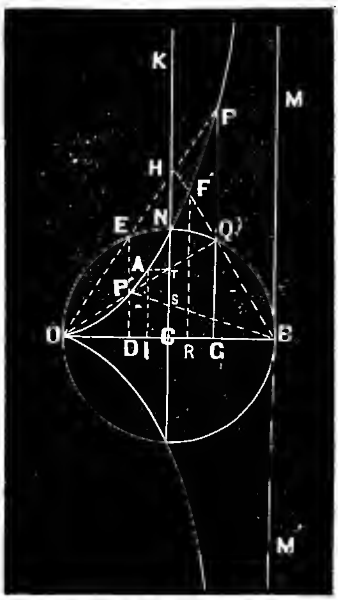

Fig. 21.

Formula.—$s=OAP=$

$$\int\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=$$

$$\int\sqrt{1+\left(\frac{(3a-x)\sqrt{x}}{\sqrt{(2a-x)^3}}\right)^2}\,dx=$$

$$a\int\sqrt{\frac{8a-3x}{(2a-x)^3}}\,dx=a\left\{\sqrt{\frac{8a-3x}{2a-x}}\right.$$

$$\left.-2+3\log_e\left[\frac{\sqrt{2a}(\sqrt{3}+2)}{\sqrt{3}\sqrt{2a-x}+\sqrt{8a-3x}}\right]\right\}$$

I. What is the length of the arc OAN, in which case $x=a$?

By formula, $s=a\left\{\sqrt{5}-\right.$

$$\left.2+3\log_e\left[\frac{\sqrt{2}(\sqrt{3}+2)}{\sqrt{3}+\sqrt{5}}\right]\right\}$$

Prob. XLIV. To find the area included between the curve and its asymptote, BM.

Formula.—$A=2\int_o^{2a} y\,dx=2\int_o^{2a}\sqrt{\frac{x^3}{2a-x}}dx=\Big[-\frac{1}{2}\sqrt{x}.\sqrt{2a-x}(a+x)-3\sin^{-1}\frac{\sqrt{2a-x}}{\sqrt{2a}}\Big]_o^{2a}=3\pi a^2$, *i. e.*, 3 times the area of the circle, *OEB*.

Note.—The name Cissoid is from the Greek κισσοειδής, like ivy, from κισσός, ivy, εἶδος form. The curve was invented by the Greek geometer Diocles, A D. 500, for the purpose of solving two celebrated problems of the higher geometry; viz., to trisect a plane angle, and to construct two geometrical means between two given straight lines. The construction of two geometrical means between two given straight lines is effected by the *cissoid*. Thus in the figure of the cissoid, *ED* and *OG* are the two geometrical means between the straight lines *OD* and *PG*: that is, $OD:ED::OG:PG$. The trisection of a plane angle is effected by the conchoid. The duplication of the cube, *i. e.*, to find the edge of a cube whose volume shall be twice that of a cube whose volume is given, may be effected by the cissoid Thus, on *KC* lay off $CH=2BC$, and draw *BH*. Let fall from the point *F*, where *BH* cuts the curve, the perpendicular *FR*. Then $RF=2BR$. Now a cube described on *RF* is twice one described on *OR*; for, since $FR=y, OR=x$, and $BR=2a-x$, we have $RF^2=\frac{OR^3}{BR}=\frac{OR^3}{\frac{1}{2}FR}$, or $\frac{1}{2}RF^3=OR^3$. $\therefore FR^3=2OR^3$ Q. E. D.

2. THE CONCHOID OF NICOMEDES.

1. ***The Conchoid*** is the locus formed by measuring, on a line which revolves about a fixed point without a given fixed line, a constant length in either direction from the point where it intersects the given fixed line.

2. $x^2y^2=(b+y)^2(a^2-y^2)$ is the equation of the *conchoid* referred to rectangular axes.

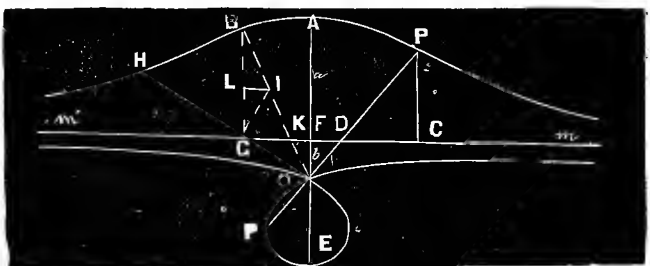

FIG. 22.

3. $\rho=b\sec\theta\pm a$ is the polar equation referred to polar co-ordinates. In this equation, θ is the angle *PO* makes with *AO*.

Prob. XLV. To find the length of an arc of the conchoid.

Formula.—$s=\int\sqrt{1+\left(\frac{dr}{d\theta}\right)^2}\,d\theta=\int\sqrt{1+\tan^2\theta\sec^2}d\theta.$

Prob. XLVI. To find the whole area of the conchoid between two radiants each making an angle θ with OA.

Formula.—$A=2\int\frac{1}{2}r^2d\theta=b^2\int(\sec\theta\pm a)^2d\theta=$ $b^2\tan\theta+2a^2\theta+3b\sqrt{a^2-b^2}$ or $b^2\tan\theta+2a^2\theta$, according as a is or is not greater than b. 1. The area above the directrix $m\,m'$ and the same radiants $=2ab\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)+a^2\theta$.

2. The area of the loop which exists when a is $>b$ is $a^2\cos^{-1}\frac{b}{a}-2ab\log\left\{\frac{b+\sqrt{a^2-b^2}}{b-\sqrt{a^2-b^2}}\right\}+b\sqrt{a^2-b^2}$.

NOTE.—The name conchoid is from the Greek, χογχοειδές, from κόγχη, shell, and εἶδος, form, and signifies shell-form. It was invented by the Greek geometer Nicomedes, about A. D. 100 for the purpose of trisecting any plane angle. The trisection of an angle may be accomplished by this curve as follows: Let AOH be any angle to be trisected. From any point, G, in one side let fall a perpendicular, GF, upon the other. Take $AF=2GO$, and with O as the fixed point, $m\,m'$ as the fixed line and PP as the revolving line of which $PD=a$ is constant, construct the arc of the conchoid, PAH. Erect BG perpendicular to $m\,m'$ and draw BO. Then is BOA one third of HOA. For bisect BK at I, and draw GI. Also draw IL parallel to GK. Since $BI=IK$, $BL=LG$ and $GI=BI=IK=GO$. By reason of the isosceles triangle BIG, we have the angle $GIO=2\angle GBO=2\angle BOA$. But $\angle GIO=\angle IOG$. $\therefore 2\angle IOA=\angle IOG$, or $IOA=\frac{1}{3}\angle HOA$.
Q. E. F.

3. THE OVAL OF CASSINI OR CASSINIAN.

1. ***The Oval of Cassini*** is the locus of the vertex of the triangle whose base is $2a$ and the product of the other sides $=m^2$.

2. $\left\{y^2+(a+x)^2\right\}\left\{y^2+(a-x)^2\right\}=m^4$ or $(x^2+y^2+a^2)^2-4a^2x^2=m^4$ is the rectangular equation of the curve, in which $2a=AB$.

3. $r^4-2a^2r^2\cos2\theta+a^2-m^4=0$ is the polar equation of the curve.

Discussion.—If a be $>m$, there are two ovals, as shown in the figure. In that case, the last equation shows that if OPP' meets the curve in P and P', we have $OP.OP'=\sqrt{a^4-m^4}$; and therefore the curve is its own inverse with respect to a circle of radius$=\sqrt[4]{a^4-m^4}$.

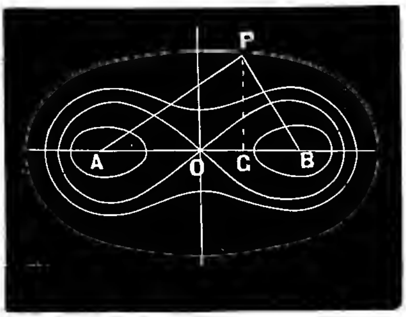

FIG. 23.

4. LEMNISCATE OF BERNOUILLI.

1. This curve is what a *cassinian* becomes when $m=a$. The above equations then reduce to

2. $(x^2+y^2)^2=2a^2(x^2-y^2)$ and

3. $r^2=2a^2\cos 2\theta$.

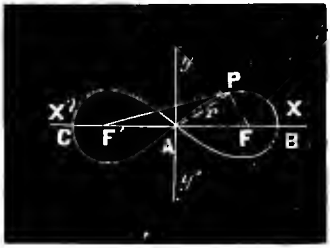

FIG. 24.

Prob. XLVII. To find the length of the arc of the Lemniscate.

Formula.—$s=\int\sqrt{1+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$

$=\int\sqrt{r^2+\frac{a^4}{r^2}\left(1-\frac{r^4}{a^4}\right)}\,d\theta=\int\frac{a^2}{r}d\theta=\int_0^a\frac{a^2dr}{\sqrt{(a^4-r^4)}}=$

$a^2\int_0^a\left[\frac{1}{a^2}+\frac{1}{2}\cdot\frac{r^4}{a^6}+\frac{1}{2}\cdot\frac{3}{4}\frac{r^8}{a^{10}}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\frac{r^{12}}{a^{14}}+\&c.\right]dr=a^2\left\{\frac{r}{a^2}+\frac{1}{2}\cdot\frac{1}{5}\frac{r^5}{a^6}\right.$ $\left.+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{9}\cdot\frac{r^9}{a^{10}}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{1}{13}\frac{r^{13}}{a^{14}}+\&c.\right\}$. When $r=a$, $s=a\left[1+\frac{1}{2}\cdot\frac{1}{5}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{9}\right.$ $\left.+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{1}{13}+\&c\right]=$ arc BPA. $\therefore$ The entire length of the curve is $4a[1+\frac{1}{2}\cdot\frac{1}{5}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{9}+\&c]$

Prob. XLVIII. To find the area of the lemniscate.

Formula.—$A=4\int\frac{1}{2}r^2d\theta=4a^2\int_0^{\frac{1}{4}\pi}\cos 2\theta\,d\theta=$ $\left[2a^2\sin 2\theta\right]_0^{\frac{1}{4}\pi}=2a^2$.

5. THE VERSIERA OR WITCH OF AGNESI.

1. *The Versiera* is the locus of the extremity of an ordinate to a circle, produced until the produced ordinate is to the ordinate itself, as the diameter of a circle is to one of the segments into which the ordinate divides the diameter, these segments being all taken on the same side.

2. Let P be any point of the curve, $PD=y$, the ordinate of the point P and $OD=x$, the abscissa. Then, by definition, $EP:EF::AO:EO$, or $x:EF::2a:y$. But $EF=\sqrt{AE\times EO}=\sqrt{(2a-y)y}$. $\therefore x:\sqrt{(2a-y)y}::2a:y$. Whence $x^2y=4a^2(2a-y)$ is the equation referred to rectangular co-ordinates.

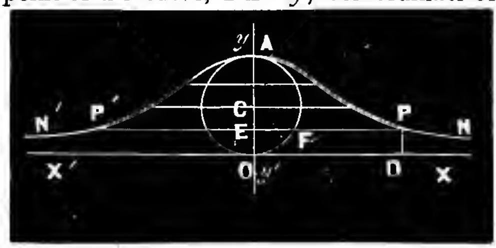

FIG. 25.

3. $r(r^2-r^2\sin^2\theta+4a^2)\sin\theta=8a^3$ is the polar equation of the curve.

Prob. XLIX. To find the length of an arc of the Versiera.

$$\textit{Formula.}\text{—}s=\int\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx=\int\sqrt{1+\frac{4x^2}{(x^2+4a^2)^4}}dx.$$

This can be integrated by series and the result obtained approximately.

Prob. L. To find the area between the curve and its asymptote.

$$\textit{Formula.}\text{—}A=2\int ydx=2\times8a^3\int_o^{2a}\frac{dx}{x^2+4a^2}=$$

$$2\left[4a^2\tan\frac{x}{2a}\right]_o^{2a}=4\pi a^2.$$

Rule.—*Multiply the area of the given circle by 4.*

NOTE.—This curve was invented by an Italian lady, Dona Maria Agnesi, Professor of Mathematics at Bologna, 1748.

6. THE LIMAÇON.

1. ***The Limaçon*** is the locus of a point P on the radius vector OP, of a circle OFE from a fixed point, O, on the circle and at a constant distance from either side of the circle.

2. $(x^2+y^2-ax)^2=b^2(x^2+y^2)$ is the rectangular equation of the curve.

3. $r=a\cos\theta\pm b$ is the polar equation. In these equations, $a=OA$ and $b=PF$.

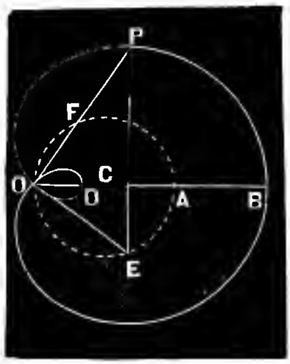

FIG. 26.

Prob. LI. To find the length of an arc of the Limaçon.

$$\textit{Formula.}\text{—}s=\int\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}d\theta=$$

$$\int\sqrt{(a\cos\theta+b)^2+a^2\sin^2\theta}\,d\theta=\int\sqrt{(a^2+b^2+2ab\cos\theta)}\,d\theta=$$

$$\int\sqrt{\left\{(a+b)^2\cos^2\frac{\theta}{2}+(a-b)^2\sin^2\frac{\theta}{2}\right\}}d\theta.$$ $\therefore$ The rectification of the Limaçon depends on that of an ellipse whose semi-axes are $(a+b)$ and $(a-b.)$

When $a=b$, the curve is the *cardioid*, the polar equation of which is $r=a(1+\cos\theta)$, and $s=\int\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}d\theta=$

$a\int\sqrt{2+2\cos\theta}\,d\theta=\pm 2a\int\cos\frac{1}{2}\theta\,d\theta=2a\int_0^{\pi}\cos\frac{1}{2}\theta\,d\theta-$

$2a\int_{\pi}^{2\pi}\cos\frac{1}{2}\theta\,d\theta=8a=$the entire length of the cardioid.

Prob. LII. To find the area of the Limaçon.

Formula.—$A=\int\frac{1}{2}r^2\,d\theta=\frac{1}{2}\int_0^{2\pi}(a\cos\theta+b)^2 d\theta=$

$\pi(\frac{1}{2}a^2+b^2)$. When $a=b$, the curve becomes a cardioid, and $A=\frac{3}{2}\pi a^2$. When $a>b$, the curve has two loops and is that in the figure. $r=a\cos\theta+b$ is the polar equation of the outer loop, and $r=a\cos\theta-b$ is the polar equation of the inner loop. The area of the inner loop is $A=\int\frac{1}{2}r^2 d\theta=\frac{1}{2}\int_0^{\cos^{-1}\frac{b}{a}}(a\cos\theta-b)^2 d\theta=$

$\frac{1}{2}(a^2+b^2)\cos^{-1}\frac{b}{a}-\frac{3}{2}b\sqrt{a^2-b^2}$.

NOTE.—This curve was invented by Blaise Pascal in 1643. When $a=2b$, the curve is called the *Trisectrix*.

7. THE QUADRATRIX.

1. The Quadratrix is the locus of the intersection, P, of the radius, OD, and the ordinate QN, when these move uniformally, so that $ON:OA::\angle BOD:\frac{1}{2}\pi$.

2. $y=x\tan\left(\frac{a-x}{a}\cdot\frac{\pi}{2}\right)$ is the rectangular equation of the curve, in which $a=OA$, $x=ON$, and $y=IN$.

3. The curve effects the quadrature of the circle, for $OC:OB::OB:$arc ADB.

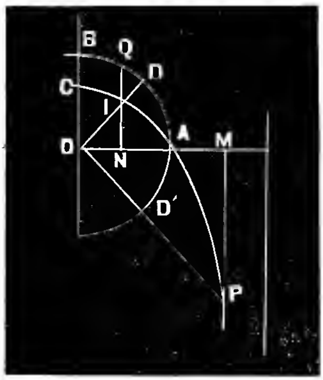

FIG. 27.

Prob. LIII. To find the area enclosed above the x—axis.

Formula.—$A=\int y\,dx=\int x\tan\left(\frac{a-x}{a}\cdot\frac{\pi}{2}\right)dx=4a^2\pi^{-1}\log 2.$

NOTE.—This curve was invented by Dinostratus, in 370 B. C.

8. THE CATENARY.

1. ***The Catenary*** is the line which a perfectly flexible chain assumes when its ends are fastened at two points as B and C in the figure.

2. $y=\frac{1}{2}a(e^{\frac{x}{a}}+e^{-\frac{x}{a}})$ is the rectangular equation of the curve, in which $a=OA$. A is the origin of co-ordinates. BAC is the catenary. $M'APM$ is the evolute of the catenary and is called the *Tractrix*. To find the equation of the curve, let A be the origin of co-ordinates. Let s denote the length of any arc AE; then, if p be the weight of a unit of length of the chain, the verticle tension at E, is sp. Let the horizontal

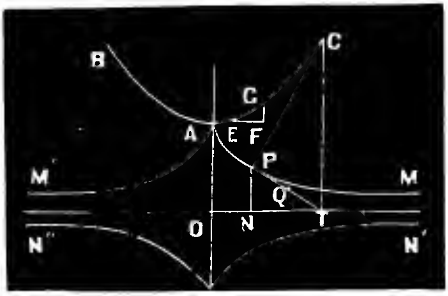

FIG. 28.

tension at E, be ap, the weight of a units of length of the chain. Let EG be a tangent at E, then, if EG represents the tension of the chain at B, EF and GF will represent respectively its horizontal and its vertical tension at B.

$\therefore\ \frac{dy}{dx}=\frac{FG}{EF}=\frac{sp}{ap}=\frac{s}{a}$. $\therefore\ \frac{s}{a}=\frac{\sqrt{ds^2-dx^2}}{dx}$. $\therefore\ x=a\int\frac{ds}{\sqrt{a^2+s^2}}=$ $a\log(s+\sqrt{a^2+s^2})+c$. Since $x=o$, when $s=o$, $c=-a\log a$.

$\therefore\ x=a\log\left(\frac{s}{a}+\sqrt{1+\frac{s^2}{a^2}}\right)$. From this equation, we find $s=$ $\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)$ which is the length of the curve measured from A.

But $\frac{dy}{dx}=\frac{s}{a}$. $\therefore\ \frac{dy}{dx}=\frac{s}{a}=\frac{1}{2}\left(e^{\frac{x}{a}}-e^{-\frac{x}{a}}\right)$.

$\therefore\ y=\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)$.

Prob. LIV. To find the area included between the Catenary, the axis of x, and two ordinates.

Formula.—$A=\int ydx=\int\frac{1}{2}a\left\{e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right\}dx=$

$\frac{1}{2}a^2\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)=a\sqrt{y^2-a^2}$. This is the area included between the axis of x, the curve and the two ordinates, $y_1=a$, $y_2=y$.

NOTE.—The form of equilibrium of a flexible chain was first investigated by Galileo, who pronounced the curve to be a parabola. His error was detected experimentally, in 1669, by Joachim Jungius, a German geometer; but the true form of the Catenary was obtained by James Bernouilli, in 1691.

9. THE TRACTRIX.

1. ***The Tractrix*** is the involute of the Catenary.

2. $x = a \log \left\{ a + \sqrt{(a^2 - y^2)} \right\} - a \log y - \sqrt{(a^2 - y^2}$, is the rectangular equation of the curve.

Prob. LV. To find the length of an arc of the tractrix.

Formula.—$s = a \log \left(\frac{a}{y}\right)$.

Prob. LVI. To find the area included by the four branches.

Formula.—$A = \int y dx = -4\int_o^a \sqrt{a^2 - y^2}\, dy = \pi a^2$.

10. THE SYNTRACTRIX.

1. ***The Syntractrix*** is the locus of a point, Q, on the tangent, PT, of the Tractrix.

2. $x = a \log \left\{ c + \sqrt{(c^2 - y^2)} \right\} - a \log y - \sqrt{(c^2 - y^2)}$ is the rectangular equation of the *Syntractrix*, in which c is QT, a constant length.

11. ROULETTES.

1. ***A Roulette*** is the locus of a point rigidly connected with a curve which rolls upon a fixed right line or curve.

(a). *Cycloids.*

1. ***The Cycloid*** is the roulette generated by a point in the circumference of a circle which rolls upon a right line.

2. ***A Prolate Cycloid*** is the roulette generated by a point without the circumference of a circle which rolls upon a right line.

3. ***A Curtate Cycloid*** is the roulette generated by a point within the circumference of a circle which rolls upon a right line.

4. $x = \text{versin}^{-1} y - \sqrt{2ry - y^2}$ is the rectangular equation of the cycloid referred to its base and a perpendicular at the left hand vertex. To produce this equation, let $AN = x$ and $PN = y$, P being any point of the curve. Let $OC = r =$ the radius of the generatrix OPL. Now $AN = AO - NO$. But by construction $AO =$ arc $PO = \text{versin}^{-1} FO$, or $\text{versin}^{-1} y$ to a radius r. $NO = PF = \sqrt{FL \times FO} = \sqrt{y(2r - y)} = \sqrt{2ry - y^2}$. $\therefore$ $x = \text{versin}^{-1} y - \sqrt{2ry - y^2}$

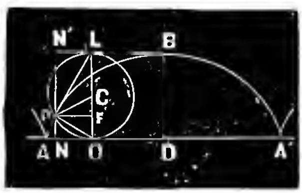

FIG. 29.

Or, we may have $x = a(\theta - \sin\theta)$, and $y = a(1 - \cos\theta)$ in which θ is the angle, PCO, through which the generatrix has rolled.

For $x=AO-NO$. But $AO=a\angle PCO=a\theta$, and $NO=PF=PC\sin\angle PCF=a\sin\theta$. $\therefore x=a\theta-a\sin\theta=a(\theta-\sin\theta)$, $y=OC-CF=a-CF$. But $CF=PC\cos\angle PCF=a\cos\theta$ $\therefore$ $y=a-a\cos\theta=a(1-\cos\theta)$.

Prob. LVII. To find the length of an arc of the cycloid.

Formula.—$s=\int\sqrt{1+\left(\frac{dx}{dy}\right)^2}dy=\int\sqrt{1+\frac{y^2}{2ry-y^2}}dy=\sqrt{2r}\int(2r-y)^{-\frac{1}{2}}dy=-2\sqrt{2r}(2r-y)^{\frac{1}{2}}+c$. Reckoning the arc from the origin, $c=4r$; and the corrected integral is $s=-2(2r)^{\frac{1}{2}}(2r-y)^{\frac{1}{2}}+4r$. When $y=2r$, $s=4r$. $\therefore$ The whole length of the cycloid is $8r=4D$, *i e.*, the length of the cycloid is 4 times the diameter of the generating circle.

Rule.—(*1*) *Multiply the corresponding chord of the generatrix by 2. To find the length of the cycloid*:

(*2*) *Multiply the diameter of the generating circle by 4.*

I. Through what distance will a rivet in the tire of a 3-ft. buggy wheel pass in three revolutions of the wheel?

By formula, $s=3(8r)=24\times1\frac{1}{2}$ft.$=36$ft.

II.
1. 3ft.=the diameter of the wheel. Then
2. 12ft.$=4\times3$ft.=distance through which it moves in 1 revolution.
3. $\therefore$ 36ft.$=3\times12$ft.=distance through which it moves in 3 revolutions.

III. $\therefore$ It will move through a distance of 36 ft.

Prob. LVIII. To find the area of a cycloid.

Formula.—$A=2\int ydx=2\int_0^{2r}\frac{y^2dy}{\sqrt{2ry-y^2}}=3r^2 versin^{-1}2=3\pi r^2$.

Rule.—*Multiply the area of the generating circle by 3.*

I. What is the area of a cycloid generated by a circle whose radius is 2ft.?

By formula, $A=3\pi r^2=3\pi2^2=12\pi=37.6992$ sq. ft.

II.
1. 2ft.=the radius of the generating circle.
2. $\pi2^2=12.5664$ sq. ft.=the area of the generating circle.
3. $3\pi2^2=37.6992$ sq. ft.=the area of the cycloid,

III. $\therefore$ The area of the cycloid is 37.6992 sq. ft.

Prob. LIX. A wheel whose radius is r rolls along a horizontal line with a velocity v′; required the velocity of any point, P, in its circumference; also the velocity of P horizontally and vertically.

Since a point in the circumference of a wheel describes, in space, a cycloid, let P, Fig. 29, be the point, referred to the axes AA' and a perpendicular at A. Let (x,y) be the coordinates of the point; then will the horizontal and vertical velocities of P be the rates of change of x and y respectively. O being the point of contact, $AO = r \text{ versin}^{-1}\frac{y}{r}$. Since the center C, is vertically over O, its velocity is equal to the rate of increase of AO. In an element of time, dt, the center C will move the distance $d\left(r \text{ versin}^{-1}\frac{y}{r}\right) = \frac{r\,dy}{\sqrt{2ry-y^2}}$. $\therefore$ Its velocity $v' =$ the distance it moves divided by the time it moves, or $v' = \frac{r\,dy}{\sqrt{2ry-y^2}} \div dt = \frac{r}{\sqrt{2ry-y^2}}\frac{dy}{dt}$. $\therefore$ $\frac{dy}{dt} = \frac{\sqrt{2ry-y^2}}{r}v' =$ the velocity vertically....(1).

From the equation of the cycloid, $x = r \text{ versin}^{-1}\frac{y}{r} - \sqrt{2ry-y^2}$, we have $dx = \frac{y}{\sqrt{2ry-y^2}}dy$. Now $dx \div dt =$ the velocity of the point horizontally. But $dx \div dt$, or $\frac{dx}{dt} = \frac{y}{\sqrt{2ry-y^2}}\frac{dy}{dt}$. Substituting the value of $\frac{dy}{dt}$, we have $\frac{dx}{dt} = \frac{y}{r}v'$ .....(2). An element of the curve $APBA'$ is ds and this is the distance the point travels in an element of time, dt. $\therefore$ $\frac{ds}{dt} =$ the velocity of the point, P. But $ds = \sqrt{dy^2 + dx^2} = \sqrt{\left(\frac{2ry-y^2}{r^2} + \frac{y^2}{r^2}\right)}\,v'dt = \sqrt{\frac{2y}{r}}\,v'dt$, since, from (1), $dy = \frac{\sqrt{2ry-y^2}}{r}\frac{v'}{dt}$ and, from (2), $dx = \left(\frac{y}{r}\cdot\frac{v'}{dt}\right)$. $\therefore$ By dividing by dt, we have $\frac{ds}{dt} = v = \sqrt{\frac{2y}{r}}\,v' =$ the velocity of the point, P......(3). From (1), (2), and (3), we have,

if $y=0$, $\frac{dy}{dt}=0$, $\frac{dx}{dt}=0$, and $\frac{ds}{dt}=0$;

if $y=r$, $\frac{dy}{dt}=v'$, $\frac{dx}{dt}=v'$, and $\frac{ds}{dt}=\sqrt{2}v'$;

if $y=2r$, $\frac{dy}{dt}=0$, $\frac{dx}{dt}=v'$, and $\frac{ds}{dt}=2v'$.

Hence, when a point of the circumference is in contact with the line, its velocity is 0; when it is in the same horizontal plane as the center, its velocity horizontally and verically is the same as the velocity of the center, and when it is at the highest point, its motion is entirely horizontal, and its velocity is twice that of the center. Since $\frac{ds}{dt}=\sqrt{\frac{2y}{r}}v'=\frac{\sqrt{2ry}}{r}v'$, we have by proportion, $\frac{ds}{dt}:v'::\sqrt{2ry}:r$. But $\sqrt{2ry}=\sqrt{(PF^2+FO^2)}=PO$.

$\therefore$ The velocity of P is to that of C as the chord PO is to the radius CO; that is, P and C are momentarily moving about O with equal angular velocity.

(*b*). *The Prolate and Curtate Cycloid.*

1. $x=a(\theta-m\sin\theta)$, $y=a(1-m\cos\theta)$ are the equations in every case.

2. The cycloid is prolate when m is >1 as $AIP'B'T'A'$, Fig. 30, and curtate when m is <1, as PB. These equations are found thus: Let $CP=ma$, and $\angle OCP=\theta$. Then $x=AN=AO-ON$. But $AO=$arc subtended by $\angle OCP=a\theta$, and $ON=PC\times\sin\angle NPC=ma\sin\angle NPC(=\angle PCL=\pi-\theta)=ma\sin(\pi-\theta)=ma\sin\theta$. $\therefore x=a\theta-ma\sin\theta=a(\theta-m\sin\theta)$. $y=PN=OC+PC\cos\angle NPC\ (=\angle PCL=\pi-\theta)=a+ma\cos(\pi-\theta)=a-ma\cos\theta=a(1-m\cos\theta)$. The same reasoning applies when we assume the point to be P'.

NOTE.—These curves are also called *Trochoids*.

Prob. LX. To find the length of a Trochoid.

Formula.—$s=\int\sqrt{dx^2+dy^2}$.

Since $x=a(\theta-m\sin\theta)$, $dx=a(1-m\cos\theta)d\theta$; and since $y=a(1-m\cos\theta)$, $dy=am\sin\theta d\theta$. $\therefore s=\int\sqrt{dx^2+dy^2}=a\int_0^\pi\sqrt{\Big[(1-m\cos\theta)^2+m^2\sin^2\theta\Big]}d\theta=a\int_0^\pi\sqrt{(1+m^2-2m\cos\theta)}d\theta=$
$4a\int_0^{\frac{1}{2}\pi}\sqrt{\Big\{(1+m)^2-4m\cos^2\varphi\Big\}}d\varphi=4a(1+m)\int_0^{\frac{1}{2}\pi}\sqrt{\left(1-\frac{4m}{(1+m)^2}\cos^2\varphi\right)}d\varphi$.

I. If a fly is on the spoke of a carriage wheel 5 feet in diameter, 6 inches up from the ground, through what distance will the fly move while the wheel makes **one revolution on a level plane?**

Let C be the center of the wheel, in the figure, and P the position of the fly at any time. Let $OC=$ the radius of the carriage wheel $=a=2\frac{1}{2}$ ft., $PC=$ 2 ft., and the angle OCP $=\theta$. Let (x,y) be the coordinates of the point P. Let F, a point at the intersection of the curve and AI be the position of the fly when the motion of the wheel commenced. Then since $x=a(\theta-m\sin\theta)$ and $y=a(1-m\cos\theta)$, we have $dx=a(1-m\cos\theta)d\theta$, and $dy=$ $a\,m\sin\theta\,d\theta$. $\therefore$ $s=\int\sqrt{(dx^2+dy^2)}=\int_0^\pi\sqrt{\left\{a^2(1-m\cos\theta)^2+a^2m^2\sin^2\theta\right\}}d\theta=a\int_0^\pi\sqrt{(1+m^2-2m\cos\theta)}d\theta=4a\int_0^{\frac{1}{2}\pi}\sqrt{\left\{(1+m)^2-4m\cos^2\varphi\right\}}d\varphi$, in which $\varphi=\frac{1}{2}\theta$. But $PC=2$ ft., and since $PC=ma=m\times2\frac{1}{2}$ ft., $m=2\div2\frac{1}{2}=\frac{4}{5}$. $\therefore$ $s=4\times2\frac{1}{2}\int_0^{\frac{1}{2}\pi}\sqrt{\left\{(1+\frac{4}{5})^2-4\times\frac{4}{5}\cos^2\varphi\right\}}d\varphi=10\int_0^{\frac{1}{2}\pi}\frac{1}{5}\sqrt{(81-80\cos^2\varphi)}d\varphi=$ * $18\int_0^{\frac{1}{2}\pi}\sqrt{(1-\frac{80}{81}\cos^2\varphi)}d\varphi=9\pi\left\{1-\frac{20}{81}-3(\frac{1}{1.2})^2(\frac{20}{81})^2-5(\frac{1.3}{1.2.3})^2(\frac{20}{81})^3-7(\frac{1.3.5}{1.2.3.4})^3(\frac{20}{81})^4-\&c\right\}=18.84$ ft.

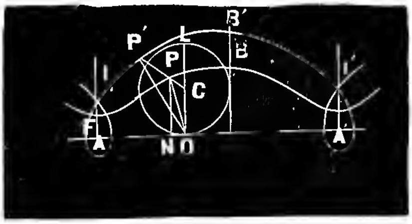

FIG. 30.

II. $\therefore$ The fly will move 18.84 ft.

Prob. LXI. To find the area contained between the trochoid and its axis.

Formula.—$A=\int y dx=2a^2\int_0^\pi(1-m\cos\theta)(1-m\cos\theta)d\theta=2a^2\int_0^\pi(1-m\cos\theta)^2d\theta=2a^2\Big(\theta-2m\sin\theta+\frac{1}{2}m^2(\theta-\sin\theta\cos\theta)\Big)_0^\pi=2a^2(\pi+\frac{1}{2}m^2\pi)$. When $m=1$, the curve is the cycloid and the area $=3\pi a^2$ as it should be.

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* When φ is replaced by $(\frac{1}{2}\pi+\varphi)$, this is an *elliptic integral* of the second kind and may be written $4aE(\frac{80}{81},\varphi)$.

(*c*). *Epitrochoid and Hypotrochoid.*

1. *An Epitrochoid* is the roulette formed by a circle rolling upon the convex circumference of a fixed circle, and carrying a generating point either within or without the rolling circle.

2. *An Hypotrochoid* is the roulette formed by a circle rolling upon the concave circumference of a fixed circle, and carrying a generating point either within or without the rolling circle.

3. $x=(a+b)\cos\theta-mb\cos\frac{a+b}{b}\theta$, $y=(a+b)\sin\theta-mb\sin\frac{a+b}{b}\theta$ are the equations of the epitrochoids.

In the figure, let C be the center of the fixed circle and O the center of the rolling circle. Let $FP'Q$ be a portion of the curve generated by the point P' situated within the rolling circle, and let $CG=x$ and $P'G=y$ be the co-ordinates of the point, P'.

Let A be the position of P when the rolling commences, and $\varphi=\angle POC$ through which it rolled. Draw OK perpendicular to CG and $P'I$ perpendicular to OK; draw DP and DP'. Let $OP'=mOP=mb$ and the angle $ACD=\theta$. Then $x=CG=CK+KG=KC+P'I$. But $CK=OC\cos\theta=(a+b)\cos\theta$ and $P'I=P'O\cos\angle OP'I=mb\cos\{\pi-(\varphi+\theta)\}=-mb\cos(\varphi+\theta)$. But $arc\ AD=arc PD$. $\therefore a\theta=b\varphi$. Whence $\varphi=\frac{a}{b}\theta$.

$\therefore P'I=-mb\cos\frac{a+b}{b}\theta$, and $x=(a+b)\cos\theta-mb\cos\frac{a+b}{b}\theta$. $y=P'G=IK=OK-OI$. But $OK=OC\sin\angle KCO=(a+b)\sin\theta$, and $OI=OP'\sin\angle OP'I=mb\sin\{\pi-(\varphi+\theta)\}=mb\sin(\varphi+\theta)=mb\sin\frac{a+b}{b}\theta$. $\therefore y=(a+b)\sin\theta-mb\sin\frac{a+b}{b}\theta$.

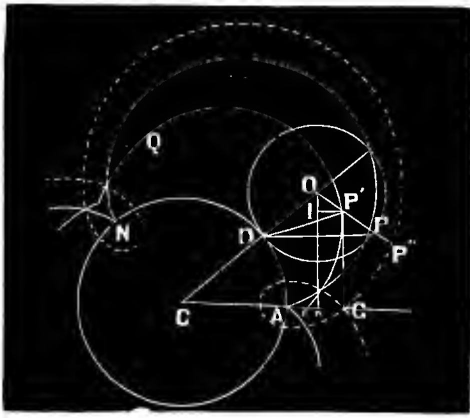


FIG. 31.

If $m=1$, the point P' will be on the circumference of the rolling circle and will describe the curve APN which is called the

Epicycloid The equations for the Epicycloid are $x=(a+b)\cos\theta-b\cos\frac{a+b}{b}\theta$, and $y=(a+b)\sin\theta-b\sin\frac{a+b}{b}\theta$. The equations for the *Hypotrochoid* may be obtained by changing the signs of b and mb, in the equations for the Epitrochoid. $\therefore x=(a-b)\cos\theta+mb\cos\frac{a-b}{b}\theta$, and $y=(a-b)\sin\theta-mb\sin\frac{a-b}{b}\theta$ are the equations for the Hypotrochoid. If $m=1$, the generating point is in the circumference of the rolling circle and the curve generated will be a *Hypocycloid*. $\therefore x=(a-b)\cos\theta+b\cos\frac{a-b}{b}\theta$, and $y=(a-b)\sin\theta-b\sin\frac{a-b}{b}\theta$ are the equations of the *Hypocycloid*.

Prob. LXII. To find the length of the arc of an epitrochoid.

Formula.—$s=\int\sqrt{dx^2+dy^2}=\int\sqrt{\left\{\left[-(a+b)\sin\theta+m(a+b)\sin\frac{a+b}{b}\theta\right]^2+\left[(a+b)\cos\theta-m(a+b)\cos\frac{a+b}{b}\theta\right]^2\right\}}d\theta$ $=(a+b)\int\sqrt{\left\{1+m^2-2m(\sin\theta\sin\frac{a+b}{b}\theta+\cos\theta\cos\frac{a+b}{b}\theta)\right\}}d\theta=(a+b)\int\sqrt{(1+m^2-2m\cos\frac{a}{b}\theta)}d\theta$. This may be expressed as an *elliptic integral*, $E(k,\varphi)$, of the second kind, by substituting $(\pi+\frac{2b}{a}\varphi)$ for θ, and then reducing.

2. By making $m=1$, we have $s=(a+b)\sqrt{2}\int\sqrt{(1-\cos\frac{a}{b}\theta)}\,d\theta$, the length of the arc of an hypocycloid.

3. By changing sign of b, the above formula reduces to $s=(a-b)\int\sqrt{(1+m^2+2m\cos\frac{a}{b}\theta)}d\theta$, which is the length of the arc of an *hypotrochoid.*

4. By making $m=1$, in the last formula, we have $s=(a-b)\sqrt{2}\int(1+\cos\frac{a}{b}\theta)^{\frac{1}{2}}d\theta$, which is the length of the arc of an *hypocycloid.*

I. A circle 2 ft. in diameter rolls upon the convex circumference of a circle whose diameter is 6 feet. What is the length of the curve described by a point 4 inches from the center of the rolling circle, the rolling circle having made a complete revolution about the fixed circle?

In Fig. 31, let O be the center of the rolling circle; C the center of the fixed circle; $CD=3$ ft.$=a$, the radius of fixed circle; $OD=1$ ft.$=b$, the radius of the rolling circle; $OP=4$ inches$=\frac{1}{3}$ of 12 inches$=mb$ the distance of the point from the center; and P the position of the point at any time after the rolling begins. Let $\theta=$ the angle ACD and $\varphi=$the angle POD through which the rolling circle has rolled. Then we have, as previously shown, the equations of the locus P,

$$x=(a+b)\cos\theta-mb\cos(\varphi+\theta)=(a+b)\cos\theta-mb\cos\frac{a+b}{b}\theta,$$

$$y=(a+b)\sin\theta-mb\sin(\varphi+\theta)=(a+b)\sin\theta-mb\sin\frac{a+b}{b}\theta.$$

From these equations, we can find dx and dy.

$\therefore$ By formula, $s=\int\sqrt{dx^2+dy^2}=6(a+b)\int_0^{\frac{1}{3}\pi}\sqrt{(1+m^2-2m\cos\frac{a}{b}\theta)}d\theta=24\int_0^{\frac{1}{3}\pi}\sqrt{(1+(\frac{1}{3})^2-\frac{2}{3}\cos 3\theta)}d\theta=8\int_0^{\frac{1}{3}\pi}\sqrt{(10-6\cos 3\theta)}d\theta$. Let $3\theta=2\psi$. Then $s=8\int_0^{\frac{1}{3}\pi}\sqrt{(10-6\cos 3\theta)}d\theta=$

$21\frac{1}{3}\int_0^{\frac{1}{2}\pi}\sqrt{(1-\frac{3}{4}\cos^2\psi)}d\psi,=21\frac{1}{3}\int_0^{\frac{1}{2}\pi}\Big[1-\frac{1}{2}.\frac{3}{4}\cos^2\psi-\frac{1}{2}.\frac{1}{4}.(\frac{3}{4})^2\cos^4\psi-\frac{1}{2}.\frac{1}{4}.\frac{3}{6}(\frac{3}{4})^3\cos^6\psi-\frac{1}{2}\,\frac{1}{4}.\frac{3}{6}.\frac{5}{8}(\frac{3}{4})^4\cos^8\psi-\&c.\Big]d\psi=$

$21\frac{1}{3}\Big\{\psi-\frac{1}{2}.\frac{3}{4}[\frac{1}{2}(\frac{1}{2}\sin 2\psi+\psi)]-\frac{1}{2}.\frac{1}{4}.(\frac{3}{4})^2[\frac{1}{8}(\frac{1}{4}\sin 4\psi+2\sin 2\psi+3\psi)]-\frac{1}{2}.\frac{1}{4}.\frac{3}{6}(\frac{3}{4})^3[\frac{1}{32}(\frac{1}{6}\sin 6\psi+\frac{3}{2}\sin 4\psi+\frac{15}{2}\sin 2\psi+10\psi)]-\&c.\Big\}_0^{\frac{1}{2}\pi},=$

$10\frac{2}{3}\pi\Big\{1-(\frac{1}{2})^2(\frac{3}{4})-\frac{1}{3}(\frac{1.3}{2.4})^2(\frac{3}{4})^2-\frac{1}{5}(\frac{1}{2}.\frac{3}{4}.\frac{5}{6})^2(\frac{3}{4})^3-\frac{1}{7}(\frac{1}{2}.\frac{3}{4}.\frac{5}{6}.\frac{7}{8})^2(\frac{3}{4})^4-\&c.\Big\}=10\frac{2}{3}\pi\times.773=26\ 9$ ft., nearly.

Remark.—When the point is on the circumference of the rolling wheel, the length of the curve generated by the point is $s=(a+b)\int\sqrt{(1+m^2-2m\cos\frac{a}{b}\theta)}d\theta=(a+b)\int\sqrt{(1+1-2\cos\frac{a}{b}\theta)}d\theta$.

If we let the conditions of the above problem remain the same, only changing the generating point to the circumference, we have for the length of the curve, $s=6\sqrt{2}(3+1)\int_0^{\frac{1}{3}\pi}\sqrt{(1-\cos 3\theta)}d\theta=$ $48\int_0^{\frac{1}{2}\pi}\sqrt{(1-\frac{1}{2}\cos^2\varphi)}d\varphi$, where $\varphi=\frac{3}{2}\theta$. Expanding this by the Binomial Theorem and integrating each term separately, $s=$ $24\pi\left\{1-(\frac{1}{2})^2(\frac{1}{2})-\frac{1}{3}(\frac{1}{2}\cdot\frac{3}{4})^2(\frac{1}{2})^2-\frac{1}{5}(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6})^2(\frac{1}{2})^3-\&c.\right\}$

I. A circle whose radius is 1 ft. is rolled on the concave circumference of a circle whose radius is 4 ft. What is the length of the curve generated by a point in the circumference of the rolling circle, the rolling circle having returned to the point of starting?

$x=(a-b)\cos\theta+b\cos\frac{a-b}{b}\theta,$

$y=(a-b)\sin\theta-b\sin\frac{a-b}{b}\theta$, are the equations of the curve which is a hypocycloid. In these equations $a=4$ and $b=1$.

$\therefore$ $x=3\cos\theta+\cos 3\theta=4\cos^3\theta$, and

$y=3\sin\theta-\sin 3\theta=4\sin^3\theta$. Whence,

$\cos\theta=\sqrt[3]{\left(\frac{x}{4}\right)}$, and $\sin\theta=\sqrt[3]{\left(\frac{y}{4}\right)}$.

$\therefore$ $\cos^2\theta+\sin^2\theta=\left(\frac{x}{4}\right)^{\frac{2}{3}}+\left(\frac{y}{4}\right)^{\frac{2}{3}}$ But $\cos^2\theta+\sin^2\theta=1$.

$\therefore$ $\left(\frac{x}{4}\right)^{\frac{2}{3}}+\left(\frac{y}{4}\right)^{\frac{2}{3}}=1$, whence,

$x^{\frac{2}{3}}+y^{\frac{2}{3}}=4^{\frac{2}{3}}$, which is the rectangular equation of the curve.

$\therefore$ By formula, $s=\sqrt{(dx^2+dy^2)}=4\int_0^a\left(\frac{x^{\frac{2}{3}}+y^{\frac{2}{3}}}{x^{\frac{2}{3}}}\right)^{\frac{1}{2}}dx=$

$4a^{\frac{1}{3}}\int_b^a x^{-\frac{1}{3}}dx=4a^{\frac{1}{3}}\left[\frac{3}{2}x^{\frac{2}{3}}\right]_o^a=6a=6\times4=24$ ft.

X. PLAIN SPIRALS.

1. ***A Plane Spiral*** is the locus of a point revolving about a fixed point and continually receding from it in such a manner that the radius vector is a function of the variable angle. Such a curve may cut a right line in an infinite number of points. This would render its rectilinear equation of an infinite degree. Hence, these loci are *transcendental.*

2. ***The Measuring Circle*** is the circle whose radius is the radius vector of the spiral, at the end of one revolution of the generating point in the positive direction.

3. ***A Spire*** is the portion of the spiral generated by any one revolution of the generating point.

1. THE SPIRAL OF ARCHIMEDES.

1. ***The Spiral of Archimedes*** is the locus of a point revolving about and receding from a fixed point so that the ratio of the radius vector to the angle through which it has moved from the polar axis, is constant.

2. $r=a\theta$ is the polar equation of this curve.

Prob. LXIII. To find the length of the spiral of Archimedes.

Formula.—$s=\int\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}d\theta=\int\sqrt{(r^2+a^2)}d\theta=a\int\sqrt{(1+\theta^2)}d\theta=\frac{1}{2}a\theta\sqrt{(1+\theta^2)}+\frac{1}{2}a\log\left\{\theta+\sqrt{(1+\theta^2)}\right\}$, which is the length of the curve measured from the origin.

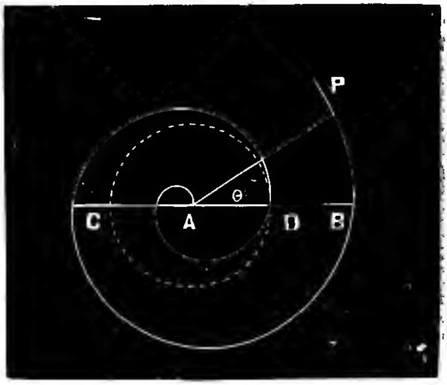


FIG. 32.

$s=a\pi\sqrt{[1+(2\pi)^2]}+\frac{1}{2}a\log\left\{2\pi+\sqrt{[1+(2\pi)^2]}\right\}$ is the length of the curve made by one revolution of the generating point.

Prob. LXIV. To find the area of the spiral of Archimedes.

Formula.—$A=\frac{1}{2}\int r^2d\theta=\frac{1}{2}a^2\int\theta^2d\theta=\frac{1}{6}a^2\theta^3=\frac{r^3}{6a}$, the area measured from the origin.

2. THE RECIPROCAL OR HYPERBOLIC SPIRAL.

1. ***The Reciprocal or Hyperbolic Spiral*** is the locus of a point revolving around and receding from a fixed point so that the inverse ratio of the radius vector to the angle through which it has moved from the polar axis, is constant.

2. $r=\frac{a}{\theta}$ is the polar equation of the *Hyperbolic Spiral.*

Prob. LXV. To find the length of the Hyperbolic Spiral.

Formula.—$s=\int\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}d\theta=\frac{a}{\theta^2}\int\sqrt{(1+\theta^2)}d\theta$
$=\theta\sqrt{(1+\theta^2)}+\log\left\{\theta+\sqrt{(1+\theta^2)}\right\}-$

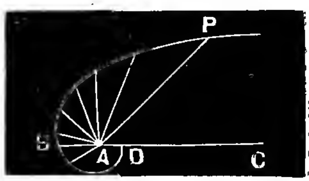


FIG. 33.

$\theta^{-1}(1+\theta^2)^{\frac{3}{2}}=\log\left\{\theta+\sqrt{(1+\theta^2)}\right\}-\theta^{-1}\sqrt{1+\theta^2}$, is the length of the spiral measured from the origin.

Prob. LXVI. To find the area of the Hyperbolic Spiral.

Formula.—$A=\frac{1}{2}\int r^2\,d\theta=\frac{1}{2}a^2\int\frac{d\theta}{\theta^2}=-\frac{a^2}{2\theta}$, the area measured from the origin. This result must be made positive since the radius vector revolves in the negative direction.

3. THE LITUUS.

1. ***The Lituus*** is the locus of a point revolving around and receding from a fixed point so that the inverse ratio of the radius vector to the square root of the angle through which it has moved, is constant.

2. $r=\frac{a}{\sqrt{\theta}}$ is the equation of the Lituus.

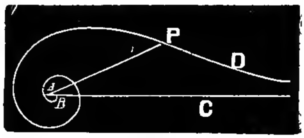


FIG. 34.

Prob. LXVII. To find the length of the Lituus.

Formula.—$s=\int\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta=\frac{1}{2}a\theta^{-\frac{3}{2}}\int\sqrt{(1+4\theta^2)}\,d.\theta=$

$$\left[-\tfrac{1}{8}a\left\{\theta^{-\frac{1}{2}}(1+\theta^2)^{\frac{3}{2}}\right\}-\tfrac{5}{16}\theta^{\frac{3}{2}}\left(\tfrac{2}{3}-\tfrac{1}{7}\theta^2-\tfrac{1}{44}\theta^4-\tfrac{1}{120}\theta^6-\&c.\right]_{\theta'}^{\theta''}.$$

Prob. LXVIII. To find the area of the Lituus.

Formula.—$A=\frac{1}{2}\int r^2\,d\theta=\frac{a^2}{2}\int\frac{d\theta}{\theta}=\frac{1}{2}a^2\log\theta.$

4. THE LOGARITHMIC SPIRAL.

1. ***The Logarithmic Spiral*** is the locus generated by a point revolving around and receding from a fixed point in such a manner that the radius vector increases in a geometrical ratio, while the variable angle increases in an arithmetrical ratio.

2. $r=a^\theta$ is the polar equation of the Logarithmic Spiral. If a is the base of a system of logarithms, this equation becomes $\theta=\log r$.

Prob. LXIX. To find the length of the Logarithmic Spiral.

Formula.—$s=\int\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta=\int\sqrt{r^2+\left(\frac{r^2}{m^2}\right)}d\theta=$

$(m^2+1)^{\frac{1}{2}}dr=\sqrt{(m^2+1)}r$, where m is the modulus of the system of logarithms.

Prob. LXX. To find the area of the Logarithmic Spiral.

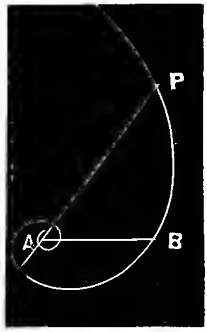


FIG. 35

Formula.—$A=\frac{1}{2}\int r^2 d\theta=\frac{m}{2}\int rdr=\frac{1}{4}mr^2$. Since $m=1$, in the Naperian System of Logarithms, $A=\frac{1}{4}r^2$, *i. e.*, the area is $\frac{1}{4}$ of the square of the radius vector.

XI. MENSURATION OF SOLIDS.

1. PARALLELOPIPEDS.

Prob. LXXI. To find the solidity of a cube, the length of its edge being given.

Formula.—$V=(\text{edge})\times(\text{edge})\times(\text{edge})=(\text{edge})^3$.

Rule.—*Multiply the edge of the cube by itself, and that product again by the edge.*

I. What is the volume of a cube whose edge is 5 feet?

By formula, $V=(\text{edge})^3=(5)^3=125$ cu. ft.

II. 1. 5 ft.=the edge of the cube.
2. $5\times5\times5=125$ cu. ft.=the volume of the cube.

III. ∴ The volume of the cube is 125 cu. ft.

Remark.—Some teachers of mathematics prefer to express the volume by saying $5\times5\times5\times1$ cu. ft. $=125\times1$ cu. ft.=125 cu. ft.

Prob. LXXII. To find the volume of a cube, having given its diagonal.

Formula.—$V=\left(\frac{d}{\sqrt{3}}\right)^3$.

Rule.—*Divide the diagonal by the square root of 3, and the cube of the quotient will be the volume of the cube.*

I. What is the volume of a cube whose diagonal is 51.9615 inches?

By formula, $V=\left(\frac{d}{\sqrt{3}}\right)^3=\left(\frac{51.9615}{\sqrt{3}}\right)^3=\left(\frac{51.9615}{1.73205}\right)^3=$ 27,000 cu. in.

II.
1. 51.9615 in.=the diagonal.
2. 30 in.=51.9615 in.$\div\sqrt{3}$=51.9615 in.$\div$1.73205=the edge of the cube.
3. ∴ 30×30×30=27,000 cu. in.=the volume of the cube.

III. ∴ The volume of the cube whose diagonal is 51.9615 in., is 27,000 cu. in.

Prob. LXXIII. To find the volume of a cube whose surface is given.

Formula.— $V=\left(\sqrt{\frac{S}{6}}\right)^3$.

Rule.—*Divide the surface of the cube by 6 and extract the squae root of the quotient. This will give the edge of the cube. The cube of the edge will be the volume of the cube.*

I. What is the volume of a cube whose surface is 294 square feet?

By formula, $V=\left(\sqrt{\frac{S}{6}}\right)^3=\left(\sqrt{\frac{294}{6}}\right)^3=(\sqrt{49})^3=7^3=$ 243 cu. in.

II.
1. 294 sq. ft.=the surface of the cube.
2. 49 sq. ft.=294 sq. ft.$\div$6=area of one side of the cube.
3. $\sqrt{49}$=7 ft.=length of the edge of the cube.
4. ∴ 7×7×7=343 cu. ft.=volume of cube.

III. ∴ 343 cu. ft. is the volume of a cube whose surface is 294 sq. ft.

Prob. LXXIV. To find the solidity of a parallelopipedon.

Formula.— $V=l\times b\times t$, where l=length, b=breadth, and t=thickness.

Rule.—*Multiply the length, breadth and thickness together.*

I. What is the volume of a parallelopipedon whose length is 24 feet, breadth 8 feet, and thickness 5 feet?

By formula, $V=l\times b\times t=24\times8\times5=960$ cu. ft.

II.
1. 24 ft.=the length.
2. 8 ft.=the breadth, and
3. 5 ft.=the thickness.
4. ∴ 24×8×5=960 cu. ft.=the volume.

III. ∴ 960 cu. ft.=the length of the parallelopipedon.

Prob. LXXV. To find the dimensions of a parallelopipedon, having given the ratio of its dimensions and the volume.

Formula.—$l=\sqrt[3]{[V\div(m\times n\times p)]}m$; $b=\sqrt[3]{[V\div(m\times n\times p)]}n$; and $t=\sqrt[3]{[V\div(m\times n\times p)]}p$, where m, n, and p are the ratios of the length, breadth, and thickness respectively.

Rule.—*Divide the volume of the parallelopipedon by the product of the ratios of the dimensions, and extract the the cube root of the quotient. This gives the G. C. D. of the three dimensions. Multiply the ratios of the dimensions by the G. C. D., and the results will be the dimensions respetively.*

I. What are the dimensions of a parallelopipedon whose length, breadth and thickness are in the ratios of 5, 4 and 3; and whose volume is 12960 cu. ft.?

By formula, $l=\sqrt[3]{[12960\div(5\times4\times3)]}5=30$ ft.; $b=\sqrt[3]{[12960\div(5\times4\times3)]}4=24$ ft.; and $t=\sqrt[3]{[12960\div(5\times4\times3)]}3=18$ ft.

II.
1. 5=the quotient obtained by dividing the length by the G. C. D. of the three dimensions.
2. 4=the quotient obtained by dividing the breadth by the G. C. D. of the three dimensions.
3. 3=the quotient obtained by dividing the thickness by the G. C. D. of the three dimensions.
4. ∴ 5×G. C. D.=the length,
5. 4×G. C. D.=the breadth, and
6. 3×G. C. D.=the thickness.
7. ∴ (5×G. C. D.)×(4×G. C. D.)×(3×G. C. D.)=60×(G. C. D.)^3=the volume of the parallelopipedon.
8. ∴ 60 (G. C. D.)^3=12960 cu. ft.
9. (G. C. D.)^3=12960÷60=216.
10. ∴ G. C. D.=$\sqrt[3]{216}$=6.
11. ∴ 5×(G. C D.)=5×6=30 ft.=the length,
12. 4×(G. C. D.)=4×6=24 ft.=the breadth, and
13. 3×(G. C. D.)=3×6=18 ft.=the thickness.

III. ∴ 30 ft., 24 ft., and 18 ft. are the dimensions of the parallelopipedon.

PROBLEMS.

1. Find the surface of a rectangular solid whose length is 12 feet, breadth 5 feet 4 inches, height 5 feet 3 inches.

2. Find the cost of papering the four walls of a room whose length is 20 feet 6 inches, breadth 15 feet 6 inches, and height 11 feet 3 inches, at 8d. a square yard.

3. A rectangular tank is 16 feet long, 8 feet wide, and 7 feet deep; how many tons of water will it hold, a cubic foot of water weighing 1,000 oz.?

4. The surface of a rectangle is 1,000 sq. in.; if its length and breadth are respectively 1 ft. 3 in. and 1 ft. 2 in., find its height.

5. The dimensions of a rectangular solid are proportional to 3, 4, and 5. If the whole surface contains 2,350 sq. in., find the length, breadth, and height.

Hint.—$2,350 \div [2(3\times4)+2(3\times5)+2(4\times5)]=25$, the greatest common divisor of the three dimensions.

6. The whole surface of a rectangular solid contains 1,224 square feet, and the four vertical faces together contain 744 square feet. If the height is 12 feet, find the length and breadth.

7. Find the surface and volume of a cube whose diagonal is 2 feet 6 inches. *Ans.* 12½ sq. ft.; 3 cu. ft. 12 cu. in., nearly.

8. Find the edge of a cubical block of lead weighing one ton, having given that a cubic foot of lead weighs 709½ lbs. *Ans.* 17.60+ inches.

9. The edges of a rectangular block of granite are proportional to 2, 3, and 5, and its volume is 101 cu. ft. 432 cu. in.; find its dimensions. *Ans.* 3 ft.; 4 ft. 6 in.; 7 ft. 6 in.

10. The diagonal of a rectangular solid is 29 inches, and its volume is 4,032 cu. in.; if the thickness is one foot, find the length and breadth. *Ans.* 21 in. and 16 in.

2. PRISMS.

Prob. LXXVI. To find the convex surface of a prism.

Formula.—$S=p\times a$, in which p is the perimeter of the base and a the altitude.

Rule.—*Multiply the perimeter of the base by the altitude.*

I. What is the convex surface of the prism $ABC—D$, if the altitude AE is 12 feet, AB, 6 feet, AC, 5 feet, and BC, 4 feet.?

By formula, $S=a\times p=12\times(9+5+4)=180$ sq. ft.

II. 1. 12 ft = the altitude of the prism.
2. 6 ft. + 5 ft. + 4 ft. = 15 ft. = the perimeter of the base.
3. ∴ $12\times15=180$ sq. ft. = the convex surface of the prism.

III. ∴ The convex surface of the prism is 180 sq. ft.

Remark.—If the entire surface is required; to the convex surface, add the area of the two bases.

Formula.—$T=S+2A$, where $2A$ is the area of the base, S the convex surface, and T the total surface.

Prob. LXXVII. To find the volume of a prism.

Formula.—$V=a\times A$, where A is the area of the base, a, the altitude.

Rule.—*Multiply the area of the base by the altitude.*

I. What is the volume of the triangular prism $ABC—D$, whose length AE is 8 feet, and either of the equal sides AB, BC, or AC, $2\frac{1}{2}$ feet?

By formula, $V=a\times A=8\times[(2\frac{1}{2})^2\frac{1}{4}\sqrt{3}]=12\frac{1}{2}\sqrt{3}=21.6506$ cu. ft.

II.
1. 8 ft.=the altitude AE.
2. $2\frac{1}{2}$ ft.=the length of one of the equal sides of the base, as AB.
3. $(2\frac{1}{2})^2\frac{1}{4}\sqrt{3}$=the area of the base ABC, by Prob. XI.
4. $\therefore 8\times(2\frac{1}{2})^2\frac{1}{4}\sqrt{3}=12\sqrt{3}=21.6506$ cu. ft. =the volume of the prism.

III. $\therefore$ 21.6506 cu. ft.=the volume of the prism.

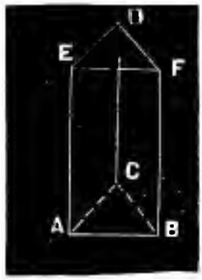


FIG. 36.

PROBLEMS.

1. A right prism stands upon a triangular base, whose sides are 13, 14, and 15 inches. If the height is 10 inches, find its volume and whole surface. *Ans.* 840 cu. in.; 4 sq. ft. 12 sq. in.

2. The weight of a brass prism standing on a triangular base is 875 lbs. If the sides of the base are 25 in., 24 in., and 7 in., find the height of the prism, supposing that 1 cu. ft. of brass weighs 8,000 oz. *Ans.* 3 ft.

3. Water flows at the rate of 30 yards per minute through a wooden pipe whose cross-section is a square on a side of 4 inches. How long will it take to fill a cubical cistern whose internal edge is 6 feet? *Ans.* $21\frac{3}{5}$ min.

4. Find the volume of a truncated prism (that is the part of a prism included between the base and a section made by a plane inclined to the base and cutting all the lateral edges), whose base is a right triangle, base 3 feet, and altitude 4 feet, and the three lateral edges 3 feet, 4 feet, and 5 feet respectively. [*Formula.*—$V=\frac{1}{3}A(e_1+e_2+e_3)$, where A is the area of the base and e_1, e_3, and e_3 the lateral edges.]

5. A square right prism, whose base is 6 inches, intersects a right equilateral triangular prism, whose base is 10 inches, in such a way that the square prism is perpendicular to one of the faces of the triangular prism, and the axes of the prisms intersect each other, and a lateral side of the square prism is parallel to the edges of the triangular prism. What is the volume common to the prisms? *Ans.* $128\sqrt{3}$ cu. in.

3. THE CYLINDER.

Prob. LXXVIII. To find the convex surface of a cylinder.

Formula.—$S=a\times C$, in which a is the altitude and C the circumference of the base.

Rule.—*Multiply the circumference of the base by the altitude.*

I. What is the convex surface of the right cylinder $AGB—C$, whose altitude EF is 20 feet and the diameter of its base AB is 4 feet?

By formula, $S=a\times C=20\times 4\pi=80\pi=251.32736$ sq. ft.

II.
1. 20 ft.=the altitude EF.
2. 4 ft.=the diameter AB of the base.
3. 12.566368 ft.$=4\pi=4\times 3.141592=$the circumference of the base.
4. $\therefore$ $20\times 12.566368=251.32736$ sq. ft.= the convex surface of the cylinder.

III. $\therefore$ The convex surface of the cylinder is 251.32736 sq. ft.

Remark.—If the entire surface is required; to the convex surface, add the area of the two bases.

Formula.—$T=S+2A=2\pi aR+2\pi R^2$

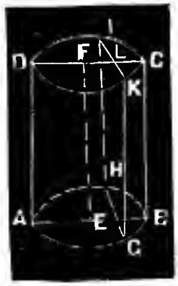


FIG. 37.

Prob. LXXIX. To find the solidity of a cylinder.

Formula.—$V=a\times A$, in which A is the area of the base.

Rule.—*Multiply the area of the base by the altitude.*

I. What is the solidity of the cylinder $AGB—C$, whose altitude FE is 8 feet and diameter AB of the base 2 feet?

By formula, $V=a\times A=8\times(1^2\pi)=8\pi=8\times 3.141592=$ 25.132736 cu. ft.

II.
1. 8 ft.=the altitude, EF.
2. 2 ft.=the diameter, AB, of the base.
3. 3.141592 sq. ft.$=\pi R^2=\pi 1^2=$area of the base.
4. $\therefore$ $8\times 3.141592=25.132736$ cu. ft.

III. $\therefore$ 25.132736 cu. ft. is the volume of the cylinder.

I. What is the volume of a slab 4 inches thick sawed from a round log 24 inches in diameter and 10 feet long?

II.
1. Let $HGB—C$ be the slab, Fig. 37,
2. $FC=FK=12$ in., the radius of the log,
3. $FE=10$ ft., the length of the log, and $LC=4$ in., the thickness of the slab.
4. Volume of $HGB—C=$area of base, HGB,$\times$altitude, FE.
5. Area of $HGB=$area of $IKC=$area of sector, $IFKC$,—area of triangle, IKF.
6. Area of sector, $IFKC$,$=\frac{1}{2}$ $FC\times$arcICK, Rule (1), Prob. XXVII.

7. Arc $ICK=\frac{1}{3}$(8 times chord IC — chord IK) = $\frac{1}{3}(8\sqrt{IL^2+LC^2}-IK)=\frac{1}{3}(8\sqrt{(FI^2-FL^2)+LC^2}-2\sqrt{FI^2-FL^2})=\frac{1}{3}(8\sqrt{(12^2-8^2)+4^2}-2\sqrt{12^2-8^2})=\frac{1}{3}(32\sqrt{6}-8\sqrt{5})=\frac{8}{3}(4\sqrt{6}-\sqrt{5})$, Rule ($b$), Prob. XXV.

8. ∴ Area of sector $ICK=\frac{1}{2}[FC\times\frac{8}{3}(4\sqrt{6}-\sqrt{5})]=\frac{1}{2}[12\times\frac{8}{3}(4\sqrt{6}-\sqrt{5})]=16(4\sqrt{6}-\sqrt{5})$ sq. in.

9. ∴ volume of slab $HBG-C=120\times16(4\sqrt{6}-\sqrt{5})=1920(4\sqrt{6}-\sqrt{5})$ cu. in.=1451.52 cu. in., nearly.

III. ∴ The volume of the slab=1451.52 cu. in.

PROBLEMS.

1. How many cubic yards of earth must be removed in constructing a tunnel 100 yards long, whose section is a semi-circle with a radius of 10 feet?

2. Find the convex surface of a cylinder whose height is three times its diameter, and whose volume is 539 cubic inches.

3. The cylinder of a common pump is 6 inches in diameter; what must be the beat of the piston if 8 beats are needed to raise 10 gallons?
Ans. 12¼ in.

4. A copper wire $\frac{1}{10}$ inches in diameter is evenly wound about a cylinder whose length is 6 inches and diameter 9.9 inches, so as to cover the convex surface. Find the length and weight of the wire, if 1 cu. in. of copper weighs 5.1 oz. *Ans.* 1,885 in., nearly; 75.5 oz.

5. A cubic inch of gold is drawn into a wire 1,000 yards long. Find the diameter of the wire. *Ans.* .006 in.

6. The whole surface of a cylindrical tube is 264 square inches; if its length is 5 inches, and its external radius is 4 inches, find its thickness. [Use $\pi=\frac{22}{7}$.] *Ans.* 1 in.

7. If the diameter of a well is 7 feet, and the water is 10 feet deep, how many gallons of water are there, reckoning 7½ gallons to the cubic foot?
Ans. 288.6345 gal.

4. CYLINDRIC UNGULAS.

1. ***A Cylindric Ungula*** (Lat. *Ungula*, a claw, or hoof) is any portion of a cylinder cut off by a plane.

Prob. LXXX. To find the convex surface of a cylindric ungula, when the cutting plane is parallel to the axis of the cylinder.

Formula.—$S=a\int ds=2a\int_0^y\frac{rdy}{(r^2-y^2)^{\frac{1}{2}}}=2ar\sin^{-1}\frac{y}{r}=a\times$ *arc* of the base.

Rule.—*Multiply the arc of the base by the altitude.*

I. What is the surface of the cylindric ungula $API-Q$, whose altitude AD is 32 feet and height AT of the arc of the base, 2 feet and cord PI of the base 12 feet?

By formula, $S=a\times arc\ PAI=a2r\sin^{-1}\frac{y}{r}=a\times 2r\sin^{-1}\left(\frac{y}{r}\right)$

$$=a\times 2\left(\frac{IT^2+AT^2}{2AT}\right)\sin^{-1}\left[IT\div\left(\frac{IT^2+AT^2}{2AT}\right)\right]=$$

$$32\left(\frac{6^2+2^2}{2}\right)\sin^{-1}\tfrac{3}{5}=640\left(\frac{17731}{86400}\pi\right)=411.84 \text{ sq. ft., nearly.}$$

The arc corresponding to the $\sin\frac{3}{5}$ is found from a table of natural sines and cosines to be $(36^\circ\ 52\frac{5}{24}'\div 360^\circ)$ of 2π or $\frac{17731}{86400}\pi$.

II.
1. 2 ft.=the height AT of the arc PAI.
2. 12 ft.= the length of the chord PI.
3. 12.87 ft.$=2\sqrt{6^2+2^2}\times\left(1+\frac{10\times 2^2}{60\times 6^2+33\times 2^2}\right)$=the length of the arc PAI, by Prob. XXV.
4. .. $32\times 12.87=411.84$ sq. ft. = convex surface $PAI—D$.

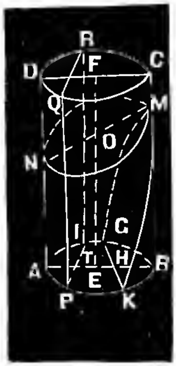


FIG. 38.

III. ∴ The convex surface of the cylindric ungula $PAI—Q$ is 411.84 sq. ft.

Remark.—r is found, by Prob. XX, formula $R=(a^2+c^2)\div 2a$.

Prob. LXXXI. To find the volume of a cylindric ungula, whose cutting plane is parallel to the axis.

$$\textit{Formula.}— V=2\int_0^y\int_0^{\sqrt{r^2-y^2}}\int_0^a dy\,dx\,dz—2ay(r^2—y^2)^{\frac{1}{2}}$$

$$=2a\int_0^y\int_0^{\sqrt{r^2-y^2}}dy\,dx—2ay(r^2—y^2)^{\frac{1}{2}}=2a\int_0^y\sqrt{(r^2—y^2)}dy—$$

$$2ay(r^2—y^2)^{\frac{1}{2}}=a\left\{y(r^2—y^2)^{\frac{1}{2}}+r^2\sin^{-1}\frac{y}{r}—2y(r^2—y^2)^{\frac{1}{2}}\right\}=$$

$a\left\{r^2\sin^{-1}\frac{y}{r}—y(r^2—y^2)^{\frac{1}{2}}\right\}$, in which y is half the chord of the base. In this formula $\left(r^2\sin^{-1}\frac{y}{r}\right)^{\frac{1}{2}}$ is the area of the sector $APEIA$, and $y(r^2—y^2)^{\frac{1}{2}}$ is the area of the triangle PEI formed by joining the center E with P and I.

Rule.—*Multiply the area of the base by the altitude.*

I. What is the volume of the cylindric ungula $PIA—D$, if PI is 12 feet, AT 2 feet, and altitude AD 40 feet?

By formula, $V=aA=a\{r^2\sin^{-1}\frac{y}{r}-y(r^2-y^2)^{\frac{1}{2}}\}=40\{10^2\sin^{-1}\frac{3}{5}-6(10^2-6^2)^{\frac{1}{2}}\}=4000\sin^{-1}\frac{3}{5}-1920=4000\left(\frac{17731}{86400}\pi\right)-1920=$ $2574.016-1920=654.016$ cu. ft.

II.
1. 40 ft.=the altitude AD.
2. 2 ft.=the height AT of the arc of the base.
3. 12 ft.=the chord PI of the base.
4. $16\frac{1}{3}$ sq. ft.$=\frac{2^2}{2\times12}+\frac{2}{3}$ of(2×12)=the area of the base, by rule, Prob. XXVIII.
5. $\therefore 40\times16\frac{1}{3}=653\frac{1}{3}$ cu. ft.=the volume of the cylindrical ungula $PIA-D$.

III. $\therefore 653\frac{1}{3}$ cu. ft.=the volume of the cylindrical ungula.

Remark.—A nearer result would have been obtained by finding the length of the arc PAI and multiplying it by half the radius. This would give the area of the sector $IEPA$. From the area of the sector subtract the area of the triangle PIE formed by joining P and I with E, and the remainder would be the area of the segment PIA.

Prob. LXXXII. To find the convex surface of a cylindric ungula, when the plane passes obliquely through the opposite sides of the cylinder.

Formula.—$S=\frac{1}{2}(a+a')2\pi r$, where a and a' are the least and greatest lengths of the ungula and $2\pi r$ the circumference of the base of the cylinder.

Rule.—*Multiply the circumference of the base by half the sum of the greatest and least lengths of the ungula.*

I. What is the convex surface of the cylindric ungula $AKBA-NM$, if AN is 8 feet, BM 12 feet and the radius BE of the base 3 feet?

By formula, $S=\frac{1}{2}(a+a')2\pi r=\pi(a+a')r=\pi(8+12)\times3=$ 188.49552 sq. ft.

II.
1. 8 ft.=the least length AN of the ungula, and
2. 12 ft.=the greatest length BM.
3. 10 ft.$=\frac{1}{2}$(8ft.+12 ft.)=half the sum of the least and greatest lengths.
4. 18.849552 ft.$=6\pi$=the circumferenc of the base.
5. $\therefore 10\times18.849552=188.49552$ sq. ft.=the convex surface.

III. ∴ 188.49552 sq. ft.=the **convex surface** of the ungula.

Prob. LXXXIII. To find the volume of a cylindric ungula, when the plane passes obliquely through the opposite sides of the cylinder.

Formula.— $V=\frac{1}{2}(a+a')\pi r^2=\frac{1}{2}\pi(a+a')r^2$.

Rule.—*Multiply the area of the base, by half the least and greatest lengths of the ungula.*

I. What is the volume of a cylindric ungula whose least length is 7 feet, greatest length 11 feet, and the radius of the base 2 feet?

By formula, $V=\frac{1}{2}(a+a')\pi r^2=\frac{1}{2}(7+11)\pi 2^2=113.097312$ cu. ft.

II.
1. 7 ft.=the least length of the ungula, and
2. 11 ft.=the greatest length.
3. 9 ft.$=\frac{1}{2}$(7 ft.+11 ft.)=half the length of the least and greatest lengths.
4. 12.566368 sq. ft.$=\pi 2^2$=the area of the base.
5. ∴ 9×12.566368=113.097312 cu. ft.=the volume of the ungula.

III. ∴ The volume of the ungula is 113.097312 cu. ft.

Prob. LXXXIV. To find the convex surface of a cylindric ungula, when the plane passes through the base and one of its sides.

$$*\textbf{Formula.}—S=2\int_0^b \frac{a}{b}(b-x)ds=2\int_0^b \frac{a}{b}(b-x)\frac{rdx}{\sqrt{2rx-x^2}}$$

$$=2r\frac{a}{b}\int_0^b \frac{b-x}{\sqrt{2rx-x^2}}dx=2r\frac{a}{b}\left[b\ \text{vers}^{-1}\frac{x}{r}+\sqrt{2rx-x^2}-r\ \text{vers}^{-1}\frac{x}{r}\right]_0^b,$$

$$=2r\frac{a}{b}\left[\sqrt{2rx-x^2}-(r-b)\text{vers}^{-1}\frac{x}{r}\right]_0^b=2r\frac{a}{b}\left[\sqrt{2rb-b^2}-\right.$$

$$\left.(r-b)\text{vers}^{-1}\frac{b}{r}\right].$$

Rule.—*Multiply the sine of half the arc of the base by the diameter of the cylinder, and from the product subtract the product of the arc and cosine; this difference multiplied by the quotient of the height divided by the versed sine will be the convex surface.*

I. What is the convex surface of the cylindric ungula ACB—

D, whose altitude BD is 28 feet, height BM of arc of base 4 feet and chord AC 16 feet?

By formula, $S=2r\frac{a}{b}\left[\sqrt{2rb-b^2}-(r-b)\text{vers}^{-1}\frac{b}{r}\right]$, $=2\times10\times\frac{28}{4}\left[\sqrt{2\times10\times4-4^2}-(10-4)\text{vers}^{-1}\frac{4}{10}\right]$, $=140\left[8-6\text{ vers}^{-1}\frac{2}{5}\right]=140\left[8-\frac{54193}{367200}2\pi\right]$, $=140[8-5.5638]=341.068$ sq. ft.

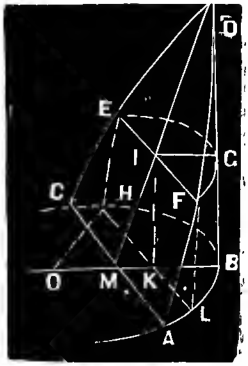


FIG. 39.

II.
1. 28 ft.=the altitude BD.
2. 4 ft.=the height BM of the arc of the base.
3. 16 ft.=the chord AC of the arc of the base.
4. 8 ft.=the sine CM of the arc.
5. 10 ft.$=(8^2+4^2)\div(2\times4)=$the radius $OC=OB$ of the base, by Prob. XX, formula $R=(a^2+c^2)\div2a$.
6. 6 ft.=10 ft.—4 ft.=cosine OM of the arc.
7. 160 sq. ft.$=20\times8=$sine multiplied by the diameter of the base.
8. 18.5438 ft.$=2\sqrt{8^2+4^2}\left(1+\frac{10\times4^2}{60\times8^2+33\times4^2}\right)=$the arc CBA, by formula of Prob. XXV.
9. ∴ 111.2628 sq. ft.$=6\times18.5438=$the arc multiplied by the cosine OM.
10. 160 sq. ft.—111.2628 sq. ft.=48.7372 sq. ft.=the difference.
11. ∴ 341.1604 sq. ft.$=(28\div4)\times48.7372$ sq. ft.=the convex surface.

III. ∴ The convex surface is 341.1604 sq. ft. nearly.

Note.—The difference in the two answers is caused by the length of the arc CBA, in the solution, only being a near approximation.

**Demonstration.*—In the figure, let $BK=x$, $BM=b$, $BD=a$, and the angle $BMD=\theta$. Then $MK=b-x$, and $IK=FL=MK\tan\theta=(b-x)\tan\theta$. But $\tan\theta=\frac{BD}{BM}=\frac{a}{b}$. ∴ $FL=\frac{a}{b}(b-x)$. Now if we take an elemennt of the arc LBH, and from it draw a line parallel to FL, we will have an element of the surface $LBHEGF$. This will be a rectangle whose length is $FL=\frac{a}{b}(b-x)$ and width an element of the arc LBH. An element of the arc is $ds=\sqrt{(dx^2+dy^2)}$. Let $HK=y$. Then $y^2=2rx-x^2$, by a property of the circle, from which we find $dy=\frac{r-x}{\sqrt{2rx-x^2}}dx$. $\therefore$ $ds=\frac{rdx}{\sqrt{2rx-x^2}}$. $\therefore$ The area of the element of the surface is $\frac{a}{b}(b-x)\frac{rdx}{\sqrt{2rx-x^2}}$, and the whole surface of $ABC-D$ is $S=2\int_0^b \frac{a}{b}(b-x)\frac{rdx}{\sqrt{2rx-x^2}}$, $=2r\frac{a}{b}\int_0^b (b-x)\frac{dx}{\sqrt{2rx-x^2}}=$ $2r\frac{a}{b}\left[\sqrt{2rx-x^2}-(r-b)\text{vers}^{-1}\frac{x}{r}\right]$, $=\frac{a}{b}\left[2r\sqrt{2rb-b^2}-2(r-b)r\text{ vers}^{-1}\frac{b}{r}\right]$. *Q. E. D.*

Prob, LXXXV. To find the volume of a cylindric ungula, when the cutting-plane passes through the base and one of its sides.

Formula.— $V=\int_0^b (b-x)dA=\frac{a}{b}\int_0^b (b-x)2\sqrt{2rx-x^2}dx$, $=2\frac{a}{b}\left[\frac{1}{3}(2rx-x^2)^{\frac{3}{2}}-(r-b)\int_0^b \sqrt{(2rx-x^2)}dx\right]=2\frac{a}{b}\left[\frac{1}{3}(2rx-x^2)^{\frac{3}{2}}\right.$ $\left.+\frac{1}{2}{(r-b)}\left\{(r-x)\sqrt{2rx-x^2}+\frac{1}{2}r^2\sin^{-1}\frac{r-x}{r}\right\}+C\right]_0^b$. When $x=0$, $V=0$. $\therefore$ $C=-\frac{1}{4}\pi r^2(r-b)$. $\therefore$ $V=\frac{a}{b}\left[\frac{2}{3}(2rb-b^2)^{\frac{3}{2}}-(r-b)\right.$ $\left.\left\{\frac{1}{2}\pi r^2-(r-b)\sqrt{2rb-b^2}-r^2\sin^{-1}\frac{r-b}{r}\right\}\right]$.

Rule.—*From $\frac{2}{3}$ of the cube of half the chord of the base,* ***sub-****tract the product of the area of the base and the difference of the radius of the base and the height of the arc of the base; this difference multiplied by the quotient of the altitude of the ungula by the height (versed sine) of the arc of the base, will give the volume.*

I. What is the volume of a cylindric ungula, whose altitude BD is 8 feet, chord AC of base 6 feet, and height BM of arc of base 1 foot?

By formula, $V=\frac{a}{b}\left\{\frac{2}{3}(2rb-b^2)^{\frac{3}{2}}-(r-b)\left[\frac{1}{2}\pi r^2-\right.\right.$

$(r-b)\sqrt{(2rb-b^2)}-r^2\sin^{-1}\frac{r-b}{r}\Big]\Big\}=8\Big\{\frac{2}{3}(2\times5\times1-1)^{\frac{3}{2}}-$

$(5-1)\Big[\frac{1}{2}\pi5^2-4\sqrt{2\times5\times1-1}-5^2\sin^{-1}\frac{5-1}{5}\Big]\Big\},=8\Big\{18-$

$4\Big[\frac{1}{2}\pi25-12-25\sin^{-1}\frac{4}{5}\Big]\Big\}=528+800\sin^{-1}\frac{4}{5}-200\pi=$

13.20394 cu. ft.

II.
1. 8 ft=the altitude *BD*.
2. 1 ft.=the altitude *BM* of the arc *ABC* of the base.
3. 6 ft.=the chord *AC* of the base.
4. 18 cu. ft.$=\frac{2}{3}$ of $3^3=\frac{2}{3}$ of the cube of the sine of half the arc of the base.
5. $4\frac{1}{12}$ sq. ft.$=\frac{1^3}{2\times6}+\frac{2}{3}$ of 6×1=area of the base, by formula, (*b*), Prob. XXVIII
6. $16\frac{1}{3}$ cu. ft.$=4\times4\frac{1}{12}$=the area of the base$\times$ *OM*, the cosine of the arc *CHB*.
7. ∴ $8(18$ cu. ft.$-16\frac{1}{3}$ cu. ft.$)=13\frac{1}{3}$ cu. ft.=the volume of the cylindric ungula *ACB—D*.

III. ∴ The volume of the cylindric ungula *ACB—D* is $13\frac{1}{3}$ cu. ft., *nearly*.

Prob. LXXXVI. To find the convex surface of the frustum of a cylindric ungula.

Formula.—$S=\frac{a}{b}\Big[2r\sqrt{2rb-b^2}-2(r-b)r\text{ vers}^{-1}\frac{b}{r}\Big]-$

$\frac{a'}{b'}\Big[2r\sqrt{2rb'-b'^2}-2(r-b')r\text{ vers}^{-1}\frac{b'}{r}\Big].$

Rule.—(1) *Conceive the section to be continued, till it meets the side of the cylinder produced; then say, as the difference of the heights of the arcs of the two ends of the ungula, is to the height of the arc of the less end, so is the height of the cylinder to the part of the side produced.*

(2) *Find the surface of each of the ungulas, thus formed, by Prob. LXXXIV., and their difference will be the convex surface of the frustum of the cylindric ungula.*

Prob. LXXXVII. To find the volume of a frustum of a cylindric ungula.

Formula.—$V=\frac{a}{b}\Big[\frac{2}{3}(2rb-b^2)^{\frac{3}{2}}-(r-b)\Big\{\frac{1}{2}\pi r^2-$

$(r-b)\sqrt{2rb-b^2}-r^2\sin^{-1}\frac{r-b}{r}\Big\}\Big]-\frac{a'}{b'}\Big[\frac{2}{3}(2rb'-b'^2)^{\frac{3}{2}}-$

$$(r-b')\left\{\tfrac{1}{2}\pi r^2-(r-b')\sqrt{(2rb'-b'^2)}-r^2\sin^{-1}\frac{r-b'}{r}\right\}\Big].$$

Rule.—*Find the volume of the ungula whose base is the the upper base of the frustum and altitude that as found by (1) of the last rule. Also the volume of the ungula whose base is the lower base of the frustum and altitude the sum of the less ungula and altitude of the frustum. Their difference will be the volume of the frustum.*

5. PYRAMID AND CONE.

Prob. LXXXVIII. To find the convex surface of a right cone.

Formula.—$S=C\times\frac{1}{2}h=2\pi r\times\frac{1}{2}\sqrt{a^2+r^2}$, where C is the circumference, h the slant height, r the radius of the base, and a the altitude.

Rule.—*Multiply the circumference of the base by the slant height and take half the product. Or, if the altitude and radius of the base are given, multiply the circumference of the base by the square root of the sum of the squares of the radius and altitude, and take half the product.*

I. What is the convex surface of a right cone whose altitude is 8 inches and the radius of whose base is 6 inches?.

By formula, $S=2\pi r\times\frac{1}{2}\sqrt{a^2+r^2}=2\pi 6\times\frac{1}{2}\sqrt{8^2+6^2}=160\pi=$ 188.495559 sq. in.

II.
1. 6 in.=the radius AD of the base, and
2. 8 in.=the altitude CD.
3. 10 in.$=\sqrt{8^2+6^2}=$the slant height CA.
4. 37.6991118 in.$=2\pi r=12\times$ 3.14159265=the circumference of the base.
4. ∴ 188.495559 sq. in=$\frac{1}{2}(10\times37.6991118)$=the convex surface of the cone.

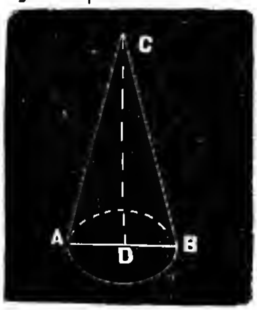


FIG 40.

III. ∴ The convex surface of the cone is 188.495559 sq. in.

Prob. LXXXIX. To find the convex surface of a pyramid.

Formula.—$S=\frac{1}{2}p\times h$, in which p is the perimeter of the base and h the slant height.

Rule.—*Multiply the perimeter of the base by the slant height and take half the product.*

I. What is the convex surface of a pentagonal pyramid whose slant height is 8 inches and one side of the base 3 inches?

By formula, $S=\frac{1}{2}p\times h=\frac{1}{2}(3+3+3+3+3)\times8=60$ sq. in.

II.
1. 8 in.=the slant height.
2. 3 in.=the length of one side of the base.
3. 5×3 in.=15 in.=the perimeter of the base.
4. ∴ $\frac{1}{2}(15\times8)=60$ sq. in.=the convex surface of the pyramid.

III. ∴ The convex surface of the pyramid is 60 sq. in.

Remark.—If the entire surface of a pyramid or cone is required, to the convex surface add the area of the base.

Formula.— $T=S+A$, where A is the area of the base and S the convex surface.

Prob. XC. To find the volume of a pyramid or a cone.

Formula.— $V=\frac{1}{3}aA=\frac{1}{3}a\times\pi r^2$, where a is the altitude and $A=\pi r^2$ the area of the base.

Rule.—*Multiply the area of the base by the altitude and take one-third of the product.*

I. What is the volume of a cone whose altitude CD is 18 inches and the radius AD of the base 3 inches?

By formula, $V=\frac{1}{3}a\times\pi r^2=\frac{1}{3}\times18\times\pi3^2=54\times3.14159265=$ 169.646 cu. in.

II.
1. 18 in.=the altitude CD, and
2. 3 in.=the radius AD.
3. 28.27433385 sq. in.$=\pi r^2=3^2\pi=$the area of the base.
4. ∴ 169.6460031 cu. in.$=\frac{1}{3}aA=\frac{1}{3}\times18\times3^2\pi=$the volume of the cone.

III. ∴ The volume of the cone is 169.6460031 cu. in.

Prob. XCI. To find the convex surface of a frustum of a cone.

Formula.—$S=\frac{1}{2}(C+C')h=\frac{1}{2}(2\pi r+2\pi r')h=\pi(r+r')\sqrt{a^2+(r-r')^2}$, in which C is the circumference of the lower base, C' the circumference of the upper base, and h,$=\sqrt{a^2+(r-r')^2}$, the slant height.

Rule.—*Multiply half the sum of the circumferences of the two bases by the slant height.*

1. What is the convex surface of the frustum of a cone whose altitude is 4 feet, radius of the lower base 4 feet, and the radius of the upper base 1 foot?

By formula, $S=\pi(r+r')\sqrt{a^2+(r-r')^2}=\pi(4+1)\sqrt{4^2+(4-1)^2}$ $=25\pi.=78.539816$ sq. ft.

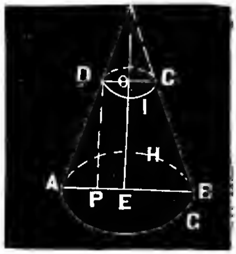


FIG. 41.

II.
1. 4 ft.= the altitude OE,
2. 4 ft.=the radius AE of the lower base, and
3. 1 ft.=the radius DO of the upper base.
4. 3 ft.$=AE-PE(=DO)=r-r'$.
5. 5 ft.$=\sqrt{(DP^2+AP^2)}=\sqrt{a^2+(r-r')^2}=\sqrt{4^2+(4-1)^2}$ $=AD$, the slant height.
6. 8π=the circumference $AGBH$ of the lower base.
7. 2π=the circumference DIC of the upper base.
8. $5\pi=\frac{1}{2}(8\pi+2\pi)$=half the sum of the circumferences.
9. $\therefore$ $5\times5\pi=25\pi=78.539816$ sq. ft.=the convex surface of the frustum.

III. $\therefore$ The convex surface of the frustum is 78.539816 sq. ft.

Remark.—If the entire surface of the frustum is required, to the convex surface add the area of the two bases.

Formula.—$T=S+A+A'=\pi(r+r')\sqrt{a^2+(r-r')^2}+\pi r^2+\pi r'^2$.

Prob. XCII. To find the convex surface of the frustum of a pyramid.

Formula.—$S=\frac{1}{2}(p+p')h$.

Rule.—*Multiply half the sum of the perimeters of the two bases by the slant height.*

I. What is the convex surface of the frustum of a pentagonal pyramid, if each side of the lower base is 5 feet, each side of the upper base 1 foot, and the altitude of the frustum 10 feet?

Before we can apply the formula, we must find the slant height. Produce FO, till $OK=OE$. Divide OK into extreme and mean ratio at H. Draw EH. Then $KO:OH::OH:KH$. $\therefore$ $OH^2=KO\times KH=KO\times(KO-OH)=KO^2-KO\times OH$; whence $OH^2+KO\times OH=KO^2$. Completing the square of this equation, $OH^2+KO\times OH+\frac{1}{4}KO^2=\frac{5}{4}KO^2$, from which $OH(=EH=EK)=\frac{1}{2}KO(\sqrt{5}-1)$. $EF^2=EK^2-KF^2=[\frac{1}{2}KO(\sqrt{5}-1)]^2-[\frac{1}{2}(KO-OH)]^2,=\frac{1}{4}KO^2(\sqrt{5}-1)^2-\left[\frac{1}{2}\left\{KO-\frac{1}{2}KO(\sqrt{5}-1)\right\}\right]^2=\frac{1}{4}KO^2(\sqrt{5}-1)^2-\frac{1}{4}KO^2(3-\sqrt{5})^2=\frac{1}{4}KO^2\Big[(\sqrt{5}-1)^2-$

$\frac{1}{4}(3-\sqrt{5})^2\Big]=\frac{1}{4}KO^2\left[\frac{10-2\sqrt{5}}{4}\right]=\frac{1}{16}KO^2(10-2\sqrt{5})$. But $EF=\frac{1}{2}EA=\frac{1}{2}s$. ∴ $\frac{1}{4}s^2=\frac{1}{16}KO^2(10-2\sqrt{5})$, and $s=\frac{1}{2}KO\sqrt{10-2\sqrt{5}}$.

∴ $KO=\frac{2s}{\sqrt{10-2\sqrt{5}}}$, where s is a side of the lower base, $=\frac{10}{\sqrt{10-2\sqrt{5}}}$. KO may be considered the radius R of a circumscribed circle of the lower base. In like manner, the radius r of the circumscribed circle of the upper base may be found to be $\frac{2s'}{\sqrt{10-2\sqrt{5}}}$, where s' is a side of the upper base, $=\frac{2}{\sqrt{10-2\sqrt{5}}}$.

OF, the apothem of the lower base, $=\sqrt{(EO^2-EF^2)}=\sqrt{\left[\left(\frac{10}{\sqrt{10-2\sqrt{5}}}\right)^2-\left(\frac{5}{2}\right)^2\right]}=\frac{5}{2}\sqrt{\left(\frac{3+\sqrt{5}}{5-\sqrt{5}}\right)}$. In like manner, $fo=\frac{1}{2}\sqrt{\left(\frac{3+\sqrt{5}}{5-\sqrt{5}}\right)}$. ∴ $IF=OF-OI(=fo)=\frac{5}{2}\sqrt{\left(\frac{3+\sqrt{5}}{5-\sqrt{5}}\right)}-\frac{1}{2}\sqrt{\left(\frac{3+\sqrt{5}}{5-\sqrt{5}}\right)}=2\sqrt{\left(\frac{3+\sqrt{5}}{5-\sqrt{5}}\right)}$. $Ff=\sqrt{(If^2+IF^2)}=\sqrt{\left\{10^2+\left[2\sqrt{\left(\frac{3+\sqrt{5}}{5-\sqrt{5}}\right)}\right]^2\right\}}=\frac{2}{5}\sqrt{650+10\sqrt{5}}$ = the slant height.

By formula, $S=\frac{1}{2}(25+5)\frac{2}{5}\sqrt{650+10\sqrt{5}}=6\sqrt{650+10\sqrt{5}}=$ 155.5795 sq. ft.

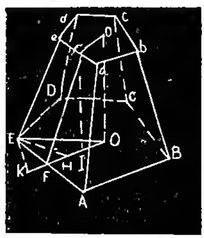


FIG. 42.

II.
1. 10 ft. = the altitude oO.
2. 5 ft. = EA, one of the equal sides of the lower base.
3. 1 ft. = ed, one of the equal sides of the upper base.
4. $\frac{2}{5}\sqrt{650+10\sqrt{5}}=fF$, the slant height.
5. 5×5 ft. = 25 ft. = the perimeter of the lower base.
6. 5×1 ft. = 5 ft. = the perimeter of the upper base.
7. ∴ $\frac{1}{2}(25+5)\frac{2}{5}\sqrt{650+10\sqrt{5}}$ = 155.5795 sq. ft. = the convex surface.

III. ∴ The convex surface of the frustum is 155.5795 sq. ft.

Prob. XCIII. To find the volume of a frustum of a pyramid or a cone.

Formula.—(*a*) $V=\frac{1}{3}a(A+\sqrt{AA'}+A')$, in which A is the area of the lower base, A' the area of the upper base and $\sqrt{AA'}$ the area of the mean base. When we have a frustum of a cone, (*b*) $V=\frac{1}{3}a(A+\sqrt{AA'}+A')=\frac{1}{3}a(\pi R^2+\sqrt{(\pi R^2\times\pi r^2)}+\pi r^2)=\frac{1}{3}a(\pi R^2+\pi Rr+\pi r^2)=\frac{1}{3}\pi a(R^2+Rr+r^2)$.

Rule.—(1) *Find the area of the mean base by multiplying the area of the upper and lower bases together and extracting the square root of the product.*

(2) *Add the upper, lower, and mean bases together and multiply the sum by $\frac{1}{3}$ the altitude.*

I. What is the solidity of a frustum of a cone whose altitude is 8 feet, the radius of the lower base 2 feet, and the radius of the upper base 1 foot?

By formula (*b*), $V=\frac{1}{3}\pi a(R^2+Rr+r^2)=\frac{1}{3}\pi 8(4+2+1)=\frac{1}{3}\times 56\pi=58.6433$ cu. ft.

II.
1. 8 ft.=the altitude.
2. 2 ft.=the radius of the lower base.
3. 1 ft.=the radius of the upper base.
4. 4π=the area of the lower base.
5. π=the area of the upper base.
6. $2\pi=\sqrt{4\pi\times\pi}$=the area of the mean base.
7. $4\pi+\pi+2\pi=7\pi$=the sum of the areas of the three bases.
8. $\therefore \frac{1}{3}\times 8\times 7\pi=58.6433$ cu. ft.=the solidity of the frustum.

III. ∴ The solidity of the frustum is 58.6433 cu. ft

PROBLEMS.

1. Find the entire surface of a right pyramid, of which the height is 2 feet and the base a square on a side of 1 ft. 8 in. *Ans.* 10 sq. ft.

2. Find the convex surface of a right pyramid 1 foot high, standing on a rectangular base whose length is 5 feet 10 inches and breadth 10 inches. *Ans.* 8 sq. ft. 128 sq. in.

3. Find the convex surface of a right pyramid having the same base and height as a cube whose edge is 10 inches. *Ans.* 223.6 sq. in.

4. Find the weight of a granite pyramid 9 feet high, standing on a square base whose side is 3 feet 4 inches, 1 cubic foot of granite weighing 165 lbs. *Ans.* 2 tons, 9 cwt. 12 lbs.

5. Find the height of a pyramid of which the volume is 623.52 cu. in., and the base a regular hexagon on a side of 1 foot. *Ans.* 5 inches.

6. The volume of a regular octahedron is 471.41 cubic feet; find the length of each edge. *Ans.* 10 feet.

7. Find the surface of a regular tetrahedron, if the perpendicular from one vertex to the opposite face is 5 inches.

8. A conical vessel is 5 inches in diameter and 6 inches deep. To what depth will a ball 4 inches in diameter sink in the vessel?

9. The ends of the frustum of a pyramid are squares whose sides are 20 inches and 4 inches, respectively. If its altitude is 15 inches, what is its convex surface? *Ans.* 110 sq. in.

10. What is the volume of a frustum of a pyramid whose upper base is 4 inches square, lower base 28 inches, and the length of the slant edges 15 inches?

11. The volume of a frustum of a cone is 407 cubic inches and its thickness is 10½ inches? If the diameter of one end is 8 inches, find the diameter of the other end. [$\pi=\frac{22}{7}$.] *Ans.* 6 inches.

12. A frustum of a pyramid has for its bases squares whose sides are respectively 0.6 m. and 0.5 m.; the altitude of the frustrum is 4 m. Find the volume.

13. The upper and lower bases of a frustum are squares whose sides are 8 inches and 28 inches, respectively, and the edges of the frustum are each 15 inches. Find the volume of the frustum; also the lateral surface.

14. A conical tent of slant height 9 feet covers a circular area 10 feet in diameter. Find the area of the canvas.

15. The radius of the base of a cone is to the altitude as 5:12 and its lateral surface is 420 sq. ft. Find the radius and altitude.

Ans. { Radius=10 feet, and altitude=24 feet.

16. The number expressing the volume of a circular cone is 7 times the number expressing its surface. Find the radius and altitude if they are in the ratio of 3 to 4.

17. A right circular cone, radius of base 3 feet and altitude 8 feet, is divided into two equal parts by a plane parallel to the base. How far does this plane cut the altitude from the base?

18. Find the dimensions of a right circular cylinder, $\frac{15}{16}$ as large as a similar cylinder, whose height is 20 feet and diameter 10 ft.

Ans. Height, $5\sqrt[3]{60}$ ft.; diameter of base, $2\frac{1}{2}\sqrt[3]{60}$ ft.

19. How many square inches of tin will be required to make a funnel, the diameters of whose top and bottom are to be 28 inches and 14 inches, respectively, and height 24 inches? *Ans.* 525 sq. in.

6. CONICAL UNGULAS.

1. ***A Conical Ungula*** (Lat. *ungula*, a claw, hoof, from *unguis*, a nail, claw, hoof) is a section or part of a cone cut off by a plane oblique to the base and contained between this plane and the base.

Prob. XCIV. To find the surface of a conical ungula.

Formula.—$S=\int_r^R s\sqrt{dx^2+dy^2}=$

$$\frac{1}{R-r}\sqrt{a^2+(R-r)^2}\int_r^R sdx=\frac{\sqrt{a^2+(R-r)^2}}{R-r}\int_r^R \left\{2\pi x-2x\cos^{-1}\left[\frac{(2R-t)r-(R+r-t)x}{R-r}\right]\right\}dx,$$

where a is the altitude of the ungula, R the radius of the base, r the radius of the upper base of the frustum from which the ungula is cut, t the distance the cutting plane cuts the base from the opposite extremity of the base, and x the radius of a section parallel to the base and at a distance $h-y$ from the base.

Prob. XCV. To find the volume of a conical ungula.

Formula.—$V=\int_r^R A dy=$

$$\frac{a}{R-r}\int_r^R \left\{ x^2 \cos^{-1}\frac{(2R-t)r-(R+r-t)x}{(R-r)x}+\frac{1}{(R-r)^2}\Big[(2R-t)r\right.$$

$$-(R+r-t)x\Big]\sqrt{\Big[-(2R-t)^2r^2+2r(2R-t)(R+r-t)x-}$$

$$\left.\overline{(2R-t)(2r-t)x^2\Big]}\right\} dx,$$

where the letters represent the same value as in the preceding problem and $dy=\left(\frac{a}{R-r}\right)dx$, since $y=\frac{a(x-r)}{R-r}$.

Prob. XCVI. To find the convex surface of a conical ungula, when the cutting plane passes through the opposite extremities of the ends of the frustum.

Formula.—$S=$

$$\frac{\pi}{R-r}\sqrt{a^2+(R-r)^2}\left\{R^2-\tfrac{1}{2}(R+r)\sqrt{Rr}\right\}.$$

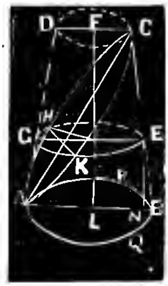


FIG. 43.

This formula is obtained by putting $t=0$, in the formula of Prob. XCIV., and integrating the result. For, in this problem, the cutting plane $AHCK$ passes through the opposite point A, and therefore the distance from A to the cutting plane is 0. $\therefore t=0$.

Rule.—*Multiply half the sum of the radii of the bases by the square root of their product and subtract the result from the square of the radius of the lower base. Multiply this difference by π times the slant height and divide the result thus obtained by the difference of the radii of the bases.*

Prob. XCVII. To find the volume of a conical ungula, when the cutting plane passes through the opposite extremities of the ends of the frustum.

Formula.—$V=\frac{\pi R^{\frac{3}{2}}a}{3(R-r)}\left(R^{\frac{3}{2}}-r^{\frac{3}{2}}\right).$

This formula is obtained by putting $t=0$, in the formula of Prob. XCV., and integrating the result.

Rule.—*Multiply the difference of the square roots of the cubes of the radii of the bases by the square root of the cube of the radius of the lower base and this product by* $\frac{1}{3}\pi$ *times the altitude.*

Divide this last product by the difference of the radii of the two bases and the quotient will be the volume of the ungula.

I. A cup in the form of a frustum of a cone is 7 in. in diameter at the top, 4 in. at the bottom, and 6 in. deep. If, when full of water, it is tipped just so that the raised edge of the bottom is visible; what is the volume of the water poured out?

By formula, $V=\frac{\pi R^{\frac{1}{2}}a}{3(R-r)}\left(R^{\frac{3}{2}}-r^{\frac{3}{2}}\right)=\frac{7}{6}\pi(49-8\sqrt{7})=$ 102.016989 cu. in.

Remark.—Fig. 43 inverted represents the form of the cup and $APBQ-C$ the quantity of water poured out, C being the tipped edge of the bottom.

I. A tank is 6 feet in diameter at the top, 8 feet at the bottom, and 12 feet deep. A plane passes from the top on one side to the bottom on the other side: into what segments does it divide the tank?

By formula, $V=\frac{\pi R^{\frac{1}{2}}a}{3(a-b)}(R^{\frac{3}{2}}-r^{\frac{3}{2}})=\frac{96\pi}{3(4-3)}(8-3\sqrt{3}),=$ $32\pi(8-3\sqrt{3})=281.87$ cu ft.

II.
1. 4 ft.$=AL$, the radius of the lower base.
2. 3 ft.$=DF$, the radius of the upper base, and
3. 12 ft.$=FL$, the altitude. Then
4. $\frac{\pi\sqrt{4^3}\times12}{3(4-3)}(\sqrt{4^3}-\sqrt{3^3})=32\pi(8-3\sqrt{3})=281.87$ sq. ft.=the volume.

III. ∴ The volume is 281.87 cu. ft.

Prob. XCVIII. To find the convex surface of a conical ungula, when the cutting plane FCE makes an angle CIB less than the angle DAB, i. e. when AI(=t) is less than DC(=2r).

$$\textbf{\textit{Formula.}}—S=\frac{1}{R-r}\sqrt{a^2+(R-r)^2}\left\{R^2\cos^{-1}\left(\frac{-R+t}{R}\right)\right.$$
$$\left.-\frac{r}{2r-t}(R-r)\sqrt{(2R-t)t}-\frac{r^2(R+r-t)}{2r-t}\sqrt{\frac{2R-t}{2r-t}}R^2\cos^{-1}\left(\frac{r-t}{r}\right)\right\}$$

This formula is obtained by integrating the formula of Prob. XCIV, recollecting that the co- efficient of x^2 is negative.

Prob. XCIX. To find the volume of a conical ungula, when the cutting plane FCE makes an angle CIB less than the angle DAB. i. e., when AI (=t) is less than CD (=2r).

Formula.— $V=\frac{a}{R-r}\Big\{\frac{1}{3}R^3\cos^{-1}\left(\frac{-R+t}{r}\right)$

$-\frac{2}{3}\left[\frac{Rr(R-r)}{t-2r}\right]\sqrt{(2R-t)t}+\frac{(R+r-t)(R-r)}{(2R-t)(t-2r)}\times$

$\left[(2R-t)t\right]^{\frac{3}{2}}+\frac{1}{3}r^3\left(\frac{2(R+r)t-4Rr-t^2}{(t-2r)^2}\right)$

$\sqrt{\frac{2R-t}{t-2r}}\cos^{-1}\left(\frac{r-t}{r}\right)$.

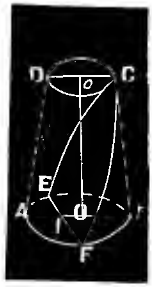


FIG. 44.

This formula is obtained by integrating the formula of Prob. XCV, recollecting that the coefficient of x^2 is negative.

Prob. C. To find the convex surface of a conical ungula, when the cutting plane FCE is parallel to the side AD, i. e., when AI(=t) is equal to DC(=2r).

Formula.—$S=\frac{1}{R-r}\sqrt{a^2+(R-r)^2}\Big\{R^2\cos^{-1}\left(\frac{-R+2r}{R}\right)$

$+2(R-2r)\sqrt{(R-r)r}-\frac{8}{3}(R-r)\sqrt{(R-r)r}\Big\}$

This formula is obtained by putting $t=2r$, in the formula of Prob. XCIV., and integrating the resulting equation.

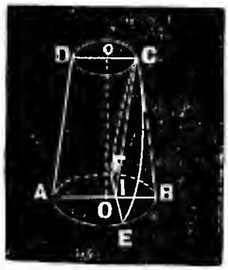


FIG 45.

Pob. CI. To find the volume of a conical ungula, when the cutting plane FCE is parallel to the side DA, i. e., when AI(=t) is equal to CD (=2r).

Formula.— $V=\frac{1}{3}a\Big\{\frac{R}{R-r}\Big[R^2\cos^{-1}\left(\frac{-R+2r}{R}\right)+$

$2(R-2r)\sqrt{(R-r)r}\Big]-\frac{16}{3}r\sqrt{(R-r)r}\Big\}$.

This formula is obtained by putting $t=2r$, in the formula of Prob. XCV., and integrating the resulting equation.

Prob. CII. To find the convex surface of a concial ungula, when the cutting plane FCE makes an angle CIB greater than the angle DAB, i. e., when AI (=t) is greater than DC, (=2r).

Formula.– $S=\frac{1}{R-r}\sqrt{a^2+(R-r)^2}\Big\{R^2\cos^{-1}\left(\frac{-R+t}{R}\right)$

$-\frac{r}{t-2r}(R-r)\sqrt{(2R-t)t}-\frac{r^2(R+r-t)}{t-2r}\sqrt{\frac{2R-t}{t-2r}}\log\Big[\{t-r+$

$(t-2r)\sqrt{\frac{t}{t-2r}}\}\div r\Big]\Big\}$.

This formula is obtained by integrating the formula of Prob. XCIV., remembering that the coefficient of x^2, which occurs in process of integrating, is positive.

Prob. CIII. To find the volume of a conical ungula, when the cutting plane FCE makes an angle CIB greater than the angle DAB, i. e., when AI(=t) is greater than DC(=2r).

Formula.— $V=\frac{a}{R-r}\left\{\frac{1}{3}R^3\cos^{-1}\left(\frac{-R+t}{R}\right)-\right.$

$\frac{2}{3}\left[\frac{Rr(R-r)}{t-2r}\right]\sqrt{(2R-t)t}+$

$\frac{(R+r-t)(R-r)}{(2R-t)(t-2r)}[(2R-t)t]^{\frac{3}{2}}+$

$\frac{1}{3}r^3\left(\frac{2(R+r)t-4Rr-t^2}{(t-2r)^2}\right)\sqrt{\frac{2R-t}{t-2r}}\times$

$\left.\log\left[\left(t-r+(t-2r)\sqrt{\frac{t}{t-2r}}\right)\div r\right]\right\}.$

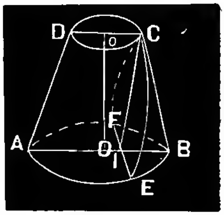


FIG. 46.

This formula is obtained by integrating the formula of Prob. XCV., regarding the coefficient of x^2 positive.

XII. THE SPHERE.

Prob. CIV. To find the convex surface of a sphere.

Formula.—$S=2\times2\pi y\sqrt{dy^2+dx^2}=4\pi R^2=\pi D^2$, where D is the diameter.

Rule.—*Multiply the square of the diameter by 3.141592.*

I. What is the surface of a sphere whose radius is 5 inches?

By formula, $S=4\pi R^2=4\pi\times25=314.1592$ sq. in.

II.
1. 5 in.=the radius.
2. 25 sq. in.= the square of the radius.
3. ∴ $4\pi\times25$ sq. in.=314.1592 sq. in.=the surface of the sphere.

III. ∴ 314.1592 sq. in.=the surface of the sphere.

NOTE.—Since πR^2 is the area of a circle whose radius is R, the area $(4\pi R^2)$ of a sphere is equal to four great circles of the sphere. The surface of a sphere is also equal to the convex surface of its circumscribing cylinder.

Prob. CV. To find the volume of a sphere, or a globe.

Formula.— $V=2\pi y^2dx=\frac{4}{3}\pi R^3=\frac{4}{3}\pi(\frac{1}{2}D)^3=\frac{1}{6}\pi D^3$.

Rule.—*Multiply the cube of the radius by* $\frac{4}{3}\pi$ *(=4.188782); or multiply the cube of the diameter by* $\frac{1}{6}\pi$*(=.5235987).*

I. What is the volume of a sphere whose diameter is 4 feet?

By formula, $V=\frac{4}{3}\pi R^3=\frac{4}{3}\pi 2^3=33.510256$ cu. ft.

II.
1. 2 ft.=the radius.
2. 8 cu. ft.$=2^3$=the cube of the radius.
3. ∴ 4.188782×8 cu. ft.=33.510256 cu. ft.=the volume of the sphere.

III. ∴ 33.510256 cu. ft.=the volume of the sphere.

Prob. CVI. To find the area of a zone.

A Zone is the curved surface of a sphere included between two parallel planes or cut off by one plane.

Formula.—$S=2\pi Ra$, in which a is the altitude of the segment of which the zone is the curved surface.

Rule.—*Multiply the circumference of a great circle of the sphere by the altitude of the segment.*

I. What is the area of a zone whose altitude is 2 feet, on a sphere whose radius is 6 feet?

By formula, $S=2\pi Ra=2\pi 6\times 2=24\pi=75.39822$ sq. ft.

II.
1. 6 ft.=the radius of the sphere.
2. 2 ft.=the altitude.
3. $12\pi=37.69911$ ft.=the circumference of a great circle of the sphere.
4. ∴ 2×37.69911=75.39822 sq. ft.=the area of the zone.

III. ∴ The area of the zone is 75.39822 sq. ft.

NOTE.—This rule is applicable whether the zone is the curved surface of the frustum of a sphere or the curved surface of a segment of a sphere.

Prob. CVII. To find the volume of the segment of a sphere.

Formula.—$V=\frac{1}{6}\pi a(3r_1^2+a^2)$ where r_1 is the radius of the base of the segment.

Rule.—*To three times the square of the radius of the base, add the square of the altitude and multiply the sum by $\frac{1}{6}\pi=.5235987$ times the altitude.*

I. What is the volume of a segment whose altitude is 2 inches and the radius of the base 8 inches?

By formula, $V=\frac{1}{6}\pi a(3r_1^2+a^2)=\frac{1}{6}\pi\times 2(3\times 64+4)=205.2406$ cu. in.

II.
1. 8 in.=the radius of the base.
2. 2 in.=the altitude of the segment.
3. 192 sq. in $=3\times 8^2$=three times the square of the radius.
4. 4 sq. in.=the square of the altitude.
5. 196 sq. in.=192 sq. in.+4 sq. in.=three times the square of the radius plus the square of the altitude.
6. $\frac{1}{6}\pi\times 2\times 196=205.2406$ cu. in.=the volume of the segment.

III. ∴ 205.2406 cu. in.=the volume of the segment.

NOTE.—From the formula $V=\frac{1}{6}\pi a(3r_1^2+a^2)$, we have $V=\frac{1}{2}\pi ar_1^2+\frac{1}{6}\pi a^3$ But $\frac{1}{2}\pi ar_1^2$ is the volume of a cylinder whose radius is r_1, and altitude $\frac{1}{2}a$, and $\frac{1}{6}\pi a^3$ is the volume of a sphere whose diameter is a. ∴ The volume of a segment of a sphere is equal to a cylinder whose base is the base of the segment and altitude half the altitude of the segment, plus a sphere whose diameter is the altitude of the segment.

Prob. CVIII. To find the volume of a frustum of a sphere, or the portion included between two parallel planes.

Formula.— $V=\frac{1}{6}\pi a[3(r_1^2+r_2^2)+a^2]=\frac{1}{2}a(\pi r_1^2+\pi r_2^2)+\frac{1}{6}\pi a^3$ *, in which r_1 is the radius of the lower base, r_2 the radius of the upper base.

Rule.—*To three times the sum of the squared radii of the two ends, add the square of the altitude; multiply this sum by .5235987 times the altitude.*

I. What is the volume of the frustum of a sphere, the radius of whose upper base is 2 feet and lower base 3 feet and altitude $\frac{1}{2}$ foot?

By formula, $V=\frac{1}{6}\pi a[3(r_1^2+r_2^2)+a^2]=\frac{1}{6}\pi\times\frac{1}{2}[3(9+4)+\frac{1}{4}]=$ 8.03839 cu. ft.

II.
1. 3 ft.=the radius of the lower base.
2. 2 ft.=the radius of the upper base.
3. 39 sq. ft.=$3(3^2+2^2)$=three times the sum of the squares of the radii of the two bases.
4. $\frac{1}{4}$ sq. ft.=the square of the altitude.
5. ∴ $\frac{1}{6}\pi\times\frac{1}{2}\times39\frac{1}{4}$=8.03839 cu. ft.=the volume of the frustum.

III. ∴ 8.03839 cu. ft.=the volume of the frustum.

Prob. CIX. To find the volume of spherical sector.

A Spherical Sector is the volume generated by any sector of a semi-circle which is revolved about its diameter.

Formula.— $V=\frac{2}{3}\pi aR^2$, where a is the altitude of the zone of the sector.

Rule.—*Multiply its zone by one-third the radius.*

---

* NOTE.—$\frac{1}{2}a(\pi r_1^2+\pi r_2^2)$=the volume of two cylinders whose bases are the upper and lower bases of the segment and whose altitude is half the altitude of the segment. $\frac{1}{6}\pi a^3$ is the volume of a sphere whose diameter is the altitude of the segment. Hence the volume of a segment of a sphere of two bases is equivalent to the volume of two cylinders whose bases are the upper and lower bases respectively of the segment and whose common altitude is the altitude of the segment, plus the volume of a sphere whose diameter is the altitude of the segment.

For a demonstration of this and the preceding formula, see *Wentworth's Plane and Solid Geometry, Bk. IX., Prob. XXXII.*

I. What is the volume of a spherical sector the altitude of whose zone is 2 meters and the radius of the sphere 6 meters?

By formula, $V=\frac{2}{3}\pi aR^2=\frac{2}{3}\pi\times2\times6^2=$ 150.7964m^3.

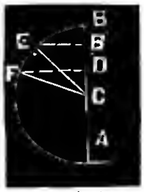


FIG. 47.

II.
1. 2m.=the altitude BD of the zone generated by the arc EF when the semicircle is revolved about AB.
2. 6m.=the radius EC of the sphere.
3. $2\pi6$m.=37.699104 m =the circumference of a great circle of the sphere.
4. $2\pi6\times2$=75.398208 m^2.=the area of the zone generated by EF, by Prob. CVI.
5. $\therefore\frac{1}{3}\times6\times75.398208=150.796416$m^3=the volume of the spherical sector.

III. $\therefore$ The volume of the spherical sector is 150.796416 m^3.

I. Find the diameter of a sphere of which a sector contains 7853.98 cu. ft., when the altitude of its zone is 6 feet.

By formula, $V=\frac{2}{3}\pi ar^2=\frac{2}{3}\pi\times6\times r^2$. $\therefore\frac{2}{3}\pi\times6\times r^2=$ 7853.98 cu. ft., or $4r^2=2500$ sq. ft., whence $2r=50$ feet, the diameter of the sphere.

II.
1. 6 ft.=the altitude of the zone.
2. $\therefore\frac{2}{3}\pi\times6\times r^2$=the volume of the sector. But
3. 7853.98 cu. ft.=the volume.
4. $\therefore\frac{2}{3}\pi\times6\times r^2$=7853.98 cu. ft.
5. r^2=625 sq. ft, by dividing by 4π.
6. $\therefore 2r$=50 ft., the diameter of the sphere.

III. $\therefore$ The diameter of the sphere is 50 feet.

Prob. CX. To find the area of a lune.

A Lune is that portion of a sphere comprised between two great semi-circles.

Formula.—$S=4\pi R^2\left(\frac{A}{360^\circ}\right)=4\pi R^2u$, where u is the quotient of the angle of the lune divided by 360°.

Rule.—*Multiply the surface of the sphere by the quotient of the angle of the lune divided by 360°*

I. Given the radius of a sphere 10 inches; find the area of a lune whose angle is 30°.

By formula, $S=4\pi R^2u=4\times\pi\times10^2\times(30^\circ\div360^\circ)=$ $\frac{1}{3}\pi10^2=104.7197$ sq. in.

II.
1. 10 in.=the radius of the sphere.
2. 30°=the angle of the lune.
3. $\frac{1}{12}=30°\div360°$=the quotient of the angle of the lune divided by 360°.
4. $4\pi 10^2=400\pi=1256.6368$ sq. in.=the surface of the sphere.
5. ∴ $\frac{1}{12}\times1256.6368$ sq. in.=104.7198 sq. in.=the area of the lune.

III. ∴ The area of the lune is 104.7198 sq. in.

Wentworth's New Plane and Solid Geometry, p. 371, Ex. 585.

Prob. CXI. To find the volume of a spherical ungula.

A Spherical Ungula is a portion of a sphere bunded by a *lune* and two great semi-circles.

Formula.— $V=\frac{4}{3}\pi R^3u$, where u is the same as in the last problem.

Rule.—*Multiply the area of the lune by one-third the radius; or, multiply the volume of the sphere by the quotient of the angle of the lune divided by 360°.*

I. What is the volume of a spherical ungula the angle of whose lune is 20°, if the radius of the sphere is 3 feet?

By formula, $V=\frac{4}{3}\pi R^3u=\frac{4}{3}\pi\times3^3\times(20°\div360°)=6.283184$ cu. ft.

II.
1. 3 ft.=the radius of the sphere.
2. $4\pi 3^2\times(20°\div360°)=6.283184$ sq. ft.=the area of the lune, by Prob CX
3. ∴ $\frac{1}{3}\times3\times6.283184=6.283184$ cu. ft.=the volume of the ungula.

III. ∴ 6.283184 cu. ft. is the volume of the ungula.

Prob. CXII. To find the area of a spherical triangle.

Formula.—$S=2\pi R^2\times(A+B+C-180°)\div360°$, in which A, B, and C are the angles of the spherical triangle.

Rule.—*Multiply the area of the hemisphere in which the triangle is situated by the quotient of the spherical excess (the excess of the sum of the spherical angles over 180°) divided by 360°.*

I. What is the area of a spherical triangle on a sphere whose diameter is 12, the angles of the triangle being 82°, 98°, and 100°?

By formula, $S=2\pi R^2\times(A+B+C-180°)\div360°=2\pi 6^2\times(82°+98°+100°-180°)\div360°=2\pi 6^2\times\frac{5}{18}=62.83184$=area.

II.
1. 6=the radius of the sphere.
2. $2\pi 6^2 = 72\pi$=the area of the hemisphere.
3. $(82°+98°+100°-180°)=100°$=the spherical excess.
4. $100° \div 360° = \frac{5}{18}$=the quotient of the spherical excess divided by 360°.
5. $\therefore \frac{5}{18} \times 72\pi = 62.83184$=the area of the spherical triangle.

III. ∴ The area of the spherical triangle is 62.83184.

(*Olney's Geometry and Trigonometry, Un. Ed., p. 238, Ex. 8.*)

Prob. CXIII. To find the volume of a spherical pyramid.

A Spherical Pyramid is the portion of a sphere bounded by a spherical polygon and the planes of its sides.

Formula.— $V = \frac{2}{3}\pi R^3 \times (E \div 360°)$, where E is the spherical excess.

Rule.—*Multiply the area of the base by one-third of the radius of the sphere*

I. The angles of a triangle, on a sphere whose radius is 9 feet, are 100°, 115°, and 120°; find the area of the triangle and the volume of the corresponding spherical pyramid.

By formula, $V = \frac{2}{3}\pi R^3 \times (E \div 360°) = \frac{2}{3}\pi R^3 \times (A+B+C-180°) \div 360° = \frac{2}{3}\pi 9^3 \times (100°+115°+120°-180°) \div 360° = \frac{31}{108}\pi 9^3 = 657.377126$ cu. ft.

II.
1. 9 ft.=the radius of the sphere.
2. $2\pi 9^2$=the area of the hemisphere in which the pyramid is situated.
3 $(100°+115°+120°-180°)=155°$=the sperical excess.
4. $\frac{31}{72} = 155° \div 360°$=the quotient of the spherical excess divided by 360°.
5. $\therefore \frac{31}{72} \times 2\pi 9^2 = \frac{31}{36} \times \pi 9^2$=the area of the base of the pyramid.
6. $\therefore \frac{1}{3} \times 9 \times \frac{31}{36} \times 2\pi 9^2 = 657.377126$ cu. ft.=the volume of the pyramid.

III. ∴ The volume of the spherical pyramid is 657.377126 cu. ft.

(*Van Amringe's Davies' Geometry and Trigonometry, p. 278, Ex. 15.*

I. Find the area of a spherical hexagon whose angles are 96°, 110°, 128°, 136°, 140°, and 150°, if the circumference of a great circle of the sphere is 10 inches.

Formula.—$S = 2\pi R^2 \frac{[T-(n-2)180°]}{360°}$, where T is the sum of the angles of the polygon and n the number of sides.

By formula, $S=2\pi R^2\times\frac{[T-(n-2)180^\circ]}{360^\circ}=2\pi\times\left(\frac{10}{2\pi}\right)^2\times$ $(96^\circ+110^\circ+128^\circ+136^\circ+140^\circ+150^\circ-(6-2)\times180^\circ)\div$ $360^\circ=\frac{50}{\pi}\times(760^\circ-720^\circ)\div360^\circ=\frac{1}{9}\frac{50}{\pi}=1.7684$ sq. in.

II.
1. $5\div\pi$=the radius of the sphere, since $2\pi R=10$ in..
2. $760^\circ=96^\circ+110^\circ+128^\circ+136^\circ+140^\circ+150^\circ$=the sum of the angles of the polygon.
3. $760^\circ-(6-2)\times180^\circ=40^\circ$=the spherical excess.
4. $\frac{1}{9}=40^\circ\div360^\circ$=the quotient of the spherical excess divided by 360°.
5. $2\pi\left(\frac{5}{\pi}\right)^2$=the area of the hemisphere on which the polygon is situated.
6. $\therefore\ \frac{1}{9}\times2\pi\left(\frac{5}{\pi}\right)^2=\frac{1}{9}\times50\div\pi=1.7684$ sq. in.

III. ∴ The area of the polygon is 1.7684 sq. in.

Wentworth's Geometry, Revised Ed., p. 374, Ex. 596.

NOTE.—A *Spherical Degree* may be defined as half of a lune whose measuring angle is 1°, the line of division being the arc of a great circle whose poles are the points of intersection of the semicircles bounding the lune. It, therefore, follows that the surface of a sphere contains 720 spherical degrees. It is easily proved in Spherical Geometry that the surface of a spherical triangle in spherical degrees equals the excess of the sum of its angles over a straight angle.

Hence, if the excess of the sum of the angles of a spherical triangle is, *e.g.*, 30°, it contains 30 spherical degrees and the surface of the triangle is $\frac{30}{720}$ or $\frac{1}{24}$ of the surface of the sphere.

PROBLEMS.

1. Find the ratio of the surface of a sphere to the surface: (i) of its circumscribed cylinder, (ii) of its circumscribed cube.

2. A cube and a sphere have equal surfaces; what is the ratio of their volumes? *Ans.* 72:100, nearly.

3. From a cubical block of rubber the largest possible rubber ball is cut. What decimal of the original solid is cut away?

4. Suppose the earth to be a perfect sphere, 8,000 miles in diameter; to what height would a person have to ascend in a balloon in order to see one-fourth of its surface? [*Formula.*—$h=\frac{2r}{n-2}$, where r is the radius of the earth, and $\frac{1}{n}$ is the part of the earth's surface visible to the observer. If the part of the earth visible to the observer is $\frac{p}{q}$, or $1/\frac{q}{p}$, $n=\frac{q}{p}$.

5. A paring an inch wide is cut from a smooth, round orange an inch and a half in diameter. What is its volume, if it is cut from the orange on a great circle of the orange? *Ans.* $\frac{1}{6}\pi$.

6. What would be the volume of a paring cut from the earth on the equator? *Ans.* $\frac{1}{6}\pi a^3$, where a is the width of the paring.

Remark.—This is a remarkable fact, since the volume of the paring is independent of the radius of the sphere.

7. If, when a sphere of cork floats in the water, the height of the submerged segment is ¾ of the radius, show that the weights of equal volumes of cork and water are as $3^4 : 4^4$.

Note.—The weight of a floating body is equal to the weight of the liquid it displaces.

8. A vertical cylindrical vessel whose internal diameter is 4 feet, is completely filled with water. If a metal sphere 25 inches in diameter is laid upon the rim of the vessel, find what weight of water will overflow.

Ans. 699 lbs., nearly.

9. A conical wine-glass $5\frac{1}{3}\sqrt{3}$ inches in diameter and 4 inches deep is filled with water. If a metal sphere $5\frac{1}{3}$ inches in diameter is placed in the vessel, what fraction of the whole contents will overflow? *Ans.* $\frac{5}{8}$.

10. Four equal spheres are tangent to each other. What is the radius of a sphere tangent to each?

11. To what depth will a sphere of ice, three feet in diameter, sink in water, the specific gravity of ice being $\frac{8}{9}$?

12. Find the volume removed by boring a 2-inch auger-hole through a 6-inch globe.

13.* What is the volume removed by chiseling a hole an inch square through an 8-inch globe?

Note.— This problem cannot be solved without the aid of the calculus.

XIII. SPHEROIDS.

1. ***A Spheroid*** is a solid formed by revolving an ellipse about one of its axes as an axis of revolution.

1. THE PROLATE SPHEROID.

1. ***The Prolate Spheroid*** is the spheroid formed by revolving an ellipse about its transverse diameter as an axis of revolution.

Prob. CXIV. To find the surface of a prolate spheroid.

Formulae.—$(a)\ S=2\int 2\pi y\,ds=2\int 2\pi y\sqrt{1+\frac{dy^2}{dx^2}}\,dx=$
$4\pi\int y\left(\frac{a^4y^2+b^4x^2}{a^4y^2}\right)^{\frac{1}{2}}dx, =\frac{4\pi}{a^2}\int_o^a\left[a^2(a^2b^2-b^2x^2)+b^4x^2\right]^{\frac{1}{2}}dx=$
$4\pi\int_o^a\frac{b}{a}(a^2-e^2x^2)^{\frac{1}{2}}dx=2\pi b^2+2\frac{\pi ab}{e}\sin^{-1}e, =2\pi b\left(b+\frac{a}{e}\sin^{-1}e\right)$, where
$e=\frac{\sqrt{a^2-b^2}}{a}=$ the eccentricity of the ellipse which generates the surface.

$$(b)\quad S=4\pi ab\left(1-\frac{e^2}{2.3}-\frac{e^4}{2.4.5}-\frac{3e^6}{2.4.6.7}-\frac{3.5e^8}{2.4.6.8.9}-\&c.\right)$$

Rule.—*Multiply the circumference of a circle whose radius is the semi-conjugate diameter by the semi-conjugate diameter increased by the product of the arc whose sine is the eccentricity into the quotient of the semi-transverse diameter divided by the eccentricity.*

I. Find the surface of a prolate spheroid whose transverse diameter is 10 feet and conjugate diameter 8 feet.

By formula (a), $S=2\pi b(b+\frac{a}{e}\sin^{-1}e)=2\pi 4(4+\frac{5}{e}\sin^{-1}e)=$ $2\pi 4\left[4+\left(5\div\frac{\sqrt{5^2-4^2}}{5}\right)\sin^{-1}\frac{\sqrt{5^2-4^2}}{5}\right]=2\pi 4[4+$ $(5\div\frac{3}{5})\sin^{-1}\frac{3}{5}]=\frac{2}{3}\pi[48+100\sin^{-1}\frac{3}{5}]=\frac{2}{3}\pi[48+100\times\frac{53093}{259200}\pi]=$ $\frac{2}{3}\pi[48+100\times .6435053]=235.3064$ sq. ft.

II.
1. $25.1327412=2\pi 4=$the circumference of a circle whose radius is the semi-conjugate diameter of the ellipse.
2. $\frac{3}{5}=\frac{\sqrt{5^2-4^2}}{5}=$the eccentricity.
3. $\frac{25}{3}$ ft.$=5$ ft$\div\frac{3}{5}=$the quotient of the semi-transverse diameter divided by the eccentricity.
4. $.6435053=$the arc (to the radius 1) whose sine is $\frac{3}{5}$, or the eccentricity.
5. 5.3625442 ft.$=\frac{25}{3}$ft. $\times .6435053=\frac{25}{3}$ ft. $\times$ the arc whose sine is $\frac{3}{5}$
6. 9.3625442 ft.$=4$ ft.$+5.3625442$ ft.$=$semi-conjugate diameter increased by said product.
7. $\therefore$ 235.3064 sq. ft.$=9.3625442\times 25.1327412=$the surface of the prolate spheroid.

III. $\therefore$ The surface of the prolate spheroid is 235.3064 sq. ft.

Prob. CXV. To find the volume of a prolate spheroid.

Formula.— $V=\int\pi y^2dx=\pi\frac{b^2}{a^2}\int_{-a}^{a}(a^2-x^2)dx=$ $\pi\frac{b^2}{a^2}\left[a^2x-\frac{1}{3}x^3\right]_{-a}^{a}=\frac{4}{3}\pi b^2a$, in which b is the semi-conjugate diameter, and a the semi-transverse diameter.

Rule.—*Multiply the square of the semi-conjugate diameter by the semi-transverse diameter and this product by* $\frac{4}{3}\pi$.

I. What is the volume of a prolate spheroid, whose semi-transverse diameter is 50 inches, and semi-conjugate diameter 30 inches.

By formula, $V=\frac{4}{3}\pi b^2a=\frac{4}{3}\pi 30^2\times 50=188495.559$ cu. in.

II. 1. 30 in.=the semi-conjugate diameter,
2. 50 in.=the semi-transverse diameter.
3. 900 sq. in=the square of the semi-conjugate diameter.
4. 45000 cu. in.=50×900=the square of the semi-conjugate diameter by the semi-transverse diameter.
5. ∴ $\frac{4}{3}\pi 45000=\frac{4}{3}\times 3.14159265\times 45000$ cu. in.= 188495.559 cu. in.=the volume of the prolate spheroid.

III. ∴ The volume of the prolate spheroid is 188495.559 cu. in.

2. THE OBLATE SPHEROID.

1. *An Oblate Spheroid* is the spheroid formed by revolving an ellipse about its conjugate diameter as an axis of revolution.

Prob. CXVI. To find the surfae of an oblate spheroid.

Formulae.—(*a*) $S=\int 2\pi x ds=2\int_{-a}^{a} 2\pi x\sqrt{1+\frac{dx^2}{dy^2}}\,dy=$ $2\pi a^2\left(1+\frac{1-e^2}{2e}\log\left\{\frac{1+e}{1-e}\right\}\right)$.

(*b*) $S=4\pi ab(1+\frac{e^2}{2.3}-\frac{e^4}{2.4.5}+\frac{3e^6}{2.4.6.7}-\frac{3.5e^8}{2.4.6.8.9}+\&c.)$

Prob. CXVII. To find the volume of an oblate spheroid.

Formula.— $V=\int \pi x^2 dy=2\int_0^b \pi\frac{a^2}{b^2}(b^2-y^2)dy=\frac{4}{3}\pi a^2 b$.

Rule.—*Multiply the square of the semi-transverse diameter by the semi.conjugate diameter and this product by $\frac{4}{3}\pi$.*

I. What is the volume of an oblate spheroid, whose transverse diameter is 100 and conjugate diameter 60?

By formula, $V=\frac{4}{3}\pi a^2 b=\frac{4}{3}\pi 50^2\times 30=314159.265$.

II. 1. $30=\frac{1}{2}$ of 60= the semi-conjugate diameter.
2. $50=\frac{1}{2}$ of 100=the semi-transverse diameter.
3. $2500=50^2$=the square of the semi-transverse diameter.
4. 75000=30×2500=the square of the semi-transverse diameter multiplied by the semi-conjugate diameter.
5. ∴ $\frac{4}{3}\pi\times 75000=314159.265$= the volume of the oblate spheroid.

III. ∴ The volume of the oblate spheroid is 314159.265.

NOTE.—Since the volume of a prolate spheroid is $\frac{4}{3}\pi b^2 a$. We may write $\frac{4}{3}\pi b^2 a=\frac{2}{3}(\pi b^2\times 2a)$. But $\pi b^2\times 2a$ is the volume of a cylinder the radius of whose base is b and altitude $2a$. ∴ The volume of a prolate spheroid is $\frac{2}{3}$ of the circumscribed cylinder. In like manner, it may be shown that the volume of an oblate spheroid is $\frac{2}{3}$ of its circumscribed cylinder.

The following is a general rule for finding the volume of a spheroid; *Multiply the square of the revolving axis by the fixed axis and this product by $\frac{1}{6}\pi$.*

Prob. CXVIII. To find the volume of the middle frustum of a prolate spheroid, its length, the middle diameter, and that of either of the ends being given.

Case I.

When the ends are circular, or parallel to the revolving axis.

Formula— $V=\frac{1}{12}\pi(2D^2+d^2)l$, where D is the middle diameter CD, d the diameter HI of an end, and l the length of the frustum.

Rule.—*To twice the square of the middle diameter add the square of the diameter of either end and this sum multiplied by the length of the frustum, and the product again by* $\frac{1}{12}\pi$, *will give the solidity.*

I. What is the volume of the middle frustum $HIGF$ of a prolate spheroid, if the middle diameter CD is 50 inches, and that of either of the ends HI or FG is 40 inches, and its length OK 18 inches?

By formula, $V=\frac{1}{12}\pi(2D^2+d^2)l=\frac{1}{12}\pi(2\times50^2+40^2)18=$ 31101.767265 cu. in.

II.
1. 50 in.=the middle diameter CD.
2. 40 in.=the diameter of either end as HI.
3. 18 in.=the length OK of the frustum.
4. 5000 sq. in.$=2\times50^2=$twice the square of the middle diameter.
5. 1600 sq. in.$=40^2=$the sqaure of the diameter of either end.
6. 5000 sq. in.+1600 sq. in.=6600 sq. in.
7. $18\times6600=118800$ cu. in.
8. ∴ $\frac{1}{12}\pi\times118800$ cu. in.=31101.767265 cu. in.=the volume.

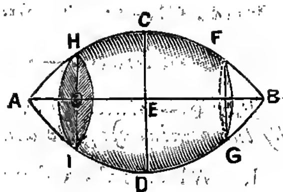


FIG. 48.

III. ∴ The volume of the frustum is 31101.767265 cu. in.

Case II.

When the ends are elliptical, or perpendicular to the revolving axis.

Formula.— $V=\frac{1}{12}\pi(2Dd+D'd')l$, where D and d are the transverse and conjugate diameters of the middle section and D' and d' the transverse and conjugate diameter of the ends and l the distance between the ends.

Rule.—(1) *Multiply twice the transverse diameter of the middle section by its conjugate diameter, and to this product add* *the product of the transverse and conjugate diameter of either of the ends.*

(2) *Multiply the sum, thus found, by the distance of the ends, or the height of the frustum, and the product again by* $\frac{1}{12}\pi$ *and the result will be the volume.*

I. What is the volume of the middle frustum of an oblate spheroid, the diameter of the middle section being 100 inches and 60 inches; those of the end 60 inches and 36 inches; and the length 80 inches?

By formula, $V=\frac{1}{12}\pi(2Dd+D'd')\,l=\frac{1}{12}\pi(2\times100\times60+60\times36)\,80=296566.44616$ cu. in.

II.
1. 100 in.=the transverse diameter FC of the middle section.
2. 60 in.=the conjugate diameter ms of the middle section.
3. 12000 sq. in.=$2\times100\times60$=twice the product of the diameters of the middle section.
4. 60 in.=the transverse diameter AB of the end.
5. 36 in.=the conjugate diameter $2(nc)$ of the end.
6. 2160 sq. in.=the product of the diameters of the end.
7. 14160 sq. in.=12000 sq. in.+2160 sq. in.
8. $80\times14160=1132800$ cu. in. = the product of said sum by the height of the frustum.
9. ∴ $\frac{1}{12}\pi\times1132800$ cu. in.=296566.44616 cu. in.=the volume of the frustum.

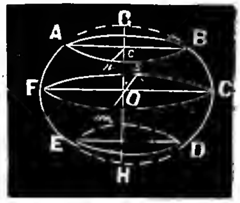


FIG. 49.

III. ∴ The volume of the frustum is 296566.44616 cu. in.

Prob. CXIX. To find the volume of a segment of a prolate spheroid

Case I.

When the base is parallel to the revolving axis.

Formula.— $V=\frac{1}{6}\pi h^2\left(\frac{d}{D}\right)^2(3D-2h)$, where h is the height of the segment, d the revolving axis, and D the fixed axis.

Rule.—(1) *Divide the square of the revolving axis by the square of the fixed axis, and multiply the quotient by the difference between three times the fixed axis and twice the height of the segment.*

(2) *Multiply the product, thus found, by the square of the height of the segment, and this product by* $\frac{1}{6}\pi$, *and the result will be the volume of the segment.*

I. What is the volume of a segment of a prolate spheroid of which the fixed axis is 10 feet and the revolving axis 6 feet and the height of the segment 1 foot?

By formula, $V=\frac{1}{6}\pi h^2\left(\frac{d}{D}\right)^2(3D-2h)=$ $\frac{1}{6}\pi\times6^2\left(\frac{6}{10}\right)^2(3\times10-2\times1)=$ 5.277875652 cu. ft.

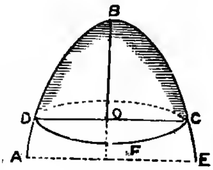


FIG. 50.

II.
1. 10 ft.=the transverse diameter $2BF$.
2. 6 ft.=the conjugate diameter AE.
3. $\frac{9}{25}=\frac{6^2}{10^2}$ =the square of the conjugate diameter divided by the square of the transverse diameter.
4. 28 ft.$=3\times10$ ft.-2×1 ft.= the difference between three times the transverse diameter and twice the height of the segment.
5. $\frac{9}{25}\times28$ ft.$=10\frac{2}{25}$ ft.=the product of said quotient by said difference.
6. $10\frac{2}{25}\times1^2=10\frac{2}{25}$ cu. ft.
7. $\therefore$ $\frac{1}{6}\pi\times10\frac{2}{25}$ cu. ft.=5.277865652 cu. ft.=the volume.

III. $\therefore$ The volume of the segment is 5.277875652 cu. ft.

Case II.

When the base is perpendicular to the revolving axis.

Formula.— $V=\frac{1}{6}\pi h^2\left(\frac{D}{d}\right)(3d-2h)$, where d is the revolving axis, D the fixed axis, and h the height of the segment.

Rule.—(1) *Divide the fixed axis by the revolving axis, and multiply the quotient by the difference between three times the revolving axis and twice the height of the segment,*

(2). *Multiply the product, thus found, by the square of the height of the segment, and this product again by* $\frac{1}{6}\pi$.

I. Required the volume of the segment of a prolate spheroid, its height being 6 inches, and the axes 50 and 30 inches respectively.

By formula, $V=\frac{1}{6}\pi h^2\left(\frac{d}{D}\right)(3d-2h)=\frac{1}{6}\pi\times6^2\left(\frac{30}{50}\right)\times$

$(3\times 30-2\times 6)=2450.442267$ cu. in.

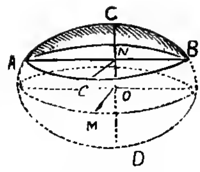


FIG. 51.

II.
1. 50 in.=the transverse diameter, or axis.
2. 30 in.=the conjugate diameter $2MO$.
3. $\frac{5}{3}=50\div 30$=the quotient of the trans verse diameter divided by the conjugate diameter.
4. 78 in.$=3\times 30$ in.-2×6in.=the difference between three times the conjugate or revolving axis, and twice the height of the segment.
5. 130 in.$=\frac{5}{3}\times 78$ in.=the product of said quotient by said difference.
6. 4680 cu. in$=130\times 6^2$=the square of the height of the segment by said product.
7. $\therefore\ \frac{1}{6}\pi\times 4680$ cu. in.=2450.442269 cu. in.=the volume of segment.

III. $\therefore$ The volume of the segment is 2450.442269 cu. in.

XIV. CONOIDS.

1. ***A Conoid*** is a solid formed by the revolution of a conic section about its axis.

1. THE PARABOLIC CONOID.

1. ***A Parabolic Conoid*** is the solid formed by revolving a parabola about its axis of abscissa.

Prob. CXX. To find the surface of a parabolic conoid, or paraboloid.

Formulae.—(*a*) $S=\int 2\pi y ds=\int 2\pi y\sqrt{1+\frac{dx^2}{dy^2}}dy=$

$\frac{2\pi y}{p}(p^2+y^2)^{\frac{1}{2}}dy,=\frac{2\pi}{3p}(p^2+y^2)^{\frac{3}{2}}+C=\frac{2\pi}{3p}\{(p^2+y^2)^{\frac{3}{2}}-p^3\}$,

where $2p$ is the latus rectum of the parabola and y the radius of the base of the conoid, or the ordinate of the parabola.

(*b*) $S=\frac{8}{3}\pi\sqrt{p}\{(p+x)^{\frac{3}{2}}-p^{\frac{3}{2}}\}$, where $2p$ is the same as above and x the altitude of the conoid, or the axis of abscissa of the parabola.

Rule.—*To the square of half the latus rectum, or principal parameter, add the square of the radius of the base of the conoid and extract the square root of the cube of the sum; from this result, subtract the cube of half the latus rectum and multiply the* *difference by* 2π, *and divide the product by one and one-half times the latus rectum.*

I. Determine the convex surface of a paraboloid whose axis is 20, and the diameter of whose base is 60.

From the equation of the parabola, $y^2=2px$, we have $30^2=2p\times20$; whence $2p=45$.

$\therefore$ By formula (a), $S=\frac{2\pi}{3p}\{(p^2+y^2)^{\frac{3}{2}}-p^3\}$

$=\frac{4\pi}{3\times45}\left\{\left[\left(\frac{45}{2}\right)^2+30^2\right]^{\frac{3}{2}}-\left(\frac{45}{2}\right)^3\right\}=$

$\frac{1}{2}\pi\times25\times(125-27)=49\times25\times3.14159265=$ 3848.45118.

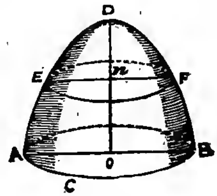


FIG. 52.

II.
1. $30=$ the radius AO of the base of the conoid.
2. $20=$the altitude OD. Then by a property of the parabola,
3. $30^2=2p\times20$, whence
4. $p=22\frac{1}{2}$, the principal parameter of the parabola.
5. $\left(\frac{15}{2}\right)^3\times125=\sqrt{\left[(22\frac{1}{2})^2+30^2\right]^3}=$the square root of the cube of the sum of the squares of half the latus rectum and the radius of the base.
6. $\left(\frac{45}{2}\right)^3=$the cube of half the latus rectum.
7. $\left(\frac{15}{2}\right)^3\times125-\left(\frac{45}{2}\right)^3=\left(\frac{15}{2}\right)^3(125-27)=$the difference between said square root and the cube of half the latus rectum.
8. $2\pi\times\left(\frac{15}{2}\right)^3(125-27)=\pi\times98\times\left(\frac{15}{2}\right)^3=2\pi$ times said difference.
9. $\therefore 2\pi98\times\left(\frac{15}{2}\right)^3\div(\frac{3}{2}\times45)=3848.45118=$the surface of the conoid.

III. $\therefore$ The surface of the conoid is 3848.45118.

Prob. CXXI. To find the volume of a parabolic conoid.

Formula.— $V=\int\pi y^2dx=\int\pi2p\,x\,dx=\pi px^2=\frac{1}{2}\pi(2px)x=\frac{1}{2}\pi y^2x$, where y is the radius of the base and x the altitude.

Rule.—*Multiply the area of the base by the altitude and take half the product.*

I. What is the volume of parabolic conoid, the radius of whose base is 10 feet and the altitude 14 feet?

By formula, $V=\frac{1}{2}\pi y^2x=\frac{1}{2}\pi 10^2\times 14=700\times\pi=2202.114855$ cu. ft.

II. 1. 10 ft.=the radius of the base.
2. 14 ft.=the altitude.
3. $\pi 10^2=314.159265$ sq. ft. the area of the base.
4. $\therefore \frac{1}{2}\times 14\times 314.159265=2202.114855$ cu. ft.=the volume of the conoid.

III. ∴ The volume of the conoid=2202.114855 cu. ft.

NOTE.—Since the volume of the conoid is $\frac{1}{2}\pi y^2x$, it is half of its circumscribed cylinder.

Prob. CXXII. To find the convex surface of a frustum of a parabolic conoid of which the radius of the lower base is R and the upper base r.

$$\textit{Formula.}—S=\int_r^R 2\pi y ds=\frac{2\pi}{3p}\left\{(p^2+R^2)^{\frac{3}{2}}-(p^2+r^2)^{\frac{3}{2}}\right\}.$$

I. What is the volume of the frustum of a parabolic conoid of which the radius of the lower base is 12 feet, the radius of the upper base 8 feet, and the altitude of the frustum 5 feet?

Since $12^2=2px'$ and $8^2=2px$, $12^2-8^2=2p(x'-x)$. Bnt $x'-x=5$ feet. ∴ $12^2-8^2=2p\times 5$, whence $2p=16$, the latus rectum.

∴ By formula, $S=\frac{2\pi}{3p}\left[(p^2+R^2)^{\frac{3}{2}}-(p^2+r^2)^{\frac{3}{2}}\right]=$

$\frac{2\pi}{3\times 8}\left[(8^2+12^2)^{\frac{3}{2}}-(8^2+8^2)^{\frac{3}{2}}\right]=\frac{\pi}{12}(832\sqrt{13}-1024\sqrt{2})=$
$\frac{16}{3}\pi[13\sqrt{13}-16\sqrt{2}]$.

Prob. CXXIII. To find the volume of the frustum of a parabolic conoid, when the bases are perpendicular to the axis of abscissa.

Formula.— $V=\frac{1}{2}\pi R^2x'-\frac{1}{2}\pi r^2x=\frac{1}{2}\pi(x'-x)(R^2+r^2)=\frac{1}{2}\pi a(R^2+r^2)$.

Rule.—*Multiply the sum of the squares of the radii of the two bases by π and this product by half the altitude.*

I. What is the volume of the frustum of a parabolic conoid, the diameter of the greater end being 60 feet, and that of the lesser end 48 feet, and the distance of the ends 18 feet?

By formula, $V=\frac{1}{2}\pi a(R^2+r^2)=\frac{1}{2}\times 18\times\pi(30^2+24^2)=9\pi(900+576)=9\times 1476\times\pi=13284\pi=41732.9177626$ cu. ft,

II.
1. 30 ft.=the radius of the larger base.
2. 24 ft.=the radius of the lesser base.
3. 18 ft.=the altitude of the frustum.
4. 900 sq. ft.=the square of the radius of the lower base.
5. 576 sq. ft.=the square of the radius of the upper base.
6. 1476 sq. ft =900 sq. ft.+576 sq. ft.=their sum.
7. ∴ $\frac{1}{2}\times18\times\pi\times1476=13284\times\pi=41732.9177626$ cu. ft.= the volume of the frustum of the conoid.

III. ∴ The volume of the frustum is 41732.9177626 cu. ft.

2. THE HYPERBOLIC CONOID.

1. ***An Hyperbolic Conoid*** is the solid formed by revolving an hyperbola about its axis of abscissa.

Prob. CXXIV. To find the surface of an hyperbolic conoid, or hyperboloid.

Formula.—$S=\int 2\pi y ds=2\pi\int y\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx=$

$$2\pi\int y\sqrt{\frac{e^2x^2-a^2}{x^2-a^2}}\,dx=2\pi\int\frac{b}{a}\sqrt{e^2x^2-a^2}\,dx=\pi\frac{b}{a}\left\{x\sqrt{e^2x^2-a^2}-\frac{a^2}{e}\log\left[x+\frac{1}{e}\sqrt{e^2x^2-a^2}\right]\right\}+C=\pi\frac{b}{a}\left\{x\sqrt{e^2x^2-a^2}-ab+\frac{a^3}{\sqrt{a^2+b^2}}\log\left(\frac{a+\frac{ab}{\sqrt{a^2+b^2}}}{x+\frac{1}{e}\sqrt{e^2x^2-a^2}}\right)\right\}.$$

Prob. CXXV. To find the volume of an hyperbolic conoid.

Formula.—$V=\frac{1}{6}\pi(R^2+d^2)h$, where R is the radius of the base, d the middle diameter, and h the altitude.

Rule.—*To the square of the radius of the base add the square of the middle diameter between the base and the vertex; and this sum multiplied by the altitude, and the product again by $\frac{1}{6}\pi$, will give the solidity.*

I. In the hyperboloid ACB, the altitude CO is 10, the radius AO of the base 12; and the middle diameter DE 15.8745; what is the volume?

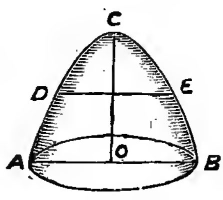


FIG. 53.

II.
1. 10=the altitude CO.
2. 12=the radius AO of the base.
3. 15.8745=the middle diameter DE.
4. $144=12^2$=the square of the radius of the base.
5. $251.99975=15.8745^2$=the square of the middle diameter.

6. 395.99975=251.99975+144=the sum of the squares of the radius of the base and the middle diameter.
7. $\therefore \frac{1}{6}\pi \times 10 \times 395.99975 = 2073.454691$=the volume.

III. $\therefore$ The volnme of the conoid is 2073.454691.

Prob. CXXVI To find the volume of the frustum of an hyperbolic conoid.

Formula.— $V=\frac{1}{6}\pi a(R^2+d^2+r^2)$, where R is the radius of the larger base, and r the radius of the lesser base, and d the middle diameter of the frustum.

Rule.—*Add together the squares of the greater and lesser semi-diameters, and the square of the whole diameter in the middle; then this sum being multiplied by the altitude, and the product again by $\frac{1}{6}\pi$, will give the solidity.*

XV. QUADRATURE AND CUBATURE OF SURFACES AND SOLIDS OF REVOLUTION.

1. CYCLOID.

Prob. CXXVII. To find the surface generated by the revolution of a cycloid about its base.

Formula.—$S=2\int 2\pi y ds=4\pi\int y\sqrt{dx^2+dy^2}=$
$4\pi\sqrt{2r}\int_0^{2r}\frac{ydy}{\sqrt{2r-y}}=\frac{64}{3}\pi r^2.$

Rule.—*Multiply the area of the generating circle by $\frac{64}{3}$.*

Prob. CXXVIII. To find the volume of the solid formed by revolving the cycloid about its base.

Formula.— $V=2\int\pi y^2dx=2\pi\int_0^{2r}\frac{y^3dy}{\sqrt{2ry-y^2}}=5\pi^2r^3=$
$\frac{5}{8}\times\pi(2r)^2\times 2\pi r.$

Rule.—*Multiply the cube of the radius of the generating circle by $5\pi^2$.*

Prob. CXXIX. To find the surface generated by revolving the cycloid about its axis.

Formula.—$S=\int 2\pi y ds=4\pi\sqrt{2r}\int y\frac{dx}{x}=8\pi r^2(\pi-\frac{4}{3}).$

Rule.—*Multiply eight times the area of the generating circle by π minus $\frac{4}{3}$.*

Prob. CXXX. To find the volume of the solid formed by revolving the cycloid about its axis.

Formulu.— $V=\int\pi y^2dx=2\pi\int y^{\frac{5}{2}}(2r-y)^{-\frac{1}{2}}dy=\pi r^3(\frac{3}{2}\pi^2-\frac{8}{3})$.

Rule.—*Multiply* $\frac{3}{4}$ *of the volume of a sphere whose radius is that of the generating circle by* $\frac{3}{2}\pi^2-\frac{8}{3}$.

Prob. CXXXI. To find the surface formed by revolving the cycloid about a tangent at the vertex.

Let P be a point on the curve, $AE=PB=y$, $EP=AB=x$, $AC=CF=r$, and the angle $ACF=\theta$. Then we shall have $AE=y=AC-CE=r-r\cos\theta$; and $AB=x=FP+EF=AF+EF=r\theta+r\sin\theta$.

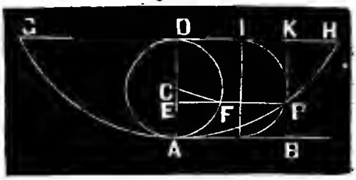


FIG. 54.

$\therefore$ ***Formula.***—$S=4\pi\int y\sqrt{dx^2+dy^2}=4\pi r\int_0^\pi(r-r\cos\theta)\sqrt{r^2(1+\cos\theta)^2+r^2\sin^2\theta}\,d\theta=4\pi r^2\int_0^\pi(1-\cos\theta)\times\sqrt{2+2\cos\theta}\,d\theta=8\pi r^2\int_0^\pi(1-\cos\theta)\cos\frac{1}{2}\theta d\theta=16\pi r^2\int_0^\pi(1-\cos^2\frac{1}{2}\theta)\times\cos\frac{1}{2}\theta d\theta=16\pi r^2\int_0^\pi(\cos\frac{1}{2}\theta-\cos^3\frac{1}{2}\theta)d\theta=16\pi r^2\Big[2\sin\frac{1}{2}\theta-\frac{2}{3}\sin\frac{1}{2}\theta\cos^2\frac{1}{2}\theta-\frac{4}{3}\sin\frac{1}{2}\theta\Big]_0^\pi=\frac{32}{3}\pi r^2$.

Rule.—*Multiply the area of the generating circle by* $\frac{32}{3}$.

Prob. CXXXII To find the volume formed by revolving a cycloid about a tangent at the vertex.

Formula.— $V'=2\int\pi y^2\,dx=2\pi\int_0^\pi(r-r\cos\theta)^2r(1+\cos\theta)d\theta=2\pi r^3\int_0^\pi(1-\cos\theta)^2(1+\cos\theta)d\theta=2\pi r^3\int_0^\pi(1-\cos\theta-\cos^2\theta+\cos^3\theta)d\theta=\pi^2r^3=$the volume generated between the curve and the tangent.

$\therefore\ V=\pi AD^2\times GH-V'=\pi(2r)^2\times2\pi r-\pi^2r^3=7\pi^2r^3$.

Rule.—*Multiply the cube of the radius of the generating circle by* $7\pi^2$.

2. CISSOID.

Prob. CXXXIII. To find the volume generated by revolving the cissoid about the axis of abscissa.

Formula.— $V=\int\pi y^2dx=\int\pi\frac{x^3}{2a-x}dx=\pi(-\frac{1}{3}x^3-ax^2-4a^2x+8a^3\log\Big(\frac{2a}{2a-x}\Big)$.

Prob. CXXXIV. To find the volume formed by revolving the cissoid about its asymptote.

Formula.— $V=2\int\pi(AR)^2dy(Fig.\ 21)=2\pi(2a-x)^2\times\frac{(3a-x)\sqrt{x}}{(2a-x)^{\frac{3}{2}}}dx=2\pi[\frac{1}{3}(2ax-x^2)^{\frac{3}{2}}+2a\int(2ax-x^2)dx]=2\pi^2a^3.$

Prob. CXXXV. To find the volume formed by revolving the Witch of Agnesi about its asymptote.

Formula.— $V=\int\pi y^2dx=\Big[\pi y^2x-4\pi a\int(2ay-y^2)^{\frac{1}{2}}dy\Big]_0^{2a}=4\pi^2a^3$

Prob. CXXXVI. To find the volume formed by revolving the Conchoid of Nicomedes about its asymptote, or axis of abscissa.

Formula.— $V=\int\pi y^2dx=\pi\int\Big[-\frac{ab^2}{\sqrt{b^2-y^2}}+y(b^2-y^2)^{\frac{1}{2}}-b^2\frac{y}{\sqrt{b^2-y^2}}\Big]dy=\pi b^2\Big(\pi a+\frac{4b}{3}\Big).$

3. SPINDLES.

(a). *The Circular, Spindle.*

A Circular Spindle is the solid formed by revolving the segment of a circle about its chord.

Prob. CXXXVII. To find the volume of a circular spindle.

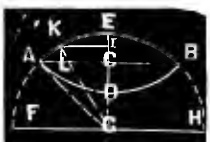


FIG. 55.

Let $AEBD$ be the circular spindle formed by revolving the segment $ACBE$ about the chord ACB. Let $AB=2a$, the length of the spindle, and $ED=2b$, the middle diameter of the spindle. Let $CI=KL=x$, the radius of any right section of the spindle, and $KI=CL=y$. Then the required volume of the spindle is $V=2\pi\int_0^a x^2dy$..(1). Let $R=(a^2+b^2)\div2b$..(2), be the radius of the circle and θ the angle AGE. Then by a property of the circle, $KI^2=(2R-EI)\times EI$, or $y^2=(2R-EI)\times EI$. But $EI=EG-IG=R-(IC+CG)=R-(x+R\cos\theta)$. $\therefore y^2=\{2R-[R-(x+R\cos\theta)\}\{R-(x+R\cos\theta)\}$ $=[R+(R\cos\theta+x)]\times[R-(R\cos\theta+x)]=R^2-(R\cos\theta+x)^2$; whence $x=\sqrt{R^2-y^2}-R\cos\theta$..(3). Substituting this value of x in (1), we have $V=2\pi\int_0^a(\sqrt{R^2-y^2}-R\cos\theta)^2dy=2\pi[R^2(1+$

$$\cos^2\theta)y-\tfrac{1}{3}y^3-2R\cos\theta\left(\tfrac{1}{2}R^2\sin^{-1}\frac{y}{R}-\tfrac{1}{2}y\sqrt{R-y^2}\right)\Big]_0^a=2\pi$$

$$\left\{2aR^2-\tfrac{4}{3}a^3-(R-b)\left[R^2\sin^{-1}\frac{a}{R}-a\sqrt{R^2-a^2}\right]\right\}.$$

Rule.—*Multiply the area of the generating segment by the path of its center of gravity.—Guildin's Rule.*

(b). *The Parabolic Spindle.*

A Parabolic Spindle is a solid formed by revolving a parabola about a double ordinate perpendicular to the axis.

Prob. CXXXVIII. To find the volume of a parabolic spindle.

Formula.— $V=2\int_0^b \pi(h-x)^2dy=2\pi\int_0^b(h^2-2hx+x^2)dy=2\pi\int_0^b(h^2dy-2h\frac{y^2}{2p}dy+\frac{y^4}{4p^2}dy)=2\pi\Big[h^2y-\frac{1}{3}\frac{h}{p}y^3+\frac{1}{20}\frac{1}{p^2}y^5\Big]_0^b=2\pi\Big[h^2y-\frac{2}{3}hxy+\frac{1}{5}x^2y\Big]_0^b=2\pi[h^2b-\frac{2}{3}hbx+\frac{1}{5}bx^2]$. But $x=h$, when $y=b$.

$\therefore V=2\pi[h^2b-\frac{2}{3}h^2b+\frac{1}{5}h^2b]=\frac{16}{15}\pi h^2b=\frac{8}{15}\times2b\times\pi h^2$.

Rule.—*Multply the volume of its circumscribed cylinder by* $\frac{8}{15}$.

I. What is the volume of a parabolic spindle whose length AC is 3 feet and height BD 1 foot?

By formula, $V=\frac{16}{15}\pi h^2b=\frac{8}{15}\pi\times1^2\times3=4.9945484$ cu. ft.

II.
1. 1 ft.=height BD of the spindle.
2. 3 ft.=length AC.
3. $\pi\times1^2\times3=9.42477795$ cu. ft. the volume of its circumscribed cylinder.
4. $\therefore$ $\frac{8}{15}\times9.42477795$ cu. ft.=4.9945484 cu. ft., the volume of the parabolic spindle.

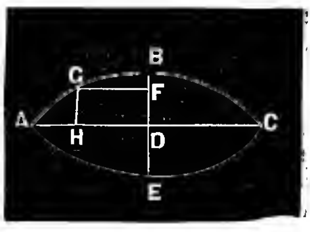


FIG. 56.

III. $\therefore$ The volume of the spindle is 4.9945484 cu. ft.

Prob. CXXXIX. To find the volume generated by revolving the arc of a parabola about the tangent at its vertex.

Let APC be an arc of a parabola revolved about AB, and let P be any point of the curve. Let $AE=PF=x$, and $AF=PF=y$. Then the area of the circle described by the line PF is πx^2.

$\therefore$ ***Formula.***— $V=2\pi\int x^2dy=2\pi\int\frac{y^4}{4p^2}dy=2\pi\times\frac{1}{4p^2}\times$

$\frac{1}{5}y^5=\frac{1}{5}\pi x^2y=\frac{1}{5}\pi h^2b$, where h=the height, and $b=CD$, the ordinate of the curve.

Rule.—*Multiply the volume of its circumscribed cylinder by* $\frac{1}{5}$.

Prob. CXL. To find the volume generated by revolving the arc APC of the parabola about BC parallel to the axis AD.

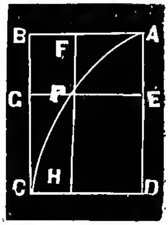


FIG. 57.

The area of the circle generated by the line GP is $\pi(b-y)^2$.

$\therefore$ ***Formula.***—$V=\pi\int(b-y)^2dx=\frac{1}{6}\pi b^2h$.

Rule.—*Multiply the volume of its circumscribed cylinder by* $\frac{1}{6}$.

NOTE.—In the last two problems, the volume considered, lies between the curve and the lines AB and BC respectively. The volume generated by the segment ACD is found by subtracting the volume found in the two problems from the volume of the circumscribed cylinders.

Prob. CXLI. To find the volume formed by revolving a semi-circle about a tangent parallel to its diameter.

Let the semi-circle be revolved about the tangent AG. Let $AC=R, PF=AG=EC=y, AF=GP=x$. Then the area of the circle generated by the line GP is πx^2. But $x^2=2R^2-2R(R^2-y^2)^{\frac{1}{2}}-y^2$; for, $FC^2=PC^2-PF^2$, or $(R-x)^2=R^2-y^2$; whence $x=R-\sqrt{R^2-y^2}$, and $x^2=2R^2-2R\sqrt{R^2-y^2}-y^2$

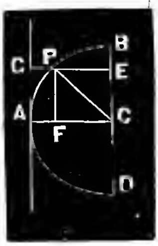


FIG. 59.

$\therefore$ ***Formula.***—$V=2\int\pi x^2dy=2\pi\int(2R^2-2R\sqrt{R^2-y^2}-y^2)dy=\frac{1}{3}\pi R^3(10-3\pi)$, which is the entire volume external to the semi-circle.

Rule.—*Multiply one-fourth of the volume of a sphere whose radius is that of the generating semi-circle by* $(10-3\pi)$.

XVI. REGULAR SOLIDS.

1. ***A Regular Solid*** is a solid contained under a certain number of similar and equal plane figures.

2. ***The Tetrahedron,*** or ***Regular Pyramid,*** is a regular solid bounded by four triangular faces.

3. ***The Hexahedron,*** or ***Cube,*** is a regular solid bounded by six square faces.

4. ***The Octahedron*** is a regular solid bounded by eight triangular faces.

5. ***The Dodecahedron*** is a regular solid bounded by twelve pentagonal faces.

6. *The Icosahedron* is a regular solid bounded by twenty equilateral triangular faces.

These are the only regular solids that can possibly be formed.

If the following figures are made of pasteboard, and the dotted lines cut half through, so that the parts may be turned up and glued together, they will represent the five regular solids.

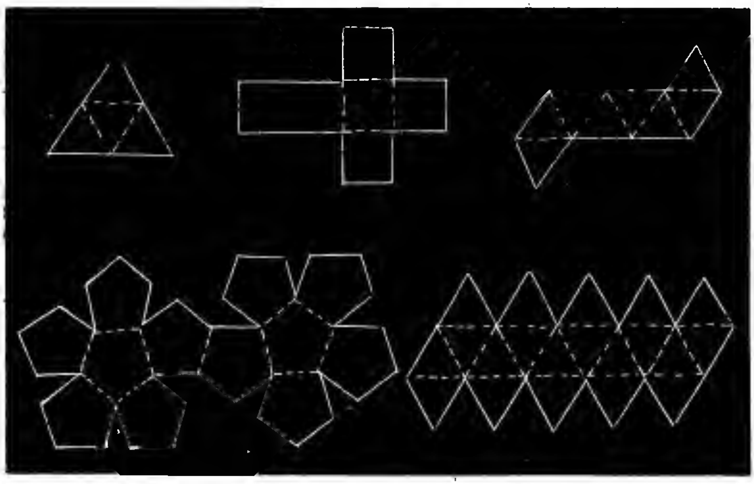


FIG. 59.

1. TETRAHEDRON.

Prob. CXLII. To find the surface of a tetrahedron.

Formula.—$S=l^2\sqrt{3}$, where l is the length of a linear side.

Rule.—*Multiply the square of a linear side by $\sqrt{3}=1.7320508$.*

I. What is the surface of a tetrahedron whose linear edge is 2 inches.

By formula, $S=l^2\sqrt{3}=2^2\sqrt{3}=4\sqrt{3}=6.9282$ sq. in.

II. 1. 2 in.=the length of a linear side.
2. 4 sq. in.$=2^2$=the square of a linear side.
3. ∴ $\sqrt{3}\times 4$ sq. in.$=1.73205\times4$ sq. in.$=6.9282$ sq. in., the surface.

III. ∴ The surface of the tetrahedron is 6.9282 sq. in.

Prob. CXLIII. To find the volume of a tetrahedron.

Formula.— $V=\frac{1}{12}\sqrt{2}\,l^3$, where l is the length of a linear side.

Rule.—*Multiply the cube of a linear side by* $\frac{1}{12}\sqrt{2}$, *or* .11785.

I. Required the solidity of a tetrahedron whose linear side is 6 feet?

By formula, $V=\frac{1}{12}\sqrt{2}\,l^3=\frac{1}{12}\sqrt{2}\times6^3=18\sqrt{2}=25.455843$ cu. ft.

II. 1. 6 ft.=the length of a linear side.
2. 216 cu. ft.=the cube of the linear side.
3. ∴ $\frac{1}{12}\sqrt{2}\times216$ cu. ft.$=\sqrt{2}\times18$ cu. ft.=25.45843 cu. ft.

III. ∴ The volume of the tetrahedron is 25.45843 cu. ft.

2. OCTAHEDRON.

Prob. CXLIV. To find the surface of an octahedron.

Formula.—$S=2\sqrt{3}\,l^2$.

Rule.—*Multiply the square of a linear side by* $2\sqrt{3}$, *i. e., by two times the square root of three.*

I. What is the surface of an octahedron whose linear side is 4 feet?

By forumla, $S=2\sqrt{3}\,l^2=2\sqrt{3}\times l^2=32\sqrt{3}=55.4256$ cu. ft.

II. 1. 4 ft.=the length of a linear side.
2. 16 sq. ft.$=4^2$=the square of the linear side.
3. ∴ $2\sqrt{3}\times16$ sq. ft.$=\sqrt{3}\times32$ sq. ft.$=1.73205\times32$ sq. ft.= 55.4256 sq. ft.

III. ∴ The surface of the octahedron is 55.4256 sq. ft.

Prob. CLXV. To find the volume of an octahedron.

Formula.—$V=\frac{1}{3}\sqrt{2}\,l^3$

Rule.—*Multiply the cube of a linear side by* $\frac{1}{3}\sqrt{2}$, *i. e., by one-third of the square root of two.*

I. What is the volume of an octahedron whose linear side is 8 inches?

By formula, $V=\frac{1}{3}\sqrt{2}\,l^3=\frac{1}{3}\sqrt{2}\times8^3=.4714045\times512=241.359104$ cu. in.

II. 1. 8 in.=the length of a linear side.
2. 512 cu. in.$=8^3$=the cube of a linear side.
3. ∴ $\frac{1}{3}\sqrt{2}\times512$ cu. in $=\frac{1}{3}\times1.4142135\times512$ cu. in.= 241.359104 cu. in.

III. ∴ The volume of the octahedron is 241.359104 cu. in.

3. DODECAHEDRON.

Prob. CXLVI. To find the surface of a dodecahedron.

Formula.—$S=15\sqrt{\left(\frac{5+2\sqrt{5}}{5}\right)}l^2=20.6457285\times l^2$.

Rule.—*Multiply the square of a linear side by* $15\sqrt{[\frac{1}{5}(5+2\sqrt{5})]}$, *or 20.6457285.*

I. What is the surface of a dodecahedron whose linear side is 3 feet?

By formula, $S=15\sqrt{\left(\frac{5+2\sqrt{5}}{5}\right)}l^2=20.6457285\times 9$
$=185.8115565$ sq. ft.

II.
1. 3 ft.=the length of a linear side.
2. 9 sq. ft.$=3^2$=square of a linear side.
3. $\therefore 15\sqrt{\left(\frac{5+2\sqrt{5}}{5}\right)}\times 9$ sq. ft$=20.6457285\times 9$ sq. ft. $=185.8115565$ sq. ft.

III. The surface of the dodecahedron is 185.8115565 sq. ft.

Prob. CXLVII. To find the volume of a dodecahedron.

Formula.—$V=5\sqrt{\left(\frac{47+21\sqrt{5}}{40}\right)}l^3=7.663115\times l^3$.

Rule.—*Multiply the cube of a linear side by* $5\sqrt{\left(\frac{47+21\sqrt{5}}{40}\right)}$, *or 7.663115.*

I. The linear side of a dodecahedron is 2 feet; what is its volume?

By formula, $V=5\sqrt{\left(\frac{47+21\sqrt{5}}{40}\right)}l^3=7.663115\times 8$
$=61.20492$ cu. ft.

II.
1. 2 ft.=the length of a linear side.
2. 8 cu. ft.$=2^2$=cube of a linear side.
3. $\therefore 5\sqrt{[\frac{1}{40}(47+21\sqrt{5})]}\times 8$ cu. ft.$=7.663115\times 8$ cu. ft. $=61.20492$ cu. ft., the volume.

III. $\therefore$ The volume of the dodecahedron is 61 20492 cu. ft.

4. ICOSAHEDRON.

Prob. CXLVIII. To find the surface of an icosahedron.

Formula.—$S=5\sqrt{3}l^2=8.66025\times l^2$.

Rule.—*Multiply the square of a linear side by* $5\sqrt{3}$, *or 8.66025.*

I. What is the surface of an icosahedron whose linear side is 5 feet.

By formula, $S=5\sqrt{3}l^2=5\sqrt{3}\times5^2=125\sqrt{3}=216.50625$ sq. ft.

II.
1. 5 ft.=length of a linear side.
2. 25 sq. ft.$=5^2$=the square of a linear side.
3. $\therefore 5\sqrt{3}\times25$ sq. ft.$=8.66025\times25$ sq. ft.=216.50625 sq. ft. =the surface.

III. ∴ The surface of the icosahedron is 216.50625 sq. ft.

Prob. CXLIX. To find the solidity of an icosahedron.

Formula.— $V=\frac{5}{6}\sqrt{[\frac{1}{2}(7+3\sqrt{5})]}l^3=2.18169\times l^3$.

Rule.—*Multiply the cube of a linear side by* $\frac{5}{6}\sqrt{[\frac{1}{2}(7+3\sqrt{5})]}$, *or 2.18169*

I. What is the volume of an icosahedron whose linear side is 3 feet?

By formula, $V=\frac{5}{6}\sqrt{[\frac{1}{2}(7+3\sqrt{5})]}l^3=2.18169\times3^3=58.90563$ cu.ft.

II.
1. 3 ft.=the length of a linear side.
2. 27 cu. ft.$=3^3$=the cube of a linear side.
3. $\therefore \frac{5}{6}\sqrt{[\frac{1}{2}(7+3\sqrt{5})]}\times27$ cu. ft.$=2.18169\times27$ cu. ft. =58.90563 cu. ft.=the volume.

III. ∴ The volume of the icosahedron is 58.90563 cu. ft.

NOTE.—The surface and volume of any of the five regular solids may be found as follows:

Rule (1).—*Multiply the tabular area by the square of a linear side, and the product will be the surface*

Rule (2).—*Multiply the tabular volume by the cube of a linear side, and the product will be the volume.*

Surfaces and volumes of the regular solids, the edge being 1.

NO. OF SIDES.	NAMES.	SURFACES.	VOLUMES.
4	Tetrahedron	1.73205	0.11785
6	Hexahedron	6.00000	1.00000
8	Octahedron	3.46410	0.47140
12	Dodecahedron	20.64578	7.66312
20	Icosahedron	8.66025	2.18169

XVII. PRISMATOID.

1. ***A Prismatoid*** is a polyhedron whose bases are any two polygons in parallel planes, and whose lateral faces are triangles determined by so joining the vertices of these bases, that each lateral edge, with the preceding, forms a triangle with one side of either base.

2. ***A Prismoid*** **is a prismatoid whose bases have the same number of sides, and every corresponding pair parallel.**

Prob. CI. To find the volume of any prismatoid.

Formula (*a*).— $V=\frac{1}{4}a(B_1+3A_{\frac{2}{3}a})=\frac{1}{4}a(B_2+3A'_{\frac{2}{3}a})$, where a is the altitude, B_1 the area of the lower base, $A_{\frac{2}{3}a}$ the area of a section distant from the lower base two-thirds the altitude, B_2 the area of the upper base, and $A'_{\frac{2}{3}a}$ the area of a section distant two-thirds the altitude from the upper base.

Remark.—This simplest Prismoidal Formula is due to Prof. George B. Halsted, A. M., Ph. D., Professor of Mathematics in the University of Texas, Austin, Texas, who was the first to demonstrate this important truth. The formula universally applies to all prisms and cylinders; also to all solids uniformly twisted, e. g. the square screw; also to the paraboloid, the right circular cone, the frustum of a paraboloid, the hyperboloid of one nappe, the sphere, prolate spheroid, oblate spheroid, frustum of a right cone, or of a sphere, spheroid, or the elliptic paraboloid, the groin, hyperboloid, or their frustums. For a complete demonstration of the Prismoidal Formula, see *Halsted's Elements of Geometry or Halsted's Mensuration.*

Rule.—(*a*) *Multiply one-fourth its altitude by the sum of one base and three times a section distant from that base two-thirds the altitude.*

Formula (*b*). $V=\frac{1}{6}a(B_1+4M+B_2)$, where a is the altitude, B_1 and B_2 the areas of the lower and upper bases respectively, and M the area of a section midway between the two bases.

Rule.—(*b*) *Add the area of the two bases and four times the mid cross-section; multiply this sum by one-sixth the altitude.*

PROBLEMS.

1. Find the weight of a steel wedge whose base measures 8 inches by 5 inches, and the height of the wedge being 6 inches; if 1 cu. in. of steel weighs 4.53 oz.?

2. Find the volume of a prismatoid of altitude 3.5 cm., the bases being rectangles whose corresponding dimensions are 3 cm. by 2 cm. and 3.5 cm. by 5 cm.

3. The base of a wedge is 4 by 6, the altitude is 5, and the edge, *e*, is 3. Find the volume.

4. A stick of timber is 12 feet long and is 4 inches by 5 inches at one end and 8 inches by 3 inches at the other. Find the number of cubic feet in the stick. *Ans.* $1\frac{17}{18}$ cu. ft.

5. How far from the larger end must the stick described in the last problem be sawed into two parts in order that the parts shall have equal solidity?

XVIII. CYLINDRIC RINGS.

1. ***A Cylindric Ring*** is a solid generated by a circle lying wholly on the same side of a line in its own plane and revolving abont that line. Thus, if a circle whose center is O be revolved about DC as an axis, it will generate a cylindric ring whose diameter is AB and inner diameter 2 BC. OC will be the radius of the path of the center O.

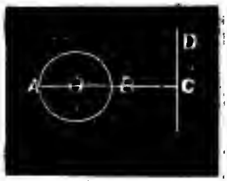


FIG. 60.

Prob. CLI. To find the area of the surface of a solid ring.

Formula.—$S=2\pi r\times 2\pi R=4\pi^2 rR$, where r is the radius of the ring, and R is the distance from the center of the ring to the center of the inclosed space.

Rule.—*Multiply the generating circumference by the path of its center. Or, to the thickness of the ring add the inner diameter and this sum being multiplied by the thickness, and the product again by 9.8697044 will give the area of the surface.*

I. What is the area of the surface of a ring whose diameter is 3 inches and the inner diameter 12 inches.

By formula, $S=4\pi^2 rR=4\pi^2\times 1\frac{1}{2}\times(1\frac{1}{2}+6)=\pi^2\times 45=9.8696044\times 45=444.132198$ sq. in.

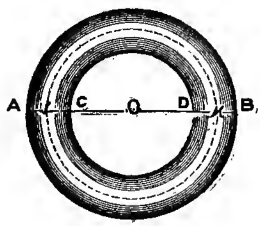


FIG. 61.

II.
1. $1\frac{1}{2}$ in.$=\frac{1}{2}$ of 3 in.=the radius r of the ring.
2. 6 in.$=\frac{1}{2}$ of 12 in.=the radius of the inclosed space.
3. 6 in.$+1\frac{1}{2}$ in.$=7\frac{1}{2}$ in. = the radius R of the center of the ring.
4. $\pi AC=\pi 3$=the circumference of a section.
5. $\pi IK=2\pi IO=2\pi 7\frac{1}{2}=\pi 15$=the path of the center.
6. ∴ $\pi 3\times\pi 15=\pi^2 45=444.132198$ sq. in.=the area of the surface of the ring.

III. ∴ The area of the surface of the ring is 444.132198 sq. in.

Prob. CLII. To find the volume of a cylindric ring.

Formula.—$V=\pi^2 r^2 R=\pi r^2\times\pi R$, where r is the radius AI of the ring, and R the distance from the center of the ring to the center of the inclosed space.

Rule.—*Multiply the area of the generating circle by the path of its center. Or, to the thickness of the ring add the inner diameter, and this sum being multiplied by the square of half the thickness, and the product again by 9.8696044, will give the volume.*

I. What is the volume of an anchorring whose inner diameter is 8 inches, and thickness in metal 3 inches?

By formula, $V=\pi^2r^2R=\pi^2\times(1\frac{1}{2})^2\times(3+8)=24.75\times 9.8696044=244.2727089$ cu. in.

II.
1. $1\frac{1}{2}$ in. $=\frac{1}{2}$ of 3 in. = the radius of the ring.
2. 8 in. = the inner diameter.
3. 4 in. + $1\frac{1}{2}$ in. = $5\frac{1}{2}$ in. = the radius R of the path of its center.
4. $\pi(1\frac{1}{2})^2$ = the area of the generating circle.
5. $2\pi(5\frac{1}{2})=\pi\times11$ = the path of its center.
6. $\therefore \pi 11\times\pi(1\frac{1}{2})^2=\pi^2\times24.75=9.86044\times24.75$ $=244.2727089$ cu. in., the volume of the ring.

III. ∴ The volume of the ring is 244.2727089 cu. in.

PROBLEMS.

1. Find the surface and volume of a ring, the radius of the inner circumference being $10\frac{1}{2}$ inches and the diameter of the cross-section $3\frac{1}{2}$ inches. *Ans.* 847 sq. in.; $741\frac{1}{8}$ cu. in.

2. Find the surface and volume of a ring, the diameters of the inner and outer circumferences being 9.8 inches and 12.6 inches respectively. *Ans.* 154.88 sq. in.; 54.21 cu. in.

3. An anchor ring, whose inner diameter is 20 inches and the diameter of a cross-section 10 inches, is cut by a plane perpendicular to its axis and at a distance of 4 inches from the center. Find the volume and surface of the segment removed.

4*. In the same ring, find the volume and surface of the segment of the ring, if the plane is passed parallel to the axis of the ring, and at a distance of 6 inches from the axis.

5*. What is the volume, if the plane be passed at a distance of 12 inches from the axis?

XIX. SIMILAR SURFACES.

Similar Plane Surfaces are surfaces having their homologous sides proportional and their homologous angles equal.

Principle.—*Similar areas are to each other as the squares of their like dimensions or as the squares of any other homologous lines.*

I. The area of a rectangle whose length is 20 rods is 120 sq. rods; what is the area of a similar rectangle whose length is 30 rods?

II.
1. 20 rods = the length of the given rectangle, and
2. 120 sq. rd. = its area.
3. 30 rods = the length of the required rectangle.
4. $\therefore 20^2:30^2::120$ sq. rd. : (?). Whence,
5. $?=(120\times30^2)\div20^2=270$ sq. rd.

III. ∴ The area of the rectangle is 270 sq. rd.

I. The area of a rectangle whose width is 7 feet, is 210 sq. ft.; what is the length of a similar rectangle whose area is 2100 sq. ft.

II.
1. 210 sq. ft.=the area of given rectangle, and
2. 7 ft.=its width. Then
3. $210 \div 7 = 30$ ft.=its length.
4. ∴ 210 sq. ft.: 2100 sq. ft. : : 30^2 :(?). Whence,
5. $? = (2100 \times 30^2) \div 210 = 300$ ft.=the length of the required rectangle.

III. The length of the required rectangle is 300 feet.

PROBLEMS.

1. The sides of a triangle are 21, 20, and 13 inches; find the area of a similar triangle whose sides are to the corresponding sides of the first as 25:3.

2. In a survey map an estate of 144 acres is represented by a quadrilateral, *ABCD*. The diagonal, *AC*, is 6 inches, and the perpendiculars from *B* and *D* on *AC* are 1.8 inches and .9 inches respectively. On what scale was the map drawn? *Ans.* 6 inches to the mile.

3. A man 6 feet in height, standing 15 feet from a lamp-post, observes that his shadow cast by the light at the top of the post is 8 feet in length; how long would his shadow be if he were to approach 8 feet nearer to the post? *Ans.* 2 ft. 4 in.

4. A man, wishing to ascertain the width of an impassable canal, takes two rods, 3 feet and 5 feet in length. The shorter he fixes vertically on one bank and then retires at right angles to the canal, until on resting the other rod vertically on the ground he sees the ends of the two rods in a line with the remote bank; if the distance between the rods is 60 feet, what is the width of the canal? *Ans.* 90 feet.

5. A man wishing to find the height of a tower, fixes a rod 11 feet in length vertically on the grouud at a distance of 80 feet from the tower. On retiring 10 feet further from the tower he sees the top of the rod in line with the top of the tower. If the observer's eye is $5\frac{1}{2}$ feet above the ground, find the height of the tower. *Ans.* 55 feet.

6. A triangle *ABC* is divided into two equal parts by a straight line *XY*, drawn parallel to the base *BC*. If *AB*=100 inches, find *AX*.

7. In a given triangle a triangle is inscribed by joining the middle points of the sides. In this inscribed triangle another similar triangle is inscribed, and so on. What fraction of the given triangle is the area of the sixth triangle so drawn?

8. (*a*) In a given square whose side is 16 inches a square is inscribed by joining the middle points of the sides of the given square; in this inscribed square a square is inscribed in like manner, and so on; find the area of the fifth square. (*b*) If the process be continued *ad infinitum* what is the sum of the areas of all the squares?

9. In a circle of a radius of 32 inches an equilateral triangle is inscribed, and in this triangle a circle. In this circle an equilateral triangle is again inscribed, and in the triangle a circle, and so on. If the process is continued, find the area of the fourth circle and find which of the circles has an area of $3\frac{1}{7}$ sq. in.? *Ans.* $50\frac{2}{7}$ sq. in.; the sixth.

10. A field of 9 acres is represented in a plan by a triangle whose sides are 25, 17, and 12 inches. On what scale is the plan drawn and what length will be represented by 80 inches? *Ans.* $\frac{1}{792}$; 1 mile.

11. The following is used by lumbermen in finding the diameter of trees at any height above the ground: If the tree casts a definite shadow on a horizontal plane, stand on the edge of the shadow and observe where the line of light from the sun to your eye strikes the tree. Then measure the shadow of the tree at the point where the shadow of your head strikes the ground. The width of the shadow is the diameter of the tree at the point where the line of light from the eye to the sun strikes it. What principle is involved?

XX. SIMILAR SOLIDS.

Solids bounded by plane surfaces are similar if the homologous edges are proportional and the homologous polyhedral angles are equal and similarly placed.

All spheres are similar solids. Cylinders generated by similar rectangles, and cones generated by similar right triangles are similar.

Principle.—*Similar solids are to each other as the cubes of their like dimensions or as the cubes of any other homologous lines.*

I. If the weight of a well proportioned man, 5 feet in height, be 125 lbs., what will be the weight of a similarly proportioned man 6 feet high?

II.
1. 5 ft.=the height of the first man, and
2. 125 lbs.=his weight.
3. 6 ft.=the height of the second man.
4. $\therefore 5^3 : 6^3 :: 125$ lbs : (?). Whence,
5. $? = (125\times6^3)\div5^3 = 216$ lbs., the weight of the second man.

III. $\therefore$The weight of the man whose height is 6 feet, is 216 lbs.

PROBLEMS.

1. The edges of two cubes are as 4:3; find the ratio of their surfaces and their volumes.

2. The surfaces of two spheres are in the ratio of 25:4; find the ratio of their volumes.

3. At what distance from the base must a cone, whose height is 1 foot, be cut by a plane parallel to the base, in order to be divided into two parts of equal volume? *Ans.* 2.47 in.

4. A right circular cone is intersected by two planes parallel to the base and trisecting the height. Compare the volumes of the three parts into which the cone is divided. *Ans.* 1:7:19.

5. The dimensions of a rectangular parallelopiped are 5, 6, and 9 feet. Find the dimensions of a similar solid, having 8 times the volume of the given parallelopiped. *Ans.* 10, 12, and 18 feet.

6. The volume of a pyramid whose altitude is 7 in., is 686 cu. in. Find the volume of a similar pyramid whose altitude is 21 inches. *Ans.* 17496 cu. in.

7. Two bins of similar form contain, respectively, 375 and 648 bushels of wheat. If the first bin is 3 ft. 9 in. wide, what is the width of the second? *Ans.* 6 feet.

THEOREM OF PAPPUS.

If a plane curve lies wholly on one side of a line in its own plane, and revolving about that line as an axis, it generates thereby a surface of revolution, the area of which is equal to the product of the length of the revolving line into the path of its center of mass; and a solid the volume of which is equal to the revolving area into the length of the path described by its center of mass.

XXI. MISCELLANEOUS MEASUREMENTS.

1. MASONS' AND BRICKLAYERS' WORK.

Masons' work is sometimes measured by the cubic foot, and sometimes by the *perch.* A perch is $16\frac{1}{2}$ ft. long, $1\frac{1}{2}$ ft. wide, 1 ft. deep, and contains $16\frac{1}{2}\times1\frac{1}{2}\times1=24\frac{3}{4}$ cu. ft.

Prob. CLIII. To find the number of perch in a piece of masonry.

Rule.—*Find the solidity of the wall in cubic feet by the rules given for the mensuration of solids, and divide the product by $24\frac{3}{4}$.*

I. What is the cost of laying a wall 20 feet long, 7 ft. 9 in. high, and 2 feet thick, at 75 cts. a perch.

II.
1. 20 ft = the length of the wall,
2. 7 ft. 9 in. = $7\frac{3}{4}$ ft. = the height of the wall, and
3. 2 ft. = the thickness.
4. ∴ $20\times7\frac{3}{4}\times2=310$ cu. ft. = the solidity of the wall.
5. $24\frac{3}{4}$ cu. ft. = 1 perch.
6. 310 cu. ft. = $310\div24\frac{3}{4}=12\frac{52}{99}$ perches.
7. 75 cts. = the cost of laying 1 perch.
8. ∴ $12\frac{52}{99}\times75$cts. = \$9.39$\frac{13}{33}$ = the cost of laying $12\frac{52}{99}$ perches.

III. ∴ It will cost \$9.39$\frac{13}{33}$ to lay $12\frac{52}{99}$ perches at 75 cts. a perch.

2. GAUGING.

Gauging is finding the contents of a vessel, in bushels, gallons, or barrels.

Prob. CLIV. To gauge any vessel.

Rule.—*Find its solidity in cubic feet by rules already given; this multiplied by 1728÷2150.42 or .83, will give the contents in bushels; by 1728÷231. will give it in wine gallons, which divided by $31\frac{1}{2}$ will give the contents in barrels.*

Prob. CLV. To find the contents in gallons of a cask or barrel.

Rule.—(*1*) *When the staves are straight from the bung to each end; consider the cask two equal frustums of equal cones, and find its contents by the rule of Prob. XCIII.*

(2). *When the staves are curved; Add to the head diameter (inside) two-tenths of the difference between the head and bung diameter; but if the staves are only slightly curved, add six-tenths of this difference; this gives the mean diameter; express it in inches, square it, multiply it by the length in inches, and this product by .0034; the product will be the contents in wine gallons.*

3. LUMBER MEASURE.

Prob. CLVI. To find the amount of square-edged inch boards that can be sawed from a round log.

Doyle's Rule.—*From the diameter in inches subtract four; the square of the remainder will be the number of square feet of inch boards yielded by a log 16 feet long.*

I. How much square-edged inch lumber can be cut from a log 32 in. in diameter, and 12 feet long?

II.
1. 32 in.=the diameter of the log.
2. 12 ft.=the length.
3. 32 in.—4 in.=28 in.=the diameter less 4.
4. 844 ft.=28^2=the square of the diameter less 4, which by the rule, is the number of feet in a log 16 ft. long.
5. 12 ft.=$\frac{3}{4}$ of 16 ft.
6. ∴ $\frac{3}{4}$ of 844 ft.=633 ft.=the number of feet of square-edged inch lumber that can be cut from the log.

III. ∴ The number of square-edged inch lumber that can be cut from a round log 32 inches in diameter and 12 ft. long is 633 ft.

4. GRAIN AND HAY.

Prob. CLVII. To find the quantity of grain in a wagon bed or in a bin.

Rule.—*Multiply the contents in cubic feet by 1728÷2150.42, or .83, and the result will be the contents in bushels.*

I. How many bushels of shelled corn in a bin 40 feet long, 16 feet wide and 10 feet high?

II.
1. 40 ft.=the length of the bin.
2. 16 ft.=the width of the bin, and
3. 10 ft.=the height of the bin.
4. ∴ 40×16×10=6400 cu. ft.=the contents of the bin in cu. ft..
5. ∴ 6400×.83 bu.=5312 bu.=the contents of the bin in bu.

III. ∴ The bin will hold 5312 bu. of shelled corn.

Rule.—(*1*) *For corn on the cob, deduct one-half for cob.*
(*2*) *For corn not "shucked" deduct two-thirds for cob and shuck.*

I. How many bushels of corn on the cob will a wagon bed hold that is $10\frac{1}{2}$ feet long, $3\frac{1}{2}$ feet wide, and 2 feet deep?

II.
1. $10\frac{1}{2}$ ft.=the length of the wagon bed,
2. $3\frac{1}{2}$ ft.=its width, and
3. 2 ft.=its depth.
4. ∴ $10\frac{1}{2}\times3\frac{1}{2}\times2=73\frac{1}{2}$ cu. ft.=contents of the wagon bed [in cu. ft
5. ∴ $73\frac{1}{2}\times.8$ bu=58.8 bu.=number of bushels of shelled corn the bed will hold.
6. ∴ $\frac{1}{2}$ of 58.8 bu=29.4 bu.=the number of bushels of corn on the cob that it will hold.

III. ∴ The wagon bed will hold 29.4 bu. of corn on the cob.

Prob. CLVIII. To find the quantity of hay in a stack, rick, or mow.

Rule.—*Divide the cubical contents in feet by 550 for clover or by 450 for timothy; the quotient will be the number of tons.*

Prob. CLXIX. To find the volume of any irregular solid.

Rule.—*Immerse the solid in a vessel of water and determine the quantity of water displaced.*

I A being curious to know the solid contents of a brush pile, put the brush into a vat 16 feet long, 10 feet wide, and 8 feet deep and containing 5 feet of water. He found, after putting in the brush, that the water rose $1\frac{1}{2}$ feet; what was the contents of the brush pile?

II.
1. 16 ft.=the length of the vat,
2. 10 ft.=the width, and
3. $1\frac{1}{2}$ ft.=the depth to which the water rose.
4. ∴ $16\times10\times1\frac{1}{2}=240$ cu. ft.=the volume of the brush pile.

III. ∴ 240 cu. ft =the volume of the brush pile.

XXII. SOLUTIONS OF MISCELLANEOUS PROBLEMS.

Prob. CLX. To find at what distance from either end, a trapezoid must be cut in two to have equal areas, the dividing line being parallel to the parallel sides.

Formula.—$d=A\div[\sqrt{\frac{1}{2}(b^2+b_1^2)}+b]=\frac{1}{2}(b+b_1)a \div[\sqrt{\frac{1}{2}(b^2+b_1^2)}+b]$, where A is the area of the trapezoid, b the lower base, and b_1, the upper base. $\sqrt{\frac{1}{2}(b^2+b_1^2)}$ is the length of the dividing line.

Rule.—*1. Extract the square root of half the sum of the squares of the parallel sides and the result will be the length of the dividing line.*

2. *Divide half the area of the whole trapezoid by half the sum of the dividing line and either end, and the quotient will be the distance of the dividing line from that end.*

I. I have an inch board 5 feet long, 17 inches wide at one end and 7 inches at the other; how far from the large end must it be cut straight across so that the two parts shall be equal?

By formula, $d=\frac{1}{2}(b+b_1)a\div[\sqrt{\frac{1}{2}(b^2+b_1^2)}+b]$ $=\frac{1}{2}(17+7)60\div[\sqrt{\frac{1}{2}(17^2+7^2)}+17]=720\div30$ $=24$ in.$=2$ ft.

FIG. 62.

II.
1. Let $ABCD$ be the board,
2. $AB=17$ in.$=b$, the width of the large end,
3. $DC=7$ in.$=b'$, the width of the small end, and
4. $HK=5$ ft.$=60$ in.$=a$, the length of the board.
5. Produce HK, AD, and BC till they meet in E. Then by similar triangles,
6. $ABE:EGL:EDC::AB^2:LG^2:DC^2$. But
7. $EGL=EDC+\frac{1}{2}(ABCD)$, or
8. $2EGL=2EDC+ABCD=EDC+EDC+ABCD$ $=EDC+EAB$.
9. $\therefore$ $EGL=\frac{1}{2}(EDC+EAB)$, *i. e.*, EGL is an arithmetic mean between EAB and EDC.
10. $\therefore$ $GL^2=\frac{1}{2}(AB^2+DC^2)=\frac{1}{2}(b^2+b'^2)$=an arithmetic mean between EAB and EDC,
11. $GL=\sqrt{\frac{1}{2}(b^2+b'^2)}=\frac{1}{2}\sqrt{2(b^2+b'^2)}$.
12. Draw CM perpendicular to AB.
13. $FL=\frac{1}{2}GL=\frac{1}{4}\sqrt{2(b^2+b'^2)}$.
14. $IL=FL-FI(=KC=\frac{1}{2}DC=\frac{1}{2}b)=\frac{1}{4}\sqrt{2(b^2+b'^2)}-\frac{1}{2}b$.
15. $CM=HK=a$.
16. $MB=\frac{1}{2}(b-b')$. Then in the similar triangles CMB and CIL,
17. $MB:IL::CM:CI$, or $\frac{1}{2}(b-b'):(\frac{1}{4}\sqrt{2(b^2+b'^2)}-\frac{1}{2}b)::a:CI$. Whence
18. $CI=a(\frac{1}{4}\sqrt{2(b^2+b'^2)}-\frac{1}{2}b)\div\frac{1}{2}(b-b')=$ $\dfrac{a(\frac{1}{2}\sqrt{2(b^2+b'^2)}-b)}{b-b'}=60\dfrac{(\frac{1}{2}\sqrt{2(17^2+7^2)}-7)}{17-7}=36$ in. $=3$ ft.
19. $\therefore$ $IM=CM-CI=5$ ft.-3 ft.$=2$ ft., the distance from the large end at which the board must be cut in two to have equal areas.

III. $\therefore$ The board must be cut in two, at a distance of 2 feet from the large end, to have equal areas in both parts.

(*R. H. A., p. 407, prob. 101.*)

Prob. CLXI. To divide a trapezoid into n equal parts and find the length of each part.

$$\textit{Formula.}—h_1=\frac{a}{b-b'}\left[\sqrt{\frac{(n-1)\,b'^2+b^2}{n}}-b\right],$$

$$h_2=\frac{a}{b-b'}\left[\sqrt{\left(\frac{(n-2)b'^2+2b^2}{n}\right)}-\sqrt{\left(\frac{(n-1)b'^2+b^2}{n}\right)}\right],$$

$$h_3=\frac{a}{b-b'}\left[\sqrt{\left(\frac{(n-3)b'^2+3b^2}{n}\right)}-\sqrt{\left(\frac{(n-2)b'^2+2b^2}{n}\right)}\right],\ \ldots$$

$$h_n=\frac{a}{b-b'}\left[b-\sqrt{\left(\frac{[n-(n-1)]b'^2+(n-1)b^2}{n}\right)}\right],$$ where b' is the width of the small end, b the width of the large end, and a the length of the trapezoid. h_1 is the length of the first part at the small end, h_2 the length of the second part, and so on.

I. A board $ABCD$ whose length BC is 36 inches, width AB 8 inches and DC 4 inches, is divided into three equal pieces. Find the length of each piece.

By formula, $h_1=\frac{a}{b-b_1}\left[\sqrt{\frac{(n-1)b_1^2+b^2}{n}}-b_1\right]=$ $\frac{36}{8-4}[\sqrt{\tfrac{1}{3}(3-1)4^2+8^2}-4]=9[\sqrt{32}-4]=36(\sqrt{2}-1)=14.911686$ in.

$h_2=\frac{a}{b-b_1}\left[\sqrt{\frac{(n-2)b_1^2+2b^2}{n}}-\sqrt{\frac{(n-1)b_1^2+2b^2}{n}}\right]=36(\sqrt{3}-\sqrt{2})$ $=11.442114$ in. $h_3=\frac{a}{b-b_1}\left[\sqrt{\frac{(n-3)b_1^2+3b^2}{n}}-\sqrt{\frac{(n-2)b_1^2+2b^2}{n}}\right]$ $=36[2-\sqrt{3}]=9.6462$ in.

FIG. 63.

II.
1. 4 in.=the width DC of the small end,
2. 8 in.=the width AB of the large end, and
3. 36 in.=the length BC of the board.
4. ∴ 216 sq. in.$=\frac{1}{2}(AB+DC)\times BC$ $=\frac{1}{2}(8+4)\times 36$=the area of the board.
5. $\frac{1}{3}$ of 216 sq. in.=72 sq. in.=the area of each piece.
6. $AK=AB-KB(=DC)=8$ in.—4 in. =4 in. In the similar triangles AKD and DCE,
7. $AK{:}DK{::}AB{:}BE$, or 4 in.:36 in.::8 in.:BE. Whence,
8. $BE=(36\times 8)\div 4=72$ in. [triangle ABE.
9. ∴ $\frac{1}{2}(AB\times BE)=\frac{1}{2}(8\times 72)=288$ sq. in.=the area of the
10. $ABE-ABCD=288$ sq. in.—216 sq. in.=72 sq. in. =area of the triangle DCE.

11. $DCE+DCGF=72$ sq. in.+72 sq. in. = 144 sq. in. =the area of the triangle FGE.
12. $DEC+DCGF+FGIH=72$ sq. in.+72 sq. in.+72 sq. in.=216 sq. in.=the area of the triangle HIE.
13. $FEG:DEC::EG^2:EC^2$, or
 144 sq. in.:72 sq. in.:: $GE^2:36^2$. Whence,
14. $GE=\sqrt{(144\times36^2)\div72}=36\sqrt{2}=50.911686$ inches.
15. ∴ $GC=GE-CE=50.911686$ in.—36 in.=14.911686 in., the length of $FGCD$. Again,
16. $DEC:HIE::EC^2:EI^2$, or
 72 sq. in.:216 sq. in.:: $36^2:EI^2$. Whence,
17. $EI=\sqrt{(216\times36^2)\div72}=36\times\sqrt{3}=62.3538$ in.
18. ∴ $GI=EI-EG=36\sqrt{3}-36\sqrt{2}=36(\sqrt{3}-\sqrt{2})$ =11.442114 in., the length of $HIGF$, and
19. $BI=EB=EI=72-36\sqrt{3}=36(2-\sqrt{3})=9.6462$ in., the length of $ABIH$.

III. ∴ $\begin{cases} BI=9.6462 \text{ in.,} \\ GI=11.442114 \text{ in., and} \\ GC=14.911686 \text{ in.} \end{cases}$

Prob. CLXII. To find the edge of the largest cube that can be cut from a sphere.

$$\textit{Formula.}—e=\sqrt{\frac{D^2}{3}}=\tfrac{1}{3}\sqrt{3}D=.57735\times D,$$ where D is the diameter of the sphere.

Rule.—*Divide the square of the diameter of the sphere by three and extract the square root of the quotient; or, multiply the diameter by .57735.*

I. What is the edge of the largest cube that can be cut from a sphere 6 inches in diameter?

By formula, $e=\sqrt{\frac{D^2}{3}}=\sqrt{\frac{36}{3}}=6\times\sqrt{\tfrac{1}{3}}=\tfrac{1}{3}\sqrt{3}\times6=.57735\times6$ =3.4641 in.

II. $\begin{cases} 1.\ 6 \text{ in.=the diameter of the sphere.} \\ 2.\ \therefore .57735\times6 \text{ in.}=3.4641 \text{ in.=the edge of the largest cube that can be cut from the sphere.} \end{cases}$

III. ∴ The edge of the largest cube that can be cut from a sphere whose diameter is 6 inches, is 3.4641 in.

Prob. CLXIII. To find the edge of the largest cube that can be cut from a hemisphere.

$$\textit{Formula.}—\ e=\sqrt{\frac{D^2}{6}}=\tfrac{1}{6}\sqrt{6}\times D=.408248\times D.$$

Rule.—*Divide the square of the diameter by 6, and extract the square root of the quotient; or, multiply the diameter by .408248.*

I. What is the edge of the largest cube that can be cut from a hemisphere, the diameter of whose base is 12 inches?

By formula, $e=\sqrt{D^2\div6}=\sqrt{\frac{144}{6}}=12\sqrt{\frac{1}{6}}=\frac{1}{6}\sqrt{6}\times12=.408248\times12=4.899176$ in.

II. $\begin{cases}\text{1. 12 in.=the diameter of the base of the hemisphere.}\\ \text{2. } \therefore .408248\times12 \text{ in.}=4.899176 \text{ in.}\end{cases}$

III. ∴ The edge of the largest cube that can be cut from a hemisphere, the diameter of whose base is 12 feet, is 4.899176 in.

Prob. CLXIV. To find the diameter or radius of the three largest equal circles that can be inscribed in a circle of a given diameter or radius.

Formula.—$d=D\div(1+\frac{2}{3}\sqrt{3})=D\div2.1557=.4641\times D$ or $r=R\div(1+\frac{2}{3}\sqrt{3})=.4641\times R$.

Rule.—*Divide the diameter or radius of the given circle by 2.1557 and the quotient will be the diameter or radius of the three largest equal circles inscribed in it; or, multiply the diameter or radius by .4641, and the result will be the diameter or radius respectively of the required circles.*

I. A circular lot 15 rods in diameter is to have three circular grass beds just touching each other and the larger boundary; what must be the distance between their centers, and how much ground is left in the triangular space about the center?

By formula, $2r=2R\div(1+\frac{2}{3}\sqrt{3})=2R\div2.1557=\frac{15}{2.1557}=6.9615242$ rd.=the distance between their centers.

Construction.—Let AHE be the circular lot, C the center, and ACE any diameter. With E as a center and radius equal to CE describe an arc intersecting the circumference of the lot in H. Draw a tangent to the lot at E and produce the radius CH to intersect the tangent at B. Bisect the angle CBE and draw the bisector GB. It will meet the radius CE in G, the center of one of the grass beds. Draw GF perpendicular to CB. Then $GF=GE$, the radius of one of the grass beds. Draw EH. Then $EH=CH=EC$, and $CH=HB$, because the triangle EHB is isosceles.

A.
1. $CE=7\frac{1}{2}$ rd.$=R$, the radius of the lot.
2. $CB=2CH=2R$.
3. $EB=\sqrt{CB^2-CE^2}=\sqrt{(2R)^2-R^2}=R\sqrt{3}$. In the similar triangles CFG and CBE,
4. $CF:FG::CE:EB$, or $CF:GF::R:R\sqrt{3}$. But
5. $CF=CB-FB(=EB)=2R-R\sqrt{3}=R(2-\sqrt{3})$.
6. $\therefore R(2-\sqrt{3}):GF::R:R\sqrt{3}$. Whence,
7. $GF=\frac{R\sqrt{3}}{2-\sqrt{3}}=R(2\sqrt{3}-3)=7\frac{1}{2}(2\sqrt{3}-3)=7\frac{1}{2}\times.4641=3.48075$ rd.=the radius.

8. $GK=2r=2R(2\sqrt{3}-3)=6.9615$ rd., the distance between their centers.

II. B.

1. $GD=\sqrt{\overline{GK}^2-\overline{DK}^2}=\sqrt{4r^2-r^2}=r\sqrt{3}$.
2. $\frac{1}{2}(IK\times GD)=\frac{1}{2}(2r\times r\sqrt{3})=r^2\sqrt{3}$=the area of the triangle IGK.
3. Area $DKF=\frac{1}{6}$ of the small circle, because the angle DKF is 60°, or $\frac{1}{6}$ of 360°.
4. $\therefore$ Area $DKF=\frac{1}{6}\pi r^2$.
5. $\frac{1}{2}\pi r^2=3$ times $\frac{1}{6}\pi r^2$ =the area of the three parts of the small circles within the triangle IGK.
6. $\therefore r^2\sqrt{3}-\frac{1}{2}\pi r^2=r^2(\sqrt{3}-\frac{1}{2}\pi)=.16125368r^2$ $=.16125368\times[R(2\sqrt{3}-3)]^2=.16125368\times(21-12\sqrt{3})R^2=.16125368\times.2153904\times R^2$ $=.03473265\times R^2=.03473265\times(7\frac{1}{2})^2=1.953712$ sq. rd.=the area of the space inclosed.

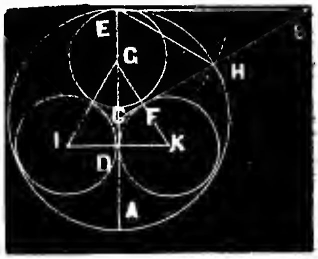


FIG. 64.

III. $\therefore$ 6.9615 rd.=the distance between their centers, and 1.953712 sq. rd.=the area inclosed about the center of the given lot. (*R. H.A. p. 407, prob. 100.*)

Prob. CLXV. Having given the area inclosed by three equal circles to find the radius of a circle that will just inclose the three equal circles.

Formula.—$R=\sqrt{\left(\frac{A}{(2\sqrt{3}-3)^2(\sqrt{3}-\frac{1}{2}\pi)}\right)}$

$=\sqrt{\left(\frac{A}{.03473265}\right)}$, where A is the area inclosed.

Rule.—*Divide the area inclosed by .03473265 and extract the square root of the quotient, and the result will be the radius of the required circle.*

Prob. CLXVI. Having given the radius a, b, c, of the three circles tangent to each other, to find the radius of a circle tangent to the three circles.

Formula.—r or $r'=\frac{abc}{2\sqrt{[abc(a+b+c)]}\mp(ab+ac+bc)}$, the minus sign giving the radius of a tangent circle circumscribing the three given circles and the plus sign giving the radius of a tangent circle inclosed by the three given circles.

NOTE.—This formula is due to Prof. E. B. Seitz, Late Professor of Mathematics in the North Missouri State Normal School, Kirksville, Mo., of whom we give a biographical sketch accompanied by his photograph.

This formula is taken from the *School Visitor, Vol. II. p. 117,* with the slight change that the plus sign is introduced for the case in which the tangent circle is inclosed by the three given circles. The problem of finding two circles tangent to three mutually tangent circles, is one supposed to have been proposed by Archimedes more than 2000 years ago, though the problem he proposed was not so general—the diameter of one of the given circles being equal to the sum of the diameters of the other two.

The problem of finding *all* circles that can be drawn within three mutually tangent circles and tangent to each of them, has been simply and elegantly solved by D. H. Davison, Minonk, Ill. The above formula led him to his wonderful solution. For a complete and elegant solution, where he has actually computed and constructed 81 circles tangent to three given circles. see *School Visitor, Vol. VI., p. 80.*

Prob. CLXVII. To find the surface common to two equal circular cylinders whose axes intersect at right angles.

Formula.—$S=16R^2$, where R is the radius of the cylinders.

Rule.—*Multiply the square of the radius of the intersecting cylinders by 16.*

I. If the radius of two equal circular cylinders, intersecting at right angles is 4 feet, what is the surface common to both?

By formula, $S=16R^2=16\times4^2=256$ sq. ft.

II. 1. 4 ft. = the radius of the cylinders.
2. 16 sq. ft. $=4^2=$ the square of the radius of the cylinders
3. ∴ 16×16 sq. ft. = 256 sq. ft. = the surface common to the two cylinders.

III. ∴ 256 sq.ft. = the surface common to the two cylinders.

Prob. CLXVIII. To find the volume common to two equal circular cylinders whose axes intersect at right angles.

Formula.—$V=\frac{16}{3}R^3$, where R is the radius of the cylinder.

Rule.—*Multiply the cube of the radius of the cylinders by* $\frac{16}{3}$.

1. A man digging a well 3 feet in diameter, came to a log 3 feet in diameter lying directly across the entire well; what was the volume of the part of the log removed?

By formula, $V=\frac{16}{3}R^3=\frac{16}{3}(\frac{3}{2})^3=18$ cu. ft.

II. 1. 3 ft. = the diameter of the log and the well.
2. $1\frac{1}{2}$ ft. = the radius.
3. $3\frac{3}{8}$ cu. ft. $=(1\frac{1}{2})^3=$ the cube of the radius.
4. ∴ $\frac{16}{3}\times3\frac{3}{8}$ cu. ft. = 18 cu.ft., the volume of the part of the log removed.

III. ∴ The volume of the part of the log removed is 18 cu.ft.

Prob. CLXIX. To find the height of an object on the earth's surface by knowing its distance, the top of the object being visible above the horizon.

Let $BF=a$ be any object, $AB=t$ a tangent to the earth's surface from the top of the object, and $FE=D$ the diameter of the earth. Then by Geometry, $AB^2=BF(BF+FE)$, or $t^2=a(a+D)$. $\therefore a=\frac{t^2}{a+D}$. But a is very small as compared with the diameter of the earth and $AB=AF$ without appreciable error.

$\therefore$ ***Formula.***—$a=\frac{AF^2}{D}=\frac{c^2}{D}$, where c is the distance to the object from the point of observation.

When $c=1$ mile, $a=\frac{1^2}{7912}=\frac{2}{3}$ ft., nearly.

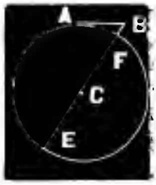


FIG. 65.

Rule.—*Multiply the square of the distance in miles by* $\frac{2}{3}$, *and the result will be the height of the object in feet.*

I. What is the height of a steeple whose top can be seen at a distance of 10 miles?

By formula, $a=\frac{c^2}{D}=\frac{10^2}{7912}=\frac{10^2}{7912}\times5280=\frac{2}{3}\times10^2=66\frac{2}{3}$ ft.

II. 1. 10 miles=the distance to the steeple.
2. $100=10^2$=the square of the distance.
3. $\therefore \frac{2}{3}$ of 100= $66\frac{2}{3}$ ft.=the height of the steeple.

III. $\therefore$ The height of the steeple is $66\frac{2}{3}$ ft.

Prob. CLXX. To find the distance to an object by knowing its height, the top only of the object being visible above the horizon.

$$\textit{Formula.}\text{—}c=\sqrt{aD}=\sqrt{\frac{aD}{5280}}=\sqrt{a\frac{7912}{5280}}=\sqrt{\tfrac{3}{2}a}.$$

Rule.—*Multiply the height of the object in feet by* $\frac{3}{2}$ *and extract the square root of the product, and the result will be the distance in miles.*

I. At what distance at sea can Mt. Aconcagua be seen, if its height is known to be 24000 feet?

By formula, $c=\sqrt{\frac{3}{2}a}=\sqrt{\frac{3}{2}\times24000}=\sqrt{36000}=190$ mi., nearly.

II. 1. 24000 ft.=the height of the mountain
2. $\frac{3}{2}\times24000=36000$.
3. $\therefore \sqrt{36000}=10\sqrt{360}=190$ mi., nearly.

III. $\therefore$ Mt. Aconcagua can be seen at a distance of 190 miles.

Prob. CLXXI. Given the sum of the hypotenuse and perpendicular, and the base, to find the perpendicular.

Formula.—$p=\frac{s^2-b^2}{2s}$, where s is the sum of the hypotenuse and perpendicular, and b the base.

Rule.—*1. From the square of the sum of the hypotenuse and perpendicular subtract the square of the base, and divide the difference by twice the sum of the hypotenuse and perpendicular.*

2. To find the hypotenuse: To the square of the sum of the hypotenuse and perpendicular, add the square of the base and divide this sum by twice the sum of the hypotenuse and perpendicular.

I. A tree 120 feet high is broken off but not severed. The top strikes the ground 34 feet from the foot of the tree; what is the height of the stump?

By formula, $p=\frac{s^2-b^2}{2s}=\frac{120^2-34^2}{2\times120}=55\frac{11}{60}$ ft., the height of the stump.

II.
1. 120 ft.=the sum of the hypotenuse and perpendicular.
2. 34 ft.=the base, or the distance the top strikes from the foot of the tree.
3. 14400 sq. ft.$=120^2$=the square of said sum,
4. 1156 sq. ft.$=34^2$=the square of the base, and
5. 14400 sq. ft.—1156 sq. ft.=13244 sq. ft.=the difference.
6. ∴ $13244\div(2\times120)=55\frac{11}{60}$ ft.=the height of the stump.

III. ∴ The height of the stump is $55\frac{11}{60}$ feet.

NOTE.—This rule is easily derived from an algebraic solution. Thus: Let x=the perpendicular, $s-x$=the hypotenuse, and b=the base. Then, $x^2+b^2=(s-x)^2$, or $x^2+b^2=s^2-2sx+x^2$, and $x=\frac{s^2-b^2}{2s}$.

Prob. CLXXII. To find at what distance from the large end of the frustum of a right pyramid, a plane must be passed parallel to the base so that the two parts shall have equal solidities.

Formula.—$h=\frac{3V}{2(B+\sqrt{BB_2}+B_2)}$, where V is the volume of the frustum, B the area of the lower base, B_2 the area of the "dividing base," and $\sqrt{BB_2}$ the area of the mean base between the "dividing base" and and lower base.

Rule.—*1. Find the volume of the frustum by Prob. XCIII.*

2. Find the dimensions of the "dividing base" by extracting the cube root of half the sum of the cubes of the homologous dimensions of the upper and lower bases. Then find the area of the "dividing base."

3. Divide half the volume of the frustum by one-third of the sum of the areas of the lower base, "dividing base," and mean base between them, and the quotient will be the length of the lower part.

I. How far from the large end must a stick of timber, 20 feet long, 5 inches square at one end and 10 inches square at the other, be sawed in two parts, to have equal solidities?

By formula, $h=\dfrac{3V}{2(B+\sqrt{(BB_2)}+B_2)}=\dfrac{3\times\frac{1}{3}a(b^2+bc+c^2)}{2\left[b^2+b\times\sqrt[3]{\left(\frac{b^3+c^3}{2}\right)}+\left(\frac{b^3+c^3}{2}\right)^{\frac{2}{3}}\right]}$

$=\dfrac{240(10^2+10\times5+5^2)}{2\left[10^2+10\times\sqrt[3]{\left(\frac{10^3+5^3}{2}\right)}+\sqrt[3]{\left(\frac{10^3+5^3}{2}\right)^2}\right]}$

$=\dfrac{42000}{2(100+25\sqrt[3]{36}+\frac{75}{2}\sqrt[3]{6})}=\dfrac{1680}{8+2\sqrt[3]{36}+\sqrt[3]{6}}$

$=\dfrac{1680}{8+6.603855+5.4513618}=\dfrac{1680}{20.0552168}=83.76883+$ in.

Construction.—Let $ABCD—E$ be the piece of timber, $ABCD$ the lower base, $EFGH$ the upper base, and OL the altitude. Prolong the edges AH, BE, CF, and DG and the altitude OL till they meet in P. Draw KL to the middle point of AD, OI to the middle point of GH and draw PIK. Let $SMNR$ be the dividing base.

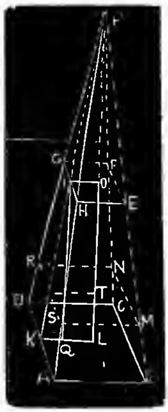


FIG 66

1. $AB=10$ in. $=b$, the side of the lower base.
2. $HE=5$ in. $=c$, the side of the upper base, and
3. $OL=20$ ft. $=240$ in. $=a$, the altitude.
4. $KQ=KL-QL(=IO)=\frac{1}{2}(b-c)=\frac{1}{2}(10$ in. -5 in$)=2\frac{1}{2}$ in. By similar triangles,
5. $KQ:QI::KL:PL$, or $\frac{1}{2}(b-c):a::\frac{1}{2}b:PL$. Whence,
6. $PL=\dfrac{ab}{b\ \ c}=40$ ft.
7. $\therefore PO=PL-OL=\dfrac{ab}{b-c}-a=\dfrac{ac}{b-c}$ $=20$ ft.
8. $v=\frac{1}{3}PO\times HE^2=\frac{1}{3}ac^2=\frac{1}{3}\times240\times5^2$ $=2000$ cu. in., the volume of the pyramid $HEFG$.
9. $V=\frac{1}{3}OL\times(AB^2+AB\times HE+HE^2)=\frac{1}{3}a(b^2+bc+c^2)=14000$ cu. in., the volume of the frustum $ABCD—E$.
10. $\therefore\ \frac{1}{2}V=\frac{1}{2}$ of 14000 cu. in. $=7000$cu. in., the volume of each part.
11. $v+\frac{1}{2}V=2000$ cu. in. $+7000$ cu. in. $=9000$ cu. in., the volume of the pyramid, $SMNR—P$, and

II. 12. $v+V=2000$ cu. in. $+14000$ cu. in. $=16000$ cu. in., the volume of the pyramid $ABCD—P$. By the principle of similar solids, [AB^3, or
13. $HEFG\text{-}P : SMNR\text{-}P : ABCD\text{-}P :: HE^3 : SM^3 :$
14. $v : v+\frac{1}{2}V : v+V :: c^3 : SM^3 : b^3$. But

15. $v+\frac{1}{2}V=\frac{1}{2}[v+(v+V)]$, i. e., $v+\frac{1}{2}V$, or $SMNRP$ is an arithmetical mean between v and $v+V$, or $HEFG-P$ and $ABCD-P$.

17. $\therefore SM^3=\frac{1}{2}(c^3+b^3)$, i. e., SM^3 is an arithmetical mean between HE^3 and AB^3, or c^3 and b^3. Whence,

18. $SM=\sqrt[3]{[\frac{1}{2}(c^3+b^3)]}=\sqrt[3]{[\frac{1}{2}(5^3+10^3)]}=\frac{5}{2}\sqrt[3]{36}=$ 8.2548188+in.

19. $SM^2=\sqrt[3]{[\frac{1}{2}(c^3+b^3)]^2}=(\frac{5}{2}\sqrt[3]{36})^2=\frac{75}{2}\sqrt[3]{6}=68.14202$ sq. in.=the area of the dividing base.

20 $\sqrt{(SM^2\times AB^2)}=SM\times AB=\frac{5}{2}\sqrt[3]{36}\times 10=25\sqrt[3]{36}=$ 82.54818 sq. in.=the area of the mean base of the part cut from the frustum.

21. $\therefore \frac{1}{3}LT(AB^2+SM\times AB+SM^2)=\frac{1}{3}LT(b^2+\sqrt[3]{[\frac{1}{2}(b^3+c^3)]}\times b+\sqrt[3]{[\frac{1}{2}(b^3+c^3)]^2})=\frac{1}{3}LT[10^2+\frac{5}{2}\sqrt[3]{(36)}\times 10+(\frac{5}{2}\sqrt[3]{36})^2]=\frac{1}{3}LT(100+82.54818+68.14202)=\frac{1}{3}LT\times 250.6902=LT\times 83.5634=$ the volume of the frustum $ABCD-M$ .But [$-M$.

23. $\frac{1}{2}V=7000$ cu. in.=the volume of the frustum $ABCD$

24. $\therefore LT\times 83.5634=7000$ cu. in. Whence,

25. $LT=7000\div 83.5634=83.76883$ in.=6 ft. 11.76883 in., the length.

III. $\therefore$ The stick must be cut in two at a distance of 83.76883 in., or 6 ft. 11.76883 in., from the large end.

NOTE.—The frustum of a cone may be divided into two equal parts in the same manner. The frustum of a pyramid or a cone can be divided into any number of equal parts on the same principle as that for dividing a trapezoid into n equal parts, Prob. CLXI.

The above problem is one that, like a wandering Jew, goes the rounds of the country, and few teachers escape having it proposed to them for solution. By a careful study of the solution here given, every teacher ought to be able not only to solve and explain this problem but a great many others of a similar kind.

I. Two poles, perpendicular to the same plane, are 40 feet and 50 feet high. At what height from the plane will lines drawn from the top of one to the base of the other, cross?

Construction.— Let $BC=50$ feet$=a$, the height of the longer pole; $AD=40$ feet $=b$, the height of the shorter; $AB=d$, the distance between the poles; and AC and BD the lines drawn from the top of BC to the base AD and the top of AD to the base BC, respectively. From F, the point of intersection of these two lines, let fall the perpendicular, FE, the length of which is required. Then

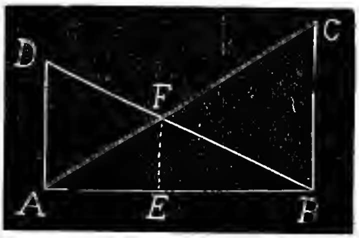


FIG. 66a.

II.
1. $AB:BC=AE:FE$, or $d:a=AE:FE$, by the similar triangles ABC and AEF.
2. $BA:AD=EB:FE$, or $d:b=d-AE:FE$, by the similar triangles, BAD and BEF.
3. $\therefore a\times AE=d\times FE$, from the first proportion, and
4. $b\times(d-AE)=d\times FE$, from the second proportion.
5. $\therefore a\times AE=b(d-AE)=bd-b\times AE$; whence
6. $AE=\dfrac{bd}{a+b}$.
7. $\therefore a\times\dfrac{bd}{a+b}=d\times FE$, by substituting in third step.
8. $\therefore FE=\dfrac{ab}{a+b}=\dfrac{50\times40}{50+40}=22\frac{2}{9}$ feet.

III. $\therefore 22\frac{2}{9}$ feet is the distance from the plane to the point where the lines cross.

Remark.— Observe that the result is independent of the distance the poles are apart.

I. James Page has a circular garden 10 rods in diameter. How many trees can be set in it so that no two shall be within 10 feet of each other and no tree within $2\frac{1}{2}$ feet of the fence?

Construction.—Let ABC be the circular garden, AC it diameter, and O its, center. Then with O as a center and radius $AO=\frac{1}{2}$of $(10\times16\frac{1}{2}$ ft.$-2\times2\frac{1}{2}$ ft.$)$, or 80 ft, discribe the circle *abcdef*, and in it describe the regular hexagon *abcdef*. Then $aO=ab=80$ ft. Begin at the center of the circle and put 8 trees 10 ft. apart on each radii, aO, bO, cO, dO. eO, and fO. Then joining these points by lines drawn parallel to the diameter of the circle as shown in the figure, their points of intersection will mark the position of the trees. Hence, the trees are arranged in hexagonal form about the center. The first hexagonal row contains 6 trees, the second, 12, the third 18, and so on. Since the radius of the circle on which the trees are placed is 80 feet and the trees 10 feet apart, there will be 8 hexagonal rows.

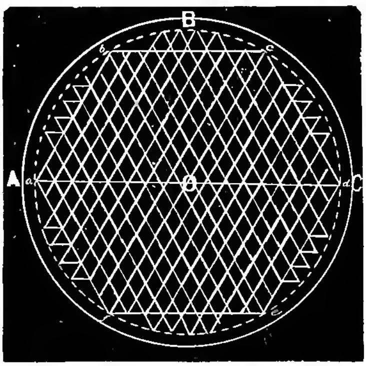


FIG. 67.

II.
1. 6=the number of trees in the first hexagonal row.
2. 12=the number of trees in the second hexagonal row.
3. 48=the number of trees in the eighth hexagonal row.
4. $\therefore 216=\frac{1}{2}(6+48)\times8=$the number of trees in the eight hexagonal rows.
5. $24=6\times4=$the number of trees at the sides of the hexagon *abcdef*.
6. $\therefore 216+24+1$, the tree at the center,$=241=$the number of trees that can be set in the garden.

III. ∴ There can be set in the garden, 241 trees.

(*Greenleaf's Nat'l Arith., p. 444, prob. 25.*)

I. There is a ball 12 feet in diameter on top of a pole 60 feet high. On the ball stands a man whose eye is 6 feet above the ball; how much ground beneath the ball is invisible to him?

Construction.—Let BE be the pole, L the center of ball, and A the position of the man's eye. Draw AFC tangent to the ball at F and draw LF and BC. Then the triangle AFL is right-angled at F.

II.
1. 60 ft.$=BE$, the length of the pole.
2. 12 ft.$=ED$, the diameter of the ball, and
3. 6 ft.$=AD$, the height of the man's eye above the ball.
4. 12 ft.$=AD+DL=AL$. Now
5. $AF=\sqrt{(AL^2-LF^2)}$ $=\sqrt{(12^2-6^2)}=6\sqrt{3}$ ft. In the similar triangles ALF and ACB,
6. $AF:LF::AB:BC$, or $6\sqrt{3}$ ft. : 6 ft. : : (6 ft.+12 ft. +60 ft.), or 78 ft. : BC.
7. ∴ $BC=(6\times78)\div6\sqrt{3}=78\div\sqrt{3}=\frac{1}{3}\times78\sqrt{3}=26\sqrt{3}$ ft.
8. ∴ $\pi BC^2=\pi(26\sqrt{3})^2=6371.1498932$ sq. ft.=the area of the circle over which the man can not see.

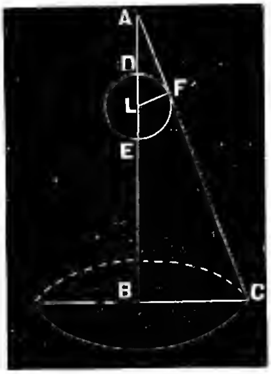


FIG. 68.

III. ∴ 6371.1498932 sq. ft.=the area of the invisible ground beneath the ball.

I. Three women own a ball of yarn 4 inches in diameter. How much of the diameter of the ball must each wind off, so that the may share equally?

II.
1. 4 in.=the diameter of the ball. Then
2. $\frac{1}{6}\pi(4)^3=\frac{32}{3}\pi$=the volume of the ball.
3. $\frac{1}{3}$ of $\frac{32}{3}\pi=\frac{32}{9}\pi$=each woman's share.
4. $\frac{32}{3}\pi-\frac{32}{9}\pi=\frac{64}{9}\pi$=the volume of the ball after the first has unwound her share. But
5. $\frac{1}{6}\pi D^3$=the volume of any sphere whose diameter is D.
6. ∴ $\frac{1}{6}\pi D^3=\frac{64}{9}\pi$. Whence,
7. $D^3=\frac{64}{9}\pi\div\frac{1}{6}\pi=\frac{128}{3}$, and
8. $D=\sqrt[3]{\frac{128}{3}}=4\sqrt[3]{\frac{2}{3}}=\frac{4}{3}\sqrt[3]{18}=\frac{4}{3}\times2.6207414=3.4943219$ in., diameter of the ball after the first unwound her share.
9. ∴ 4 in.—3.4943219 in=.5056781 in., what the diameter was reduced by the first woman.
10. $\frac{64}{9}\pi-\frac{32}{9}\pi=\frac{32}{9}\pi$, the volume of the ball after the second had unwound her share.

11. ∴ $\sqrt[3]{(\frac{32}{9}\pi\div\frac{1}{6}\pi)}=4\sqrt[3]{\frac{1}{3}}=\frac{4}{3}\sqrt[3]{9}=\frac{4}{3}\times2.0800837$ $=2.5734448$ in., the diameter of the ball after the second woman unwoud her share.
12. ∴ 3.4943219 in.—2.5734448 in.=.7208771 in., what the diameter was reduced by the second woman.

III. ∴ The diameter was diminished .5056781 in. by the first woman,
.7208771 in. by the second woman, and
2.7734448 in. by the third woman.

(*Milne's Prac. Arith., p. 335, prob. 8.*)

NOTE.—The following are the formulas to divide a sphere into n equal parts, the parts being concentric:

$$D-D_1=\sqrt[3]{\left(\frac{n-1}{n}\right)}D;\ D_1-D_2=\left[\sqrt[3]{\left(\frac{n-1}{n}\right)}-\sqrt[3]{\left(\frac{n-2}{n}\right)}\right]D;\ D_2-D_3=\left[\sqrt[3]{\left(\frac{n-2}{n}\right)}-\sqrt[3]{\left(\frac{n-3}{n}\right)}\right]D;$$

$$D_8-D_4=\left[\sqrt[3]{\left(\frac{n-3}{n}\right)}-\sqrt[3]{\left(\frac{n-4}{n}\right)}\right]D,$$ and so on, where D is the diameter of the sphere; D_1, the diameter after the first part is taken off; D_2, the diameter after the second part is taken off; and so on. Then $D-D_1$, D_1-D_2, &c, are portions of the diameter taken off by each part.

I. A park 20 rods square is surrounded by a drive which contains $\frac{19}{100}$ of the whole park; what is the width of the drive?

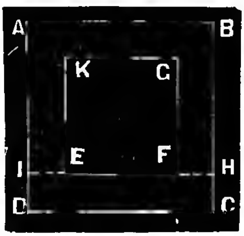


FIG. 69.

II.
1. 20 rd.=AD=DC, a side of the park.
2. 400 sq. rd.=20^2=the area of the park $ABCD$.
3. $\frac{19}{100}$ of 400 sq. rd.=76 sq. rd.=the area of the path.
4. 400 sq. rd.—76 sq. rd.=324 sq. rd. =the area of the square $EFGH$.
5. $EF=\sqrt{(324)}=18$ rd., the side of the square $EFGH$.
6. ∴ $IH-EF$=20 rd.—18rd.=2 rd., twice the width of the path.
7. ∴ 1 rd.=$\frac{1}{2}$ of 2 rd.=the width of the path.

III. ∴ The width of the path is 1 rod.

I. My lot contains 135 sq. rd., and the breadth is to the length as 3 to 5; what is the width of a road which shall extend from one corner half around the lot and occupy $\frac{1}{3}$ of the ground.

Construction.—Let $ABCD$ be the lot, and $DABSNR$ the road. Produce AB, till BE is equal to AD. Then AE is equal to $AB+AD$. On AE, construct the square $AEFG$, and on EF and GF respectively, lay off EI and FK equal to AB. Then construct the rectangles $BEIH$, $ILKF$, and $KMDG$. They will each be equal to $ABCD$, for their lengths and widths are equal to the length and width of $ABCD$. Continue the road around the square. Then the area of the road around the square is four times the area of the road $DABSNR$.

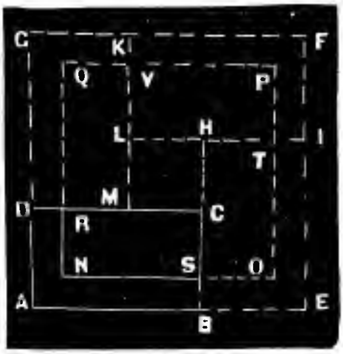


FIG. 70.

II.
1. $\frac{3}{8}$=the width AD of the lot. Then
2. $\frac{5}{8}$=the length AB.
3. $\frac{5}{8}\times\frac{3}{8}$=135 sq. rd., the area of the lot.
4. $\frac{1}{8}\times\frac{3}{8}=\frac{1}{5}$ of 135 sq. rd. =27 sq. rd., and
5. $\frac{3}{8}\times\frac{3}{8}=(\frac{3}{8})^2$=3 times 27 sq.rd.=81 sq. rd.
6. ∴ $\frac{3}{8}=\sqrt{81}$=9 rd., the width AD,
7. $\frac{1}{8}=\frac{1}{3}$ of 9 rd.=3 rd., and
8. $\frac{5}{8}$=5 times 3 rd.=15 rd., the length AB.
9. 15 rd+9 rd.=24 rd.= AE, the side of the square $AEFG$.
10. ∴ 576 sq. rd.=24^2= the area of the square $AEFG$.
11. 33¾ sq. rd.=¼ of 135 sq. rd.=the area of the road $DABSNR$.
12. ∴ 135 sq. rd.=4×33¾ sq. rd.=the area of the road around the square. Then
13. 576 sq. rd.—135 sq. rd.=441 sq. rd., the area of the square $NOPQ$.
14. ∴ 21rd.=$\sqrt{441}$=NO, a side of the square $NOPQ$.
15. AE—NO=24 rd.—21 rd.=3rd.=twice the width of the road.
16. ∴ 1½ rd.=24¾ ft.=½ of 3 rd.=the width of the road.

III. ∴ The width of the road is 24¾ ft.

(*R. H. A., p. 407, prob. 99.*)

I. The length and breadth of a ceiling are as 6 to 5; if each dimension were one foot longer, the area would be 304 sq. ft.; what are the dimensions?

Construction.—Let $ABCD$ be the ceiling, AB its width and BC its length. Let $AIGE$ be the ceiling when each dimension is increased one foot. On BC, lay off BK equal to AB and draw LK parallel to AB. Then $ABKL$ is a square whose side is the width of the ceiling.

II.

1. $\frac{5}{5}=AB$, the width of the ceiling. Then
2. $\frac{6}{5}=BC$, the length, and
3. $\frac{6}{5}\times\frac{5}{5}=AB\times BC=$the area of the ceiling.
4. $\frac{6}{5}\times1=BC\times BI$, BI being 1 foot,=the area of the rectangle $BCHI$.
5. $\frac{5}{5}\times1=DC\times CF$, CF being 1 foot,=the area of the rectangle $DCFE$.
6. 1 sq. ft.$=1^2=$the area of the square $CFGH$.
7. $\therefore\frac{6}{5}\times\frac{5}{5}+\frac{6}{5}\times1+\frac{5}{5}\times1+1$ sq. ft. =the area of $AIGE$. But

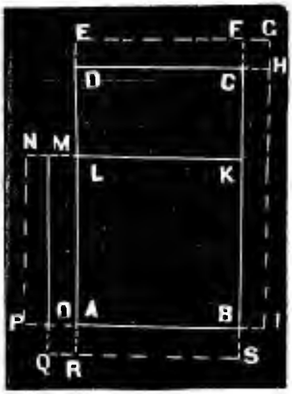


FIG. 71.

8. $\frac{6}{5}\times1+\frac{5}{5}\times1=\frac{11}{5}\times1=$rectangle $BIHC+$rectangle $DCFE$, i. e., it equals a rectangle whose length is $\frac{6}{5}+\frac{5}{5}$, or $\frac{11}{5}$, and width 1 ft.
9. $\therefore\frac{6}{5}\times\frac{5}{5}+\frac{6}{5}\times1+\frac{5}{5}\times1+1$ sq. ft.$=\frac{6}{5}\times\frac{5}{5}+\frac{11}{5}\times1+1$ sq. ft. =the area of $AIGE$. But
10. 304 sq. ft.=the area of $AIGE$.
11. $\therefore\frac{6}{5}\times\frac{5}{5}+\frac{11}{5}\times1+1$ sq. ft.=304 sq. ft. Whence,
12. $\frac{6}{5}\times\frac{5}{5}+\frac{11}{5}\times1=303$ sq. ft=the area of $AIHCFE$. But
13. $\frac{11}{5}=\frac{11}{5}\times\frac{5}{5}$, in which $\frac{11}{5}$ is $\frac{11}{5}$ ft.; for a rectangle whose length is $\frac{11}{5}$, and the width 1 ft., has the same area as a rectangle whose width is $\frac{11}{5}$ ft. and length 1, or $\frac{5}{5}$.
14. $\therefore\frac{6}{5}\times\frac{5}{5}+\frac{11}{5}\times\frac{5}{5}=303$ sq. ft., in which $\frac{11}{5}$ is $\frac{11}{5}$ ft.
15. $\frac{1}{5}\times\frac{5}{5}+\frac{11}{30}\times\frac{5}{5}=50\frac{1}{2}$ sq. ft.$=\frac{1}{6}$ of $(\frac{6}{5}\times\frac{5}{5}+\frac{11}{5}\times\frac{5}{5})=\frac{1}{6}$ of 303 sq. ft.,
16. $\frac{5}{5}\times\frac{5}{5}+\frac{11}{6}\times\frac{5}{5}=252\frac{1}{2}$ sq. ft.$=5\times(\frac{1}{5}\times\frac{5}{5}+\frac{11}{30}\times\frac{5}{5})=5\times50\frac{1}{2}$ sq. ft. But
17. $\frac{5}{5}\times\frac{5}{5}=$the area of the square $ABKL$, and
18. $\frac{11}{6}\times\frac{5}{5}=$the area of the rectangle $ALNP$ whose length AL is $\frac{5}{5}$ and width LN $\frac{11}{6}$ ft.
19. $\frac{1}{2}$ of$(\frac{11}{6}\times\frac{5}{5})=\frac{11}{12}\times\frac{5}{5}=$half the rectangle $ALNP=$ the rectangle $OMNP$, which put to the side AB of the square $ABKL$ as in the figure.
20. $\frac{5}{5}\times\frac{5}{5}+\frac{11}{6}\times\frac{5}{5}=252\frac{1}{2}$ sq. ft.=the area of $SRAOMK$.
21. $\frac{121}{144}$ sq. ft.$=(\frac{11}{12})^2=$the area of the square $RQOA$, since AR is $\frac{11}{12}$ ft.
22. $\therefore\frac{5}{5}\times\frac{5}{5}+\frac{11}{6}\times\frac{5}{5}+\frac{121}{144}$ sq. ft.$=252\frac{1}{2}$ sq. ft.$+\frac{121}{144}$ sq. ft. $=\frac{36481}{144}$ sq. ft.=the area of $(SRAOMK+RQOA)$. =the area of $SQMK$.
23. $\frac{5}{5}+\frac{11}{12}$ ft.$=\sqrt{\frac{36481}{144}}=\frac{191}{12}$ ft.=the side SK of the square $SQMK$.
24. $\frac{5}{5}=\frac{191}{12}$ ft.$-\frac{11}{12}$ ft.$=\frac{180}{12}$ ft.$=15$ ft.$=SK-SB=BK$ $=AB$, the width of the ceiling.

25. $\therefore \frac{1}{5}=\frac{1}{5}$ of 15 ft.=3 ft., and
26. $\frac{6}{5}=6$ times 3 ft.=18 ft.=BC, the length of the ceiling.

III. $\therefore$ { 15 ft.=the width of the ceiling, and 18 ft.=the length.

Remark.—In this solution there is but one algebraic operation; viz., extracting the square root of the trinomial expression, $(\frac{5}{5}\times\frac{5}{5}+\frac{11}{6}\times\frac{5}{5}+\frac{121}{144}$ sq. in.), in step 23. This might have been omitted and then the solution would have been purely arithmetical; for, the area of the square $SQMK$ being known, as shown by step 22, its side SK could have been found by simply extracting the square root of its area, $\frac{36841}{144}$ sq. ft. Then by subtracting SB, which is $\frac{11}{12}$ ft., from SK, we would get $BK(=AB)$, the width of the ceiling.

The following solution is quite often given in the schoolroom:
$304\div(5\times6)=10+$. $\sqrt{10}=3+$.
$5\times3=15$, the width and $6\times3=18$, the length.

I. A tin vessel, having a circular mouth 9 inches in diameter, a bottom $4\frac{1}{2}$ inches in diameter, and a depth of 10 inches, is $\frac{1}{4}$ part full of water; what is the diameter of a ball which can be put in and just be covered by the water?

Construction.—Let $ABCD$ be a vertical section of the vessel, AB the top diameter, DC the bottom diameter, and EF the altitude. Produce AD, BC, and EF till they meet in G. Draw MC parallel to EF. In the triangle ACB inscribe the largest circle IEP and let Q be its center. Draw the radius IQ. Now

1. $AE=\frac{1}{2}AB=R=4\frac{1}{2}$ in.=the radius of the mouth.
2. $CF=\frac{1}{2}DC=r=2\frac{1}{4}$ in., the radius of the bottom, and
3. $EF=a=10$ in., the altitude of the vessel.
4. $MB=EB-EM(=FC)=R-r=4\frac{1}{2}$ in. $-2\frac{1}{4}$ in.$=2\frac{1}{4}$ in. In the similar triangles BMC and BEG,
5. $MB:MC::EB:EG$, or $R-r:a::R:EG$. Whence,
6. $EG=\dfrac{aR}{R-r}=\dfrac{10\times4\frac{1}{2}}{4\frac{1}{2}-2\frac{1}{4}}=20$ in., the altitude of the triangle AGB.
7. $IQ=\dfrac{2\triangle AGB}{AB+AG+BG}=\dfrac{AB\times EG}{AB+AG+BG}$ $=\dfrac{2Ra}{AB+BG+BG}$. But

FIG. 72.

8. $AG=BG=\sqrt{(EB^2+EG^2)}=\sqrt{[R^2+(2a)^2]}=\sqrt{[(4\frac{1}{2})^2+20^2]}=\sqrt{(420\frac{1}{4})}=20\frac{1}{2}$ in.
9. $\therefore IQ=\dfrac{4Ra}{2R+2\sqrt{(R^2+a^2)}}=\dfrac{4\frac{1}{2}\times20}{4\frac{1}{2}+20\frac{1}{2}}=3\frac{3}{5}$ in., the radius of the largest sphere that can be put in the vessel or in

the cone AGB.

10. $\frac{4}{3}\pi(IQ)^3=\frac{4}{3}\pi\left(\frac{R2a}{R+\sqrt{(R^2+4a^2)}}\right)^3=\frac{4}{3}\pi(3\frac{3}{5})^3=\frac{7776}{125}\pi$
=the volume of the largest sphere that can be put in the cone AGB.

11. $\frac{1}{3}EG\times\pi EB^2=\frac{1}{3}\pi 2aR^2=\frac{1}{3}\pi\times20\times(4\frac{1}{2})^2=135\pi$, the volume of the cone AGB.

12. $\therefore \frac{1}{3}\pi 2aR^2-\frac{4}{3}\pi\left(\frac{2aR}{R+\sqrt{(R^2+2a^2)}}\right)^3=\frac{1}{3}\pi 2aR^2\times$
$\left[1-\frac{2a^2R}{[R+\sqrt{(R^2+2a^2)}]^3}\right]=135\pi-\frac{7776}{125}\pi=\frac{9099}{125}\pi=$
the quantity of water in the cone which will just cover the largest ball that can be put in the cone AGB.

II. 13. $\frac{1}{3}\pi FG\times FC^2=\frac{1}{3}\pi ar^2=\frac{1}{3}\pi\times10\times(2\frac{1}{4})^2=\frac{135}{8}\pi$, the volume of the cone DGC.

14. $\therefore \frac{1}{3}\pi ar^2+\frac{1}{4}$ of the volume of the vessel $=\frac{135}{8}\pi+\frac{1}{4}$ of the volume of the vessel=the quantity of water in the cone necessary to cover the required ball. But

15. $\frac{1}{3}\pi a(R^2+Rr+r^2)=\frac{1}{3}\pi10[(4\frac{1}{2})^2+4\frac{1}{2}\times2\frac{1}{4}+(2\frac{1}{4})^2]$
$\frac{945}{8}\pi$, the volume of the vessel, by Prob. XCIII.

16. $\therefore \frac{1}{3}\pi ar^2+\frac{1}{4}$ of the volume of the vessel $=\frac{1}{3}\pi ar^2+\frac{1}{4}$ of $\frac{1}{3}\pi a(R^2+Rr+r^2)=\frac{1}{3}\pi a[r^2+\frac{1}{4}(R^2+Rr+r^2)]=\frac{135}{8}$
$\pi+\frac{1}{4}$ of $\frac{945}{8}\pi=\frac{1485}{32}\pi$, the quantity necessary to cover the required ball.

17. $\therefore$ The quantity of water necessary to cover the largest ball: the quantity of water necessary to cover the required ball : : (radius)^3 of largest ball : (radius)^3 of required ball. Hence,

18. $\frac{1}{3}\pi 2aR^2\left[1-\frac{4a^2R}{[R+\sqrt{(R^2+4a^2)}]^3}\right]:\frac{1}{3}\pi a[r^2+\frac{1}{4}(R^2+$
$Rr+r^2)]::\left(\frac{R2a}{R+\sqrt{(R^2+4a^2)}}\right)^3:HO^3$, or

19. $\frac{9099}{125}\pi:\frac{1485}{32}\pi::(3\frac{3}{5})^3:HO^3$. Whence,

20. $\frac{3}{5}\sqrt[3]{(337)}:\frac{3}{2}\sqrt[3]{(\frac{55}{2})}::3\frac{3}{5}:HO$. Whence,

21. $HO=\sqrt[3]{\left(\frac{R[r^2+\frac{1}{4}(R^2+Rr+r^2)]}{1-\frac{2a^2R}{[R+\sqrt{(R^2+4a^2)}]^3}}\right)}=[\frac{3}{2}\sqrt[3]{(\frac{55}{2})}\times3\frac{3}{5}]$
$\div\frac{3}{5}\sqrt[3]{337}=9\sqrt[3]{(\frac{55}{674})}$, and

22. $18\sqrt[3]{(\frac{55}{674})}=6.1967+$in., the diameter of the required ball.

III. $\therefore$ The diameter of the required ball is 6.1967+ in.

I. I have a garden in the form of an equilateral triangle whose sides are 200 feet. At each corner stands a tower; the height of the first tower is 30 feet, the second 40 feet, and the third 50 feet. At what distance from the base of each tower must a ladder be placed, so that without moving it at the base it may just reach the top of each, and what is the length of the ladder?

Construction.—Let ABC be the triangular garden and AD, BE, and CF the towers at the corners. Connect the tops of the towers by the lines ED and DF. From G and H, the middle points of DE and DF, draw GM and HN perpendicular to DE and DF, and at M and N draw perpendiculars to AB and AC in the triangle ABC, meeting at O. Then O is equally distant from D and E. For, since M is equally distant from D and E, and MO perpendicular to the plane $ABED$, every point of MO is equally distant from D and E. For a like reason, every point of NO is equally distant from D and F; hence, O their point of intersection, is equally distant from D, E, and F and is, therefore, the point where the ladder must be placed. Draw DI and DJ parallel to AB and AC, GK and HL perpendicular to AB and AC, MP perpendicular to AC, and OR parallel to NP. Draw the lines OB, OC, and OA, the required distances from the base of the ladder to the bases of the towers. Draw EO, the length of the ladder.

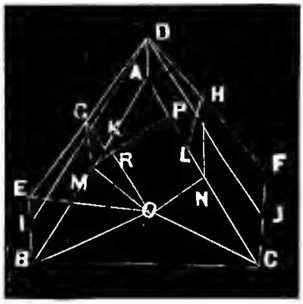


FIG. 73.

1. $AB=BC=AC=200$ ft. $=s$, the side of the triangle.
2. $FC=50$ ft. $=a$, the height of the first tower,
3. $EB=40$ ft. $=b$, the height of the second tower, and
4. $AD=30$ ft. $=c$, the height of the third tower. Let
5. $h=\sqrt{[AB^2-(\frac{1}{2}AC)^2]}=\sqrt{[s^2-(\frac{1}{2}s)^2]}=\frac{1}{2}\sqrt{3}s=100\sqrt{3}$ ft. $=$ the perpendicular from B to the side AC.
6. $EI=BE-BI(=AD)=(b-c)=40$ ft. -30 ft. $=10$ ft.
7. $GK=\frac{1}{2}(EB+AD)=\frac{1}{2}(b+c)=\frac{1}{2}(40\text{ ft.}+30\text{ ft.})=35$ ft. In the similar triangles DIE and GKM,
8. $DI:IE::GK:KM$, or $s:b-c::\frac{1}{2}(b+c):KM$.
9. $\therefore KM=\frac{b^2-c^2}{2s}=\frac{40^2-30^2}{2\times200}=1\frac{3}{4}$ ft.,
10. $AM=AK+KM=\frac{1}{2}s+\frac{b^2-c^2}{2s}=\frac{s^2+b^2-c^2}{2s}$ $=101\frac{3}{4}$ ft., and
11. $BM=AB-AM=s-\frac{s^2+b^2-c^2}{2s}=\frac{s^2+c^2-b^2}{2s}$ $=98\frac{1}{4}$ ft. In like manner,
12. $HL=\frac{1}{2}(a+c)=\frac{1}{2}(50\text{ ft.}+30\text{ ft.})=40$ ft.,

II. A.

13. $LN=\dfrac{a^2-c^2}{2s}=4$ ft.,

14. $AN=AL+LN=\frac{1}{2}s+\dfrac{a^2-c^2}{2s}=\dfrac{s^2+a^2-c^2}{2s}=104$ ft.

15. $NC=AC-AN=s-\dfrac{s^2+a^2-c^2}{2s}=\dfrac{s^2+c^2-a^2}{2s}=$ 96 ft. By similar triangles,

16. $AB:AL::AM:AP$, or $s:\frac{1}{2}s:(s^2+b^2-c^2)\div 2s:AP$. Whence,

17. $AP=(s^2+b^2-c^2)\div 4s=50\frac{7}{8}$ ft.

18. $\therefore PL=AL-AP=[\frac{1}{2}s-(s^2+b^2-c^2)\div 4s]=(s^2+c^2-b^2)\div 4s=49\frac{1}{8}$ ft.

19. $RO=PN=PL+LN=(s^2+c^2-b^2)\div 4s+(a^2-c^2)\div 2s=(s^2+2a^2-b^2-c^2)\div 4s=53\frac{1}{8}$ ft. By similar triangles,

20. $AB:BL::AM:MP$, or $s:\frac{1}{2}\sqrt{3}s::(s^2+b^2-c^2)\div 2s:MP$. Whence,

21. $MP=[(s^2+b^2-c^2)\div 4s]\times\sqrt{3}=50\frac{7}{8}\sqrt{3}$ ft. By similar triangles,

22. $MP:AP::RO:RM$, or $[(s^2+b^2-c^2)\div 4s]\sqrt{3}:(s^2+b^2-c^2)\div 4s::(s^2+2a^2-b^2-c^2)\div 4s:RM$.

23. $RM=(s^2+2a^2-b^2-c^2)4\sqrt{3}s=[(s^2+2a^2-b^2-c^2)\div 12s]\sqrt{3}=17\frac{17}{24}\sqrt{3}$ ft. Again

24. $MP:MA::RO:OM$, or $[(s^2+b^2-c^2)\div 4s]\sqrt{3}:(s^2+b^2-c^2)\div 2s::(s^2+2a^2-b^2-c^2)\div 4s:OM$.

25. $\therefore OM=(s^2+2a^2-b^2-c^2)\div 2\sqrt{3}s=[(s^2+2a^2-b^2-c^2)\div 6s]\sqrt{3}=35\frac{5}{12}\sqrt{3}$ ft.

26. $ON=RP=MP-RM=[(s^2+b^2-c^2)\div 4s]\sqrt{3}-(s^2+2a^2-b^2-c^2)\div 12s]\sqrt{3}=[(s^2-a^2+2b^2-c^2)\div 6s]\sqrt{3}=33\frac{1}{6}\sqrt{3}$ ft. Then

27. $OC=\sqrt{(ON^2+NC^2)}=\sqrt{\left[\left(\dfrac{s^2-a^2+2b^2-c^2}{6s}\sqrt{3}\right)^2+\left(\dfrac{s^2+c^2-a^2}{2s}\right)^2\right]}=\sqrt{[(33\frac{1}{6}\sqrt{3})^2+96^2]}=\sqrt{12516\frac{1}{12}}$ $=111.8796+$ft.

28. $OA=\sqrt{(ON^2+AN^2)}=\sqrt{\left[\left(\dfrac{s^2-a^2+2b^2-c^2}{6s}\sqrt{3}\right)^2+\left(\dfrac{s^2+c^2-a^2}{2s}\right)^2\right]}=\sqrt{[(33\frac{1}{6}\sqrt{3})^2+104^2]}$ $=\sqrt{14116\frac{1}{12}}=118.8111+$ft.

29. $OB=\sqrt{(OM^2+MB^2)}=\sqrt{\left[\left(\dfrac{s^2+2a^2-b^2-c^2}{6s}\sqrt{3}\right)^2+\left(\dfrac{s^2+c^2-b^2}{2s}\right)^2\right]}=\sqrt{[(35\frac{5}{12}\sqrt{3})^2+(98\frac{1}{4})^2]}$ $=\frac{1}{4}\sqrt{214657\frac{1}{3}}=115.8278+$ft.

B.

1. $OE=\sqrt{(BE^2+OB^2)}=\sqrt{[(\frac{1}{4}\sqrt{214657\frac{1}{3}})^2+40^2]}$, $=\sqrt{(13416\frac{1}{2}+1600)}=\sqrt{15016\frac{1}{12}}=122.5402+$ft.=the length of the ladder.

III. ∴

1. 111.8796+ft.=the distance from base of the ladder to the base of the tower FC,
2. 118.8111+ft.=the distance from the base of the ladder to the base of the tower AD.
3. 115.8278+ft.=the distance from the base of the ladder to the base of the tower BE, and
4. 122.5402+ft.=the length of the ladder.

(*Greenleaf's Nat'l Arith., p. 444, prob. 38.*)

Remark.—When the sides of the triangle are unequal, proceed in the same manner as above. In some cases the base of the ladder will fall without the triangle.

I. At the extremities of the diameter of a circular garden stands two trees, one 20 feet high and the other 30 feet high. At what point on the circumference must a ladder be placed so that without moving it at the base it will reach to the top of each tree, the diameter of the garden being 40 feet.

Construction.—Let ABC be the circular garden and AC its diameter, and let AF and CD be the two trees at the extremities of the diameter. Connect the tops of the trees by the line FD and from the middle point E of FD let fall the the perpendicular EH. Draw EG perpendicular to FD. Then all points of EG are equally distant from FD. At G draw BG perpendicular to AC. Then all points of BG are equally distant from F and D. Hence, B is the required point.

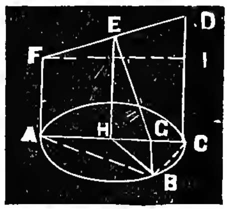


FIG. 74.

II.

1. $AC=2R=40$ft., the diameter of the garden.
2. $CD=a=30$ ft., the height of the tree CD, and
3. $AF=b=20$ ft., the height of the tree AF.
4. $DI=DC-CI\ (=AF)=a-b$ $=40$ ft.—30 ft.$=10$ ft.
5. $EH=\frac{1}{2}(CD+AF)=\frac{1}{2}(a+b)$ $=\frac{1}{2}(40\text{ ft.}+30\text{ ft.})=35$ ft. By similar triangles,
6. $FI:ID::EH:HG$, or $2R:a-b::\frac{1}{2}(a+b):HG$. Whence,
7. $HG=\dfrac{a^2-b^2}{4R}=8\frac{3}{4}$ ft.
8. $GB=\sqrt{(BH^2-HG^2)}=\sqrt{\left[R^2-\left(\dfrac{a^2-b^2}{4R}\right)^2\right]}=$ $\dfrac{\sqrt{[16R^4-(a^2-b^2)^2]}}{4R}=\frac{15}{4}\sqrt{23}$ ft.
9. $\therefore AB=\sqrt{(AG^2+GB^2)}=\sqrt{[(AH+HG)^2+(GB^2)]}=$ $\sqrt{\left[\left(R+\dfrac{a^2-b^2}{4R}\right)^2+\dfrac{16R^4-(a^2-b^2)^2}{16R^2}\right]}=5\sqrt{46}$ ft. $=34.91165$ ft., nearly, and

10. $BC=\sqrt{(GC^2+GB^2)}=\sqrt{\left[\left(R-\frac{a^2-b^2}{4R}\right)^2+\frac{16R^4-(a^2-b^2)^2}{16R^2}\right]}=\frac{5}{4}\sqrt{82}$ ft. $=11.31942$ ft.

III. $\therefore$ { 34.91165 ft. the distance from the smaller tree, and 11.31942 ft. the distance from the larger tree.

I. Seven men bought a grindstone 5 feet in diameter; what part of the diameter must each grind off so that they may share equally?

Construction.—Let AB be the diameter of the grind stone, O its center, and AO its radius. From A draw any indefinite line AN and on it lay off any convenient unit of length seven times, beginning at A. Let P be the last point of division. Draw OP, and from the other points of division draw lines parallel to OP, intersecting the radius AO, in the points f, e, d, c, b, and a. Then the radius is divided into seven equal parts. On radius AO, as a diameter describe a semi-circumference AOK, and at a, b, c, d, e, and f, erect perpendiculars intersecting the semi-circumference in M, L, K, I, H, and G. Then with

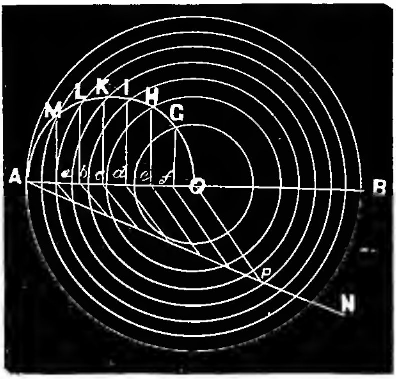


FIG. 75.

O as a center and radii equal the chords MO, LO, KO, IO, HO, and GO, describe the circles as shown in the figure. Then each man's share will be the area lying between the circumferences of these circles. For, the chord $GO^2=Gf^2+fO^2$ and, by a property of the circle, $Gf^2=Af\times fO$. $\therefore$ $GO^2=Af\times fO+fO^2$, $=(Af+fO)fO=AO\times fO=\frac{1}{7}AO^2$. In like manner $HO^2=AO\times eO=\frac{2}{7}AO^2$, $KO^2=AO\times dO=\frac{3}{7}AO^2$, &c.

1. $AB=D=5$ ft., the diameter of the grind stone.
2. $AO=R=2\frac{1}{2}$ ft., the radius.
3. $\therefore$ $\pi R^2=\pi\times(2\frac{1}{2})^2=6\frac{1}{4}\pi=$the area of the stone.
4. $\frac{1}{7}$ of $\pi R^2=\frac{1}{7}\pi R^2=\frac{1}{7}$ of $6\frac{1}{4}\pi=\frac{25}{28}\pi=$each man's share.
5. $6\frac{1}{4}\pi-\frac{25}{28}\pi=\frac{75}{14}\pi=$the area of the stone after the first has ground off his share.
6. $\therefore$ $\sqrt{(\frac{75}{14}\pi\div\pi)}=\frac{5}{14}\sqrt{42}=2.31455+$ft., the radius MO.
7. $2(AO-MO)=2(2\frac{1}{2}$ ft.$-\frac{5}{14}\sqrt{42}$ ft.$)=2(2\frac{1}{2}$ ft.-2.31455 ft.$)=.3709$ ft., part of the diameter the first grinds off.
8. $6\frac{1}{4}\pi-\frac{2}{7}$ of $6\frac{1}{4}\pi=\frac{125}{28}\pi=$the area after the second grinds off his share.

9. $\therefore \sqrt{(\frac{125}{28}\pi \div \pi)} = \frac{5}{2}\sqrt{\frac{5}{7}} = 2.112875$ ft., the radius LO. Then

10. $2(MO - LO) = 2(5\sqrt{\frac{3}{14}} - \frac{5}{2}\sqrt{\frac{5}{7}}) = 2(2.31455$ ft.$-$ 2.112875 ft.$) = .40335$ ft., the part of the diameter the second grinds off.

11. $6\frac{1}{4}\pi - \frac{3}{7}$ of $6\frac{1}{4}\pi = \frac{25}{7}\pi =$ the area after the third has ground off his share.

12. $\therefore \sqrt{(\frac{25}{7}\pi \div \pi)} = \frac{5}{2}\sqrt{\frac{4}{7}} = 5\sqrt{\frac{1}{7}} = 1.889822+$ ft., the radius KO. Then,

13. $2(LO - KO) = 2(\frac{5}{2}\sqrt{\frac{5}{7}} - 5\sqrt{\frac{1}{7}}) = 2(2.112875$ ft.$-$ 1.8899822 ft$) = .446106$ ft., the part of the diameter the third grinds off

II.

14. $6\frac{1}{4}\pi - \frac{4}{7}$ of $6\frac{1}{4}\pi = \frac{75}{28}\pi =$ the area after the fourth has ground off his his share.

15. $\therefore \sqrt{(\frac{75}{28}\pi \div \pi)} = \frac{5}{2}\sqrt{\frac{3}{7}} = 1.636634$ ft., the radius IO. Then

16. $2(KO - IO) = 2(5\sqrt{\frac{1}{7}} - \frac{5}{2}\sqrt{\frac{3}{7}}) = 2(1.889822$ ft.$-$ 1.636634 ft.$) = .506376$ ft., the part of the diameter the fourth grinds off.

17. $6\frac{1}{4}\pi - \frac{5}{7}$ of $6\frac{1}{4}\pi = \frac{50}{28}\pi =$ the area after the fifth grinds off his share.

18. $\therefore \sqrt{(\frac{50}{28}\pi \div \pi)} = \frac{5}{2}\sqrt{\frac{2}{7}} = 1.336306$ ft., the radius HO. Then

19. $2(IO - HO) = 2(\frac{5}{2}\sqrt{\frac{3}{7}} - \frac{5}{2}\sqrt{\frac{2}{7}}) = 2(1.636634$ ft.$-$ 1.336306 ft.$) = .600656$ ft., the part of the diameter the fifth grinds off.

20. $6\frac{1}{4}\pi - \frac{6}{7}$ of $6\frac{1}{4}\pi = \frac{25}{28}\pi =$ the area after the sixth grinds off his share.

21. $\therefore \sqrt{(\frac{25}{28}\pi \div \pi)} = \frac{5}{2}\sqrt{\frac{1}{7}} = .949911$ ft., the radius GO. Then

22. $2(HO - GO) = 2(\frac{5}{2}\sqrt{\frac{2}{7}} - \frac{5}{2}\sqrt{\frac{1}{7}}) = 2(1.336306$ ft.$-$ .944911 ft.$) = .782790$ ft., the part of the diameter the sixth grinds off.

23. $2 \times .944911$ ft.$= 1.889822$ ft., the diameter of the part belonging to the seventh man.

I. J. A. M., having a wooden ball 2 feet in diameter, bored a hole 1 foot in diamer through the center. What is the volume bored out?

Construction.—Let $ABCDEF$ be a great circle of the ball and let $ACDF$ be a verticle section of the auger hole. Draw the diameter BOE and the radius AC. Then the volume bored out consists of a cylinder, of which $ACDF$ is a vertical section, and two spherical segments, of which ACB and FDE are vertical sections.

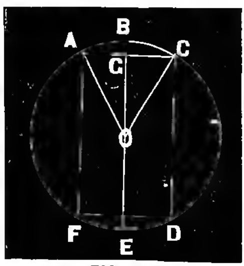


FIG. 76

1. $BE = 2$ feet $= 2R$, the radius of the ball, and
2. $AO = 1$ foot $= 2r$, the radius of the auger hole.

II.
3. $\frac{1}{2}AF=OG=\sqrt{(AO^2-AG^2)}=\sqrt{(R^2-r^2)}=\frac{1}{2}\sqrt{3}$.
4. $\therefore AF=2\sqrt{(R^2-r)^2}=\sqrt{3}$, the length of the cylinder.
5. $\therefore V=\pi r^2\times(\sqrt{3})=\frac{1}{4}\pi\sqrt{3}$, the volnme of the cylinder, and
6. $2V'=2(\frac{1}{2}BG\times\pi AG^2+\frac{1}{6}\pi BG^3)=[R-\sqrt{(R^2-r^2)}]\times\pi r^2+\frac{1}{3}\pi[R-\sqrt{(R^2-r^2)}]^3$
7. $\frac{1}{4}\pi(1-\frac{1}{2}\sqrt{3})+\frac{1}{3}\pi(1-\frac{1}{2}\sqrt{3})^3=\frac{1}{12}\pi(16-9\sqrt{3})$, the volume of the two spherical segments.
8. $\therefore V+2V'=\frac{1}{4}\pi\sqrt{3}+\frac{1}{12}\pi(16-9\sqrt{3})=\frac{1}{6}\pi(8-3\sqrt{3})$, $=1.46809$ cu. ft.$=2536.85952$ cu. in.

III. $\therefore$ The volume bored out is 2536.85952 cu. in.

I. What is the diameter of the largest circular ring that can be put in a cubical box whose edge is 1 foot?

Construction.—Let $ABCD-E$ be the cubical box. Let I, K, L, M, N, and P, be the middle points of the edge CF, GF, GH, HA, AB, and BC respectively. Connect these points by the lines KI, KL, LM, MN, NP, and PI. Then $IKLMNP$ is a regular hexagon, and the largest ring that can be put in the box will be the inscribed cricle of the hexagon.

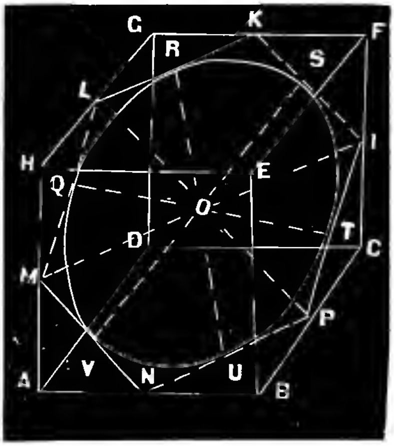


FIG. 77.

II.
1. $AB=12$ in.$=e$, the edge of the cube.
2. $AN=AM=\frac{1}{2}AB=6$ in.$=\frac{1}{2}e$.
3. $\therefore MN=ML=MO=\sqrt{(AN^2+AM^2)}=\sqrt{(2AN^2)}=AN\sqrt{2}=\frac{1}{2}\sqrt{2}e$, the side of the hexagon.
4. $MQ=\frac{1}{2}ML=\frac{1}{2}$ of $\frac{1}{2}\sqrt{2}e=\frac{1}{4}\sqrt{2}e$. Then
5. $OR=\sqrt{(MO^2-MQ^2)}=\sqrt{[(\frac{1}{2}\sqrt{2}e)^2-(\frac{1}{4}\sqrt{2}e)^2]}=\frac{1}{4}\sqrt{6}e$ the radius of the circle.
6. $\therefore 2OR=2\times(\frac{1}{4}\sqrt{6}e)=\frac{1}{2}\sqrt{6}e=\frac{1}{2}\sqrt{6}\times12=6\sqrt{6}=14.6969382$ in., the diameter.

III. $\therefore$ The diameter of the largest circular ring that can be put in a cubical box whose edge is 1 foot, is 14.6969382 in.

I. A fly takes the shortest route from a lower to the opposite upper corner of a room 18 feet long, 16 feet wide, and 8 feet high. Find the distance the fly travels and locate the point where the fly leaves the floor.

Construction.—Let $FABE—D$ be the room, of which AB is the length, AF the width, and AD the height; and let F be the position of the fly, and C the opposite upper corner to which it is to travel. Conceive the side $ABCD$ to revolve about AB until it comes to a level with the floor and takes the position of $ABC'D'$. Then the shortest path of the fly is the diagonal FC' of the rectangle $FD'C'E$, and P will be the point where the fly leaves the floor.

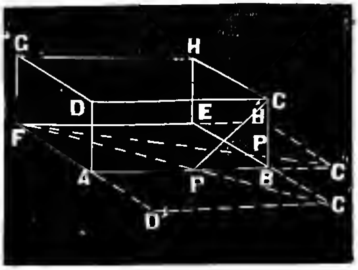


FIG. 78.

II.
- A.
  1. $AB=a=18$ ft., the length of the room,
  2. $AF=b=16$ ft., the width, and
  3. $AD=h=8$ ft., the height.
  4. $FD'=FA+AD'=b+h=16$ ft.$+8$ ft.$=24$ ft. Then
  5. $FC'=\sqrt{(FD'^2+D'C'^2)}=\sqrt{[(b+h)^2+a^2]}$, $=\sqrt{[(16+8)^2+18^2]}=30$ feet, the length of the path of the fly.
- B.
  1. $FD':D'C'::AF:AP$, from the similar triangles $C'D'F$ and PAF, or
  2. $b+h : a :: b : AP$. Whence, $AP=\frac{ab}{b+a}=\frac{18\times16}{16+8}$ $=12$ feet, the distance from A to where the fly leaves the floor.

III. ∴ { 30 feet is the distance the fly travels, and 12 feet is the distance from A to where it leaves the floor.

Remark.—If we conceive the side $BCHE$ to revolve about EH until it is level with the floor, the path of the fly will be FC'' and the length of this is $\sqrt{[(a+h)^2+b^2]}$. But $\sqrt{[(a+h)^2+b^2]}>\sqrt{[(b+h)^2+a^2]}$, because, by expanding the terms under the radicals, it will be seen that the terms are the same, except $2ah$ and $2bh$, and since a is greater than b, FC' is less than FC''. When $a=b$, $FC'=FC''$.

I. How many acres are there in a square tract of land containing as many acres as there are boards in the fence inclosing it, if the boards are 11 feet long and the fence is 4 boards high?

1. $\frac{(side)^2}{160}$=number of acres in the tract, the side being expressed in rods.
2. $4\times16\frac{1}{2}\times side$=number of feet in the perimeter of the field.

I. 3. $\therefore 4\times\left[\frac{4\times16\frac{1}{2}\times side}{11}\right]$=number of boards in the fence inclosing the tract.

4. $\therefore \frac{(side)^2}{160}=4\left[\frac{4\times16\frac{1}{2}\times side}{11}\right]=24\times side.$ Whence,

5. $(side)^2=160\times24\times side=3840\times side.$

6. $\therefore side=3840$ rods=12 miles.

7. $\therefore (3840)^2\div160=92160$=number of acres in the tract.

III. ∴ There are 92160 A. in the tract.

(*Milne's Pract. Arith., p. 362, prob. 71.*)

SECOND SOLUTION.

II.
1. 16=number of acres comprised between two panels of fence on opposite sides of the field.
2. 1A.=43560 sq. ft.
3. 16 A.=16×43560 sq. ft.=696960 sq. ft.
4. 11ft.=the width of this strip comprised between the two panels.
5. ∴ 12 mi.=63360 ft.=$696960\div11$, the length of the strip, which is the width of the field.
6. 144 sq. mi.=$(12)^2$=the area of the field.
7. 1 sq. mi.=640 A.
8. 144 sq. mi.=144×640 A=92160 A.

III. ∴ There are 92160 A. in the tract.

Explanation—Since for every board in the fence there is an acre of land in the tract for 4 boards, or one panel of fence there would be 4 A. Now a panel on the opposite side of the field would also indicate 4 A. Hence, between two panels on opposite sides of the field there would be comprised a tract 11 ft. wide and containing 8 A. But this would make boards on the *other* two sides of the field have no value. Now the boards on the other two sides having as much value as the boards on the first two sides, it follows that we must take twice the area of the rectangle included between two opposite panels for the area comprised between two opposite panels in the entire tract. Hence, between two opposite panels in the tract there are comprised 16 A. The length of this rectangle is $16\times43560\div11=63360$ ft.=12 mi., which is the length of the side of the tract.

In any problem of this kind, we may find the length of a side in miles, by multiplying the number of boards in the height of the fence by 33 and divide the product by the length of a board, expressed in feet.

THIRD SOLUTION.

Construction.—Let $ABCD$ be the square tract of land; O, its center; and EF, a panel of the fence. Draw OE and OF. Draw OI perpendicular to AB. Then, by the conditions of the problem, the area of the triangle, EOF, is 4 acres,= 640 sq. rds., or 174240 sq. ft.

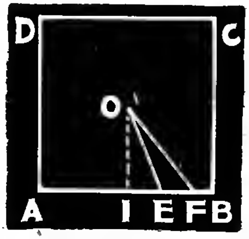


FIG. 78a.

II.
1. 11 = number of feet in the length of the panal, EF.
2. $\frac{1}{2}(EF\times OI)=\frac{1}{2}(11\times OI)=$ the number of square feet in the area of the triangle, EOF.
3. 174240=the number of square feet in the area of the triangle, EOF.
4. $\therefore\ \frac{1}{2}(11\times OI)=174240$, and
5. $OI=\frac{2}{11}$ of 174240=31680=the number of feet in the length of the altitude, OI, of the triangle EOF, which is half the length of a side of the field.
6. ∴ 63360=the number of feet in the length of a side.
7. 63360 feet=12 miles.
8. ∴ 144 sq. mi.$=12^2\times1$ sq. mi.=the area of the tract in sq. miles.
9. 1 sq. mi.=640 A.
10. 144 sq. mi.=144×640 A.=92160 A.

III. ∴ The tract contains 92160 acres.

I. How many acres are there in a rectangular tract of land containing as many acres as there are boards in the fence enclosing it, the fence being 6 boards high, the boards being a rod long, and the length of the field being $1\frac{1}{2}$ times its width?

Construction.—Let $ABCD$ be the tract; AB, its length; BC, its breadth; O, its center; EF, a panel in its length; and GH, a panel in its width. Draw the lines OE, OF, OG, and OH. Draw OI and OK perpendicular to AB and AD, respectively.

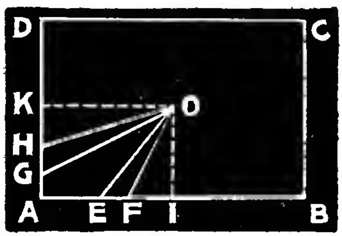


FIG. 78b.

Then, since $AB=1\frac{1}{2}$ times BC, KO, the half of AB,= $1\frac{1}{2}$ times IO, the half of BC. Hence, the area of triangle, GOH,$=1\frac{1}{2}$ times the area of triangle, EOF. Now, if the tract were a square, the area of the triangle, EOF, would=6 A., or, 960 sq. rds. But, since it is a rectangle whose length is $1\frac{1}{2}$ times its breadth, the area of triangle, $EOF,=\frac{1+1\frac{1}{2}}{2(1\times1\frac{1}{2})}$ of 960 sq. rds.=800 sq. rds. Hence, the area of triangle, GOH,$=1\frac{1}{2}$ times 800 sq. rds.=1200 sq. rds.

II.
1. ∴ $\frac{1}{2}(EF\times IO)=\frac{1}{2}(1\times IO)=800$ sq. rds.
2. ∴ $IO=2\times800$ rds.=1600 rds., and
3. therefore, $BC=3200$ rds.
4. ∴ $AB=1\frac{1}{2}$ times 3200 rds.=4800 rds.
5. ∴ $3200\times4800\div160=96000$ A., the area of the tract.

III. ∴ 96000 A.=the area of the tract.

Remark.—The above solution is derived from the algebraic solution of the generalized problem. Thus, if the length of the tract is to the breadth as $m:n$, the algebraic solution shows that the area of the triangle, $EOF,=\frac{m+n}{2mn}$ of the same triangle, were the tract a square.

I. A horse is tied to one corner of a rectangular barn 40 feet long and 30 feet wide; what is the area of the surface over which the horse can range if the rope with which he is tied, is 80 feet long?

Construction.—Let $ABCD$ be the barn; AB, its length; BC, its width; A, the corner to which the horse is tied; and AK, the length of the rope. With A as a center and a radius equal to AK,= 80 feet, describe $\frac{3}{4}$ of a circumference, beginning at K and ending at G. (All of this is not shown in the figure as it would make the figure unnecessarily large). With D as a center and DK,= 70 feet, describe the quadrant, KPE. With B as a center and BG,=50 feet, as a radius, describe the quadrant, GPF. Then draw the lines, DP and BP, and prolong AB to G, AD to K, BC to F, and DC to E. Draw PH and PI perpendicular to DE and BF, respectively.

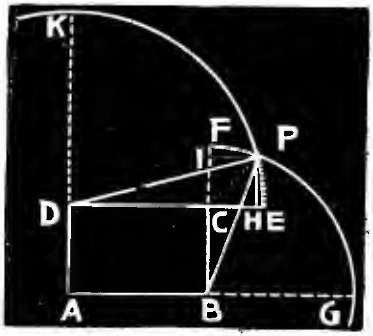


FIG. 78c.

1. Then $\frac{3}{4}$ of a circle whose radius is AK, or $\frac{3}{4}\pi AK^2$,+sector KDP+sector GBP+triangle, DCP,+triangle, BCP,=the area over which the horse can range. But
2. $\frac{3}{4}\pi AK^2=\frac{3}{4}\pi 80^2=4800\pi$=the number of square feet in the area of $\frac{3}{4}$ of the circle, radius AK.
3. Area of sector, KDP,=area of quadrant, KDE,—area of sector, PDE,$=\frac{1}{4}\pi KD^2$—area of sector, PDE,$=\frac{1}{4}\pi 50^2$—area of sector, PDE,$=625\pi$—area of sector, PDE. But
4. area of sector, $PDE,=\frac{1}{2}PD\times$arc PE.
5. Arc, $PE,=\frac{1}{3}(8\times$chord, $PE,-2\times PH)$, by Prob. XXV, Rule b.

6. $(DC+CH)^2+HP^2=DP^2$, or $DC^2+2DC\times CH+CH^2+HP^2=DP^2$. Also,
7. $(BC+CI)^2+IP^2=BP^2$, or $BC^2+2BC\times CI+CI^2+IP^2=BP^2$.
8. $\therefore DC^2-BC^2+2(DC\times CH-BC\times CI)=DP^2-BP^2$, by subtracting step 6 from step 5.
9. $\therefore CH=\{\frac{1}{2}[(DP^2-BP^2)-(DC^2-BC^2)]+BC\times CI\}\div DC$, by solving the equation in step 7, with respect to $CH,=\{\frac{1}{2}[900-700]+30\times CI\}\div 40$, by substituting the numbers for DP, BP, DC, and $BC,=2\frac{1}{2}+\frac{3}{4}CI$.
10. $\therefore (BC+2\frac{1}{2}+\frac{3}{4}CI)^2+IP^2(=CH^2=[2\frac{1}{2}+\frac{3}{4}CI]^2)=BP^2$, by substituting for CH, in step 6, $2\frac{1}{2}+\frac{3}{4}CI$, as found in step 8.
11. $\therefore (32\frac{1}{2}+\frac{3}{4}CI)^2+(2\frac{1}{2}+\frac{3}{4}CI)^2=40^2$, or
12. $1056\frac{1}{4}+48\frac{3}{4}CI+\frac{9}{16}CI^2+6\frac{1}{4}+3\frac{3}{4}CI+\frac{9}{16}CI^2=1600$.
13. $\therefore \frac{9}{8}CI^2+52\frac{1}{2}CI=537\frac{1}{2}$, or
14. $CI^2+46\frac{2}{3}CI=477\frac{7}{9}$.

II. 15. $CI^2+46\frac{2}{3}CI+(23\frac{1}{3})^2=477\frac{7}{9}+(23\frac{1}{3})^2=1022\frac{2}{9}$ by making the first side of the last equation a perfect square.
16. $\therefore CI+23\frac{1}{3}=\pm\sqrt{1022\frac{2}{9}}=\pm6\frac{2}{3}\sqrt{23}$, or $CI=6\frac{2}{3}\sqrt{23}-23\frac{1}{3}$.
17. $\therefore CH=2\frac{1}{2}+\frac{3}{4}(6\frac{2}{3}\sqrt{23}-23\frac{1}{3})=5\sqrt{23}-15$.
18. $HE=CE-CH=10-(5\sqrt{23}-15)=25-5\sqrt{23}=5(5-\sqrt{23})$.
19. The chord, $PE,=\sqrt{PH^2+HE^2}=\sqrt{CI^2+HE^2}=\sqrt{\{2766\frac{2}{3}-561\frac{1}{9}\sqrt{23}\}}=1\frac{2}{3}\sqrt{\{996-202\sqrt{23}\}}$.
20. The arc, $PE,=\frac{1}{3}(8\times\text{chord, } PE,-2\times PH)=\frac{1}{3}[8\times1\frac{2}{3}\sqrt{\{996-202\sqrt{23}\}}-2(6\frac{2}{3}\sqrt{23}-23\frac{1}{3})]$ ft. $=\frac{40}{9}[\sqrt{\{996-202\sqrt{23}\}}-\sqrt{23}+3\frac{1}{2}]$ ft. $=17.773$ ft., by Prob. XXV, Rule (*b*).
21. Area of sector, $PBE,=\frac{1}{2}(\text{arc, } PE,\times PD)=25\times\text{arc, } PE,$ $=444.325$ sq. feet.
22. Area of sector, $PBF,=\frac{1}{2}(\text{arc, } PF,\times PB)=20\times\text{arc, } PF,$ $=20\times14.63=286.6$ sq. ft., the arc, PF, being found in the same way as the arc, PE, was found.
23. Area of triangle, $DCP,=\frac{1}{2}[DC\times PH(=CI)]=\frac{1}{2}[50\times(6\frac{2}{3}\sqrt{23}-23\frac{1}{3})]=207.624$ sq. ft.
24. Area of triangle, $BCP,=\frac{1}{2}[BC\times PI(=CH)]=\frac{1}{2}[40\times5(\sqrt{23}-3)]=179.574$ sq. ft.
25. $\therefore$ Area over which the horse can range $=\frac{3}{4}\pi80^2+(\frac{1}{4}\pi50^2-444.325)+(\frac{1}{4}\pi40^2-286.6)+207.624+179.574=(5825\pi-730.925)=16956.093$ sq. ft.

III. $\therefore$ The horse can range over 16956.093 sq. ft.

NOTE.—A solution of this problem is given by the author in the *American Mathematical Monthly*, Vol. VII. A solution of the generalized problem was given by Professor G. B. M. Zerr, in Vol. VIII, of the same journal. When the barn is a square, the solution is much shorter and simpler. The generalized solution may be solved by the same method as pursued in the above problem. By using Rule (*b*) of Prob. XXV for finding the length of the arcs, PE and PF, a very close approximation can be secured if the arc is not greater than 30° or 40°.

I. What is the area of the largest square that can be inscribed in a semi-circle whose diameter is 10 feet?

Formula.—Side of square, $s,=\frac{2}{5}\sqrt{5}R$.

Rule.—*Multiply the radius of the given circle, by* $\frac{2}{5}$ *of the square root of* 5; *the result will be a side of the inscribed square.*

Construction.—Let $ABDF$ be the semi-circle, and C, its center. With B as a center and AB, the diameter of the given semi-circle, as radius, describe the quadrant AI. Draw BI perpendicular to AB. Draw CI. From D, let fall the perpendicular DH to AB and draw DF parallel to AB. From F drop FG perpendicular to AB. Then $GHDF$ is the required square.

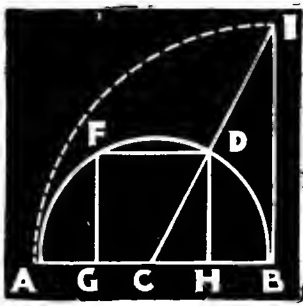


FIG. 78d.

Proof.

1. $CI=\sqrt{CB^2+BI^2}=\sqrt{CB^2+(2CB)^2}=\sqrt{5}BC$.
2. $CD:CI=DH:IB$, by the similar triangles CDH and CIB, or,
3. $CB:\sqrt{5}CB=DH:2CB$, by substituting for CD, its equal, CB; for CI, its equal, $\sqrt{5}CB$; and for BI, its equal, $2BC$.
4. $\therefore DH=\frac{2}{5}\sqrt{5}BC$.
5. $CH=\sqrt{CD^2-DH^2}=\sqrt{BC^2-\frac{4}{5}BC^2}=\frac{1}{5}\sqrt{5}BC$.
6. $\therefore GH=\frac{2}{5}\sqrt{5}BC$.
7. $GH=DH,=FD=FG$.
8. $\therefore HDFG$ is a square, the sides being equal and perpendicular to each other.

III. $\therefore$ The side, DH, of the inscribed square $=\frac{2}{5}\sqrt{5}CB=\frac{2}{5}\sqrt{5}\times10=4\sqrt{5}$ feet.

I. What is the edge of the largest cube that can be inscribed in a hemisphere whose diameter is 12 feet?

Formula.—$e=\frac{1}{3}\sqrt{6}R$.

Rule.—*Multiply the radius by* $\frac{1}{3}$ *of the square root of* 6.

Construction.— Let $KLM-N$ be the hemisphere and $ABCD-F$ the inscribed cube, vertices E, F, G, and H being in the curved surface of the hemisphere. Let O be the center of the hemisphere. Draw the radius, OF, also the line, OB.

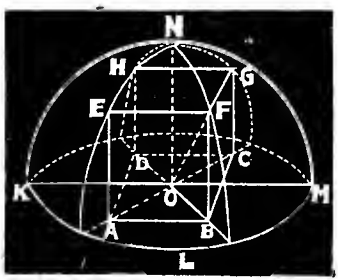


FIG. 78e.

II.
1. $OB=\frac{1}{2}OD=\frac{1}{2}\sqrt{BC^2+CD^2}=\frac{1}{2}\sqrt{2}e$, where e is the edge of the cube.
2. $OF=\sqrt{OB^2+BF^2}=\sqrt{\frac{1}{2}e^2+e^2}=\frac{1}{2}\sqrt{6}e$, or
3. $\frac{1}{2}\sqrt{6}e=R$.
4. $\therefore e$, the edge of the cube, $=\frac{1}{3}\sqrt{6}R=\frac{1}{3}\sqrt{6}\times6=2\sqrt{6}$ feet.

III. $\therefore$ The edge of the required cube is $2\sqrt{6}$ feet.

I. How many stakes can be driven down upon a space 15 feet square, allowing no two to be nearer each other than $1\frac{1}{4}$ ft.?

A. By Rectangular Arrangement.

II.
1. $13=15\div1\frac{1}{4}+1$, = the number of stakes that can be put in each row.
2. $169=(13)^2$, the number that can be placed in the given space.

B. By Triangular-Rectangular Arrangement.

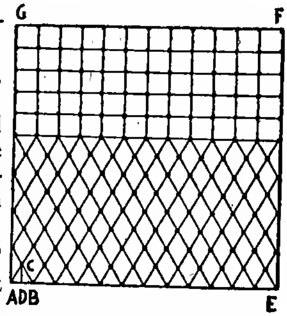


FIG. 78f.

II.
1. Place 13 stakes on the base line, AE.
2. Place 12 stakes in the second row, setting each of the twelve directly over the centers of the twelve spaces in the base, as in the figure.
3. Then CD, the width of the strip between the first and second row, $=\sqrt{AC^2-AD^2}=\sqrt{(1\frac{1}{4})^2-[\frac{1}{2}(1\frac{1}{4})]^2}=\frac{5}{8}\sqrt{3}=1.0825+$ ft.
4. $\therefore 1\frac{1}{4}$ ft. $-\frac{5}{8}\sqrt{3}$ ft. $=\frac{5}{8}(2-\sqrt{3})$ft. $=.1674+$ft., the gain in width, by using strips $\frac{5}{8}\sqrt{3}$ ft. wide, instead of $1\frac{1}{4}$ ft. wide.
5. $\therefore$ To gain $1\frac{1}{4}$ ft., we must take $1\frac{1}{4}$ ft. $\div\frac{5}{8}(2-\sqrt{3})$ ft., or $2(2+\sqrt{3})$ strips, $=7.46+$ strips.
6. $\therefore$ In 7 strips, $1\frac{1}{4}$ ft. wide, we can have 8 strips $\frac{5}{8}\sqrt{3}$ ft. wide.
7. $\therefore$ On these 8 strips we can place 9 rows of stakes,—5 rows of 13 each, and 4 rows of 12 each, or 113 stakes in all.
8. On the remaining 6 strips, $1\frac{1}{4}$ ft. wide, we can place 5 rows of 13 stakes in a row, or 65 stakes.
9. $\therefore$ 113 stakes + 65 stakes = 178 stakes.

C. By Isosceles-Triangle Method.

1. Place 12 stakes on the base line, AE, placing the first stake at A, and the remaining 11 in such a way that the 12 stakes occupy $11\frac{1}{2}$ equal spaces.

II.

2. Begin at I, and place a row of 12 stakes exactly as the first row was placed.
3. Begin at H, and place a third row of 12 stakes exactly as in the last, and so on.
4. By this method we can place 15 rows of 12 stakes in a row, in all, 180 stakes. This is possible; for
5. $AB=15$ ft.$\div 11\frac{1}{2}=1\frac{7}{23}$ ft.
6. $\therefore CD=\sqrt{AC^2-AD^2}=\sqrt{(1\frac{1}{4})^2-[\frac{1}{2}(1\frac{7}{23})]^2}=\frac{5}{92}\sqrt{385}$ ft.
7. $\therefore 14\times\frac{5}{92}\sqrt{385}$ ft., $=\frac{35}{46}\sqrt{385}$ ft. $=14.92+$ ft., which is less than the side of the given square.
8. Hence, we can have 15 rows of 12 stakes in a row, or 180 stakes.

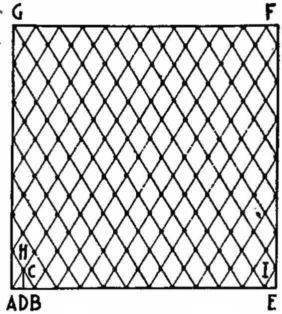


FIG. 78g.

III. $\therefore$ 169 stakes can be placed on the square by rectangular arrangement; 178, by triangular-rectangular arrangement; and 180, by isosceles-triangle arrangement.

I. Two poles, AD and BC, are a certain distance apart; from the top of BC to the foot of AC is 100 feet, and from the top of AC to the foot of BC is 70 feet; and from the point where these two lines cross to the plane is 20 feet. What is the height of each pole, and how far apart are they?

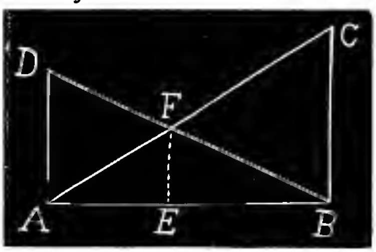


FIG. 78h

Let $AC=100$ feet$=a$; $BD=80$ feet$=b$; $FE=20$ feet$=c$; $BC=x$; and $AD=y$. Then

II.

1. $FE=c=\dfrac{xy}{x+y}$, from problem, page 365.
2. $\therefore y=\dfrac{cx}{x-c}$.
3. $AC^2-BC^2=AB^2$, and $BD^2-AD^2=AB^2$.
4. $\therefore AC^2-BC^2=BD^2-AD^2$, or $a^2-x^2=b^2-y^2$.
5. $\therefore y^2=x^2-(a^2-b^2)$.
6. $\therefore x^4-2cx^3-(a^2-b^2)x^2+2c(a^2-b^2)x-(a^2-b^2)c^2=0$, by substituting for y in step 5, its value in step 2.
7. $\therefore x^4-40x^3-3600x^2+144000x-1440000=0$; whence
8. $x=66.473+$ feet, by solving the last equation by Horner's method. The remaining quantities are now easily found.

III. $\therefore$ The height of the longer pole is 66.473+ feet.; etc.

I. How many acres in a circular tract of land, containing as many acres as there are boards in the fence inclosing it, the fence being 5 boards high, the boards 8 feet long, and bending to the arc of a circle?

Construction.—Let C be the center of of the circular tract, $AB=AC=R$, the radius, and the arc $AB=8$ feet. Then the area of the sector is 5 A.=217800 sq. ft.

II.
1. 5 A.$=5\times43560$ sq. ft.$=217800$ sq. ft., the area of the sector ABC.
2. $\frac{1}{2}(AB\times AC)=\frac{1}{2}(8\times AC)=4AC=$area of the sector ABC.
3. $\therefore$ $4AC=217800$ sq. ft. Whence,
4. $AC=217800\div4=54450$ ft.$=3300$ rods, the radius of the circle.
5. $\pi\times(3300)^2\div160=68062.5\pi$ =number of acres in tract.

III. $\therefore$ There are 68062.5π A., in the tract.

I. What is the length of a thread wrapped spirally around a cylinder 40 feet high and 2 feet in diameter, the thread passing around 10 times?

II.
1. 2π ft.$=ABCA$ (*Fig. 79*), the circumference of the cylinder.
2. 4 ft.$=40$ ft.$\div10=AF$, the distance between the spires.
3. $\sqrt{[(2\pi)^2+4^2]}=2\sqrt{[\pi^2+4]}$ ft.$=AEF$, the length of one spire.
4. $\therefore$ $10\times2\sqrt{[\pi^2+4]}$ft. $=20\sqrt{[\pi^2+4]}$ ft.$=74.4838$ ft., the entire length of the thread.

III. $\therefore$ The entire length of the thread=74.4838 ft.

Remark.—Each spire is equivalent to the hypotenuse of a right angled triangle whose base is $ABCA$ and altitude AF. This may be clearly shown by covering a cylinder with paper and tracing the position of the thread upon it. Then cut the paper along the line AFK and spread it upon a plane surface. AEF will then be seen to be the hypotenuse of a right-angled triangle whose base is $ACBA$ and altitude AF.

I. A thread passes spirally around a cylinder 10 feet high and 1 foot in diameter. How far will a mouse travel in unwinding the thread if the distance between the coils is 1 foot?

Construction.—Let $ACB—K$ be a portion of the cylinder and $ADEFGK$ a portioa of the thread. Let A be the position of the mouse when the unwinding begins, P its position at any time afterwards, APN a portion of the path it describes, and PD the portion of the thread unwound. Draw DC parallel to HB and draw OD and OC. Then

II.
1. $AB=2R=1$ foot, the diameter of the cylinder.
2. $a=10$ ft., the altitude. Let
3. $\theta=$the angle AOC,
4. $s=AN$, the length of a portion of the curve,
5. $x=OL$, and
6. $y=PL$. Then
7. $PC=$arc $AC=R\theta$,
8. $OM=R\cos\theta$,
9. $ML=IP=CP\cos\angle CPI$ $=R\theta\cos(\frac{1}{2}\pi-\angle PCI)$ $=R\theta\sin\angle PCI=R\theta\sin\theta$.
10. $x=OM+ML=R\cos\theta+R\theta\sin\theta$, and
11. $y=PL=IM=CM-CI$ $=R\sin\theta-CP\cos\theta=$ $R\sin\theta-R\theta\cos\theta$.
12. $dx=R\theta\cos\theta\, d\theta$, by differentiating in 10,
13. $dy=R\theta\sin\theta\, d\theta$, by differentiating in 11. Now
14. $s=\int\sqrt{[dx^2+dy^2]}$.
15. $\therefore s=\int[(R\theta\cos\theta\, d\theta)^2+(R\theta\sin\theta\, d\theta)^2]^{\frac{1}{2}}=R\int\theta d\theta$ $=\frac{1}{2}R\theta^2$. But
16. $\theta=2\pi$, when one spire is unwound, and
17. $\theta=10\times2\pi=20\pi$, when the unwinding is complete.
18. $\therefore s=\frac{1}{2}R\theta^2=\frac{1}{2}\times\frac{1}{2}(20\pi)^2=100\pi^2=989.96044$ ft., the distance the mouse travels to unwind the thread.

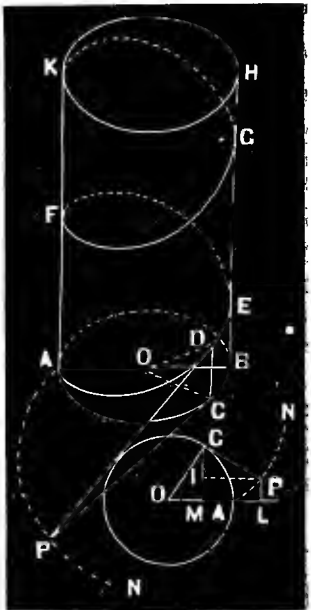


FIG. 79:

III. $\therefore$ The mouse will travel 989.96044 ft. to unwind the thread.

I. What is the length of a thread winding spirally round a cone, whose radius is R and altitude a, the thread passing round n times and intersecting the slant height at equal distances apart?

Let P be any point of the thread, $(x, y, z,)$ the co-ordinates of the point; and, let the angle $PFC'(=DOC)=\theta$, $BO=a$, the altitude, $DO=R$, the radius of the base of the cone, and $r=$the radius of the cone at the point P. Then the equations of the thread are: $x=r\cos\theta$...... (1), $y=r\sin\theta$..... (2), and $z=\frac{a}{2\pi n}\theta$....(3). From the similar triangles DEP and DOB, $r=\frac{R}{a}(a-z)=R\left(1-\frac{\theta}{2\pi n}\right)$...(4). Now the distance between P and its consecutive position is $\sqrt{(dz^2+dx^2+dy^2)}=$ $\sqrt{\left[1+\left(\frac{dx}{dz}\right)^2+\left(\frac{dy}{dz}\right)^2\right]}dz$. $\therefore s=\int\sqrt{\left[1+\left(\frac{dx}{dz}\right)^2\right.}$

$+\left(\frac{dy}{dz}\right)^2\Big]dz\ldots.(5)$. Substituting the value of r in (1) and (2), and differentiating, we have $dx=-\frac{R}{2\pi n}\Big[\cos\theta+(2\pi n-\theta)\sin\theta\Big]d\theta$ and $dy=-\frac{R}{2\pi n}\Big[\sin\theta-(2\pi n-\theta)\cos\theta\Big]d\theta$.

From (3), we have $dz=\frac{a}{2\pi n}d\theta$.

Substituting these values of dx, dy, and dz in (5), we have

$$s=\int_0^{2\pi n}\frac{a}{2\pi n}\sqrt{\left\{1+\frac{R^2[\cos\theta+(2\pi n-\theta)\sin\theta]^2}{a^2}+\frac{R^2[\sin\theta-(2\pi n-\theta)\cos\theta]^2}{a^2}\right\}}\,d\theta=$$

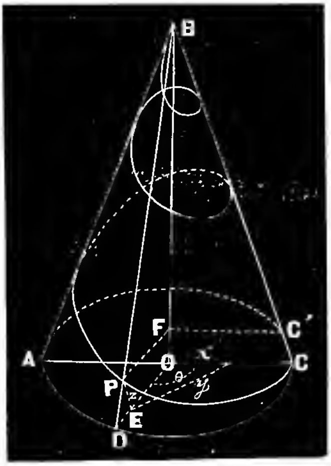


FIG. 80.

$$\int_0^{2\pi n}\frac{a}{2\pi n}\sqrt{\Big[1+\frac{R^2}{a^2}+\frac{R^2}{a^2}(2\pi n-\theta)^2\Big]}\,d\theta=\int_0^{2\pi n}\frac{1}{2\pi n}\sqrt{[a^2+R^2+R^2(2\pi n-\theta)^2]}\,d\theta=\frac{1}{2\pi n}\left\{-\frac{R(2\pi n-\theta)}{2}\sqrt{\Big[\frac{a^2+R^2}{R^2}+(2\pi n-\theta)^2\Big]}-R\left(\frac{a^2+R^2}{R^2}\right)\log_e\Big[(2\pi n-\theta)+\sqrt{\left((2\pi n-\theta)^2+\frac{a^2+R^2}{R^2}\right)}\Big]\right\}_0^{2\pi n}$$

$$=\tfrac{1}{2}\sqrt{(a^2+R^2+4\pi^2n^2R^2)}+\frac{a^2+R^2}{4\pi nR}\log_e\Big[\frac{2\pi nR+\sqrt{(a^2+R^2+4\pi^2n^2R^2)}}{\sqrt{(a^2+R^2)}}\Big]$$

$$=\tfrac{1}{2}\sqrt{(h^2+4\pi^2n^2R^2)}+\frac{h^2}{4\pi nR}\log_e\Big[\frac{2\pi nR+\sqrt{(h^2+4\pi^2n^2R^2)}}{h}\Big],$$

where $h=\sqrt{(a^2+R^2)}$, the slant height.

NOTE.—This solution was prepared for the *School Visitor*, by the author.

I. A thread makes n equidistant spiral turns around a cone whose slant height is h, and radius of the base r. The cone stands on a horizontal plane and the string is unwound with the lower end in contact with the plane, the part unwound being always tense. Find the length of the *trace* of the end of the string on the plane.

Let MH be the part unwound at any time, H being the point in contact with the cone, and $BM=u$, the trace on the plane up to this time. Put arc $BE=x$, $AH=y$, E being the point in the circumference of the base in the line AH. Let NI be the position of the string at the next instant, D and I being homologous points with E and H. Draw HK parallel to ED. Then h : DE : : AK : HK, or $\frac{HK}{ED}=\frac{AK}{h}$ .... (1). Now since the arc BE $=x$, is proportional to the distance the point of contact of the thread with the cone has ascended, $x:h-y::2\pi rn:h$, or $\frac{x}{2rn\pi}=\frac{h-y}{h}$. $\therefore \frac{dx}{dy}=-\frac{2\pi rn}{h}$ ..... (2). This is negative since y decreases as x increases. It is evident from the figure that $\frac{ED}{IK}=\frac{dx}{dy}=-\frac{2\pi rn}{h}$. By similar triangles, $IK:HK::HE:ME$, that is, form (1) and (2), we get $\frac{ME}{h-y}=\frac{HK}{IK}$ $=\frac{ED}{IK}\times\frac{y}{h}=\frac{dx}{dy}\times\frac{y}{h}=-\frac{2\pi rn}{h^2}y$ ..... (3). Therefore, $ME=-\frac{2\pi rn}{h^2}(h-y)y$ ..... (4). Put $ME=t$. Then $\frac{dt}{dy}=-\frac{2\pi rn}{h^2}(h-2y)$ .... (5). By similar figures $r:ME::ED:MP=\frac{ME\times ED}{r}=-ME\times\frac{2\pi rn}{h}\times IK$.

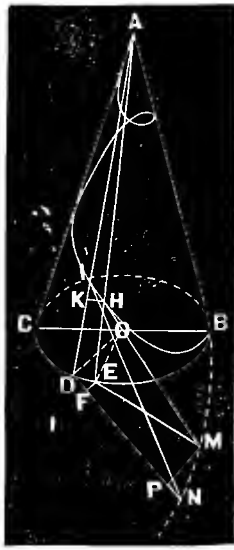


FIG. 81.

From (3), put $MP=v$, then $\frac{MP}{IK}=\frac{dv}{dy}=\frac{4\pi^2n^2r}{h^3}(h-y)y$ .... (6). Equation (5) gives the entire addition to the line ME which consists of $NP+FD$, since $PF=ME$. Consequently, NP $=\frac{dt}{dy}-\frac{dx}{dy}=-\frac{2\pi rn}{h^2}(h-2y)+\frac{2\pi rn}{h}=\frac{4\pi rn}{h^2}y$ .... (7). Now MN^2 $=MP^2+NP^2$ in the limit. Therefore $\left(\frac{du}{dy}\right)^2=\frac{16\pi^2n^2r^2y^2}{h^4}$ $\left(1+\frac{\pi^2n^2}{h^2}(h-y)^2\right)$.... (8). $\sqrt{(8)}=(9)$, $\frac{du}{dy}=\frac{4\pi rny}{h^2}\times$ $\sqrt{\left(1+\frac{n^2\pi^2}{h^2}(h-y)^2\right)}$, the integral of which is u, the length of the trace. Put $h-y=z$, and $\frac{h^2}{n^2\pi^2}=a^2$. Then $u=\frac{4r}{a^2h}\int_o^h(h-z)$ $\sqrt{(a^2+z^2)}dz$ .... (10). Or $u=\frac{4ar}{3h}+\left(\frac{2rh}{3a^2}-\frac{4r}{3h}\right)\sqrt{(a^2+h^2)}$ $+2r\log_e\left[\frac{h+\sqrt{(a^2+h^2)}}{a}\right]$ ..... (11). Write for h, its equal,

$an\pi$, in (11) and we have (12), $u=\frac{4r}{3n\pi}+\frac{2r}{3}\left(n\pi-\frac{2}{n\pi}\right)$ $\sqrt{(1+n^2\pi^2)}+2r\log_e[n\pi+\sqrt{(1+n^2\pi^2)}]$.

This result is independent of h, the cone's slant height, but involves n the number of turns of the thread.

NOTE.—This solution is by Prof. Henry Gunder and is taken from the *School Visitor*, Vol. 9, p. 199. Prof. Gunder stands in the very front rank of Ohio mathematicians. He has contributed some very fine solutions to difficult problems proposed in the *School Visitor* and the *Mathematical Messenger*. He is of a very retiring disposition and does not make any pretentions as a mathematician. But that he possesses superior ability along that line, his solutions to difficult problems will attest. Prof. Gunder was born at Arcanum, O, Sept. 15th, 1837. He passed his boyhood on a farm and it was while following a plow or chopping winter wood, that difficult problems were solved and hitherto unknown fields of thought explored. He became Principal of the Greenville High School in 1867. After seven years' work here, he became Superintendent of the Public Schools of North Manchester, Ind. After five years' work at this place he became Superintendent of schools of New Castle, Ind. In 1890, Prof. Gunder was elected professor of Pedagogy in the Findlay (Ohio) College.

I. A woman printed 10 lbs. of butter in the shape of a right cone whose base is 8 inches and altitude 10 inches. Having company for dinner, she cut off a piece parallel to the altitude and containing $\frac{1}{3}$ of the diameter. What was the weight of the part cut off?

Construction.—Let ABC—G be the cone, AC the diameter and OG the altitude. Let E be the point where the cutting plane intersected the the diameter, F the corresponding point in the slant height, and $DLFKB$ the section formed by the intersection of the cone and the cutting plane. Through F pass a plane parallel to the base ABC and anywhere between this plane and the base, pass a plane $NLMK$. Then,

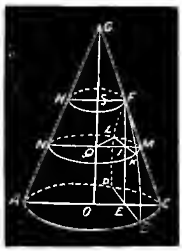


FIG. 82.

1. $AC=2R=8$ in., the diameter of the base,
2. $OG=a=10$ in., the altitude, and
3. $OE=OC-EC=R-\frac{1}{3}AC=R-\frac{2}{3}R=\frac{1}{3}R=1\frac{1}{3}$ in.$=c$, the distance of the cutting plane from the altitude. Let
4. $GQ=x$, the distance of the plane $MLNK$ from the vertex G. By similar triangles,
5. $OC:OG::EC:EF$, or $R:a::R-c:EF$. Whence,
6. $EF=\frac{a(R-c)}{R}=6\frac{2}{3}$ in. By similar triangles,
7. $GO:OC::GQ:QM$, or $a:R::x:QM$. Whence,
8. $QM=LQ=\frac{Rx}{a}$. Now,

9. *area* of LKM=*area* of $LQKM$—*area* of LKQ. But

10. *area* of $LQKM=2\left(\frac{1}{2}LQ^2\cos^{-1}\frac{QI}{LQ}\right)=\left(\frac{Rx}{a}\right)^2\times\cos^{-1}\left(\frac{ac}{Rx}\right)$, and

11. *area* of $LKQ=\frac{1}{2}(LK\times QI)=\frac{1}{2}(2LI\times c)=LI\times c=c\sqrt{(NI\times IM)}=c\sqrt{[(Rx\div a+c)\times(Rx\div a-c)]}=(c\div a)\sqrt{(R^2x^2-c^2a^2)}$.

12. ∴ *Area* of the segment $LKM=\frac{R^2x^2}{a^2}\cos^{-1}\left(\frac{ac}{Rx}\right)-(c\div a)\sqrt{(R^2x^2-c^2a^2)}$.

13. $\left(\frac{R^2x^2}{a^2}\cos^{-1}\left(\frac{ac}{Rx}\right)-\frac{c}{a}\sqrt{(R^2x^2-a^2c^2)}\right)dx$ = an element of volume of the part cut off.

II. 14. ∴ $V=\int_{\frac{ac}{R}}^{a}\left(\frac{R^2x^2}{a^2}\cos^{-1}\left(\frac{ac}{Rx}\right)-\frac{c}{a}\sqrt{(R^2x^2-a^2c^2}\right)dx$

$=\frac{1}{3}a\left\{R^2\cos^{-1}\left(\frac{c}{R}\right)-2c\sqrt{(R^2-c^2)}+\frac{c^3}{R}\times\log_e\left[\frac{R+\sqrt{(R^2-c^2}}{c}\right]\right\}=\frac{10}{3}\left\{4^2\cos^{-1}(\frac{1}{3})-2\times 1\frac{1}{3}\sqrt{[4^2-(1\frac{1}{3})^2]}+\frac{1}{27}\times4^2\log_e\left[\frac{4+\sqrt{[4^2-(1\frac{1}{3})^2]}}{1\frac{1}{3}}\right]\right\}$,

$=\frac{10}{3}\left\{4^2\cos^{-1}(\frac{1}{3})-\frac{64}{9}\sqrt{2}+\frac{16}{27}\log_e[2+\frac{2}{3}\sqrt{2}]\right\}$,

$=\frac{10}{3}\left\{4^2\times\frac{114257}{291600}\pi-\frac{64}{9}\sqrt{2}+\frac{16}{27}\log_e[2+\frac{2}{3}\sqrt{2}]\right\}$,

$=\frac{10}{3}\left\{19.6938154-10.0562976+.6396202\right\}=$ 34.223792 cu. in., the volume of the part cut off.

15. $\frac{1}{3}a\pi R^2=\frac{1}{3}\times10\times4^2\times\pi=53\frac{1}{3}\pi$ cu. in., the volume of the whole cone.

16. 10 lbs.=the weight of the whole cone. Hence, by proportion,

17. $53\frac{1}{3}\pi$ cu. in.:34.223792 cu. in.::10 lbs.:(?=2.04258 lbs.)

III. ∴ The weight of the part cut off is 2.04258 lbs.

I. After making a circular excavation 10 feet deep and 6 feet in diameter, it was found necessary to move the center 3 feet to one side; the new excavation being made in the form of a right cone having its base 6 feet in diameter and its apex in the surface of the ground· Reqired the total amount of earth removed.

Construction.—Let $ABC—F$ be the cylindrical excavation first made, AC the diameter, HO the altitude. Let A be the center of the conical excavation, GAH its diameter, and AF, an element of the cylinder, the altitude. Pass a plane at a distance x from O and parallel to the base of the excavation. Let figure II. represent the section thus formed, the letters in this section corresponding to the homologous points in the base represented by the same letters in the base of the excavation. An element of the earth removed in the conical excavation is $(area\ BAKGNB)dx$. The whole volume removed in the conical part of the excavation is

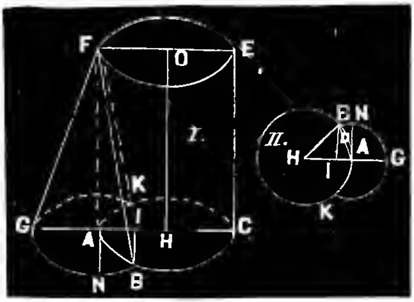


FIG. 88.

$$\int_0^a (area\ BAKGNB)dx.$$ For let

1. $HO=a=10$ ft., the altitude of the excavation,
2. $HA=r=3$ ft., the radius of the cylindrical and the conical parts.
3. $AB=AN=\frac{rx}{a}$. This is found from the proportion of similar triangles.
4. $BI^2=(rx\div a-AI)(rx\div a+AI)$. Also
5. $BI^2=(2r-AI)AI$.
6. $\therefore (2r-AI)AI=(rx\div a-AI)(rx\div a+AI)$. Whence,
7. $AI=rx^2\div 2a^2$,
8. $BI=\frac{rx}{2a^2}\sqrt{(4a^2-x^2)}$,
9. $HI=r-\frac{rx^2}{2a^2}=r(1-\frac{x^2}{2a^2})$. Now
10. *area* of $BDAKGNB=2($ *area* of $BDAN+$*area* of $NAG)$. But
11. $\frac{1}{4}\pi(r^2x^2\div a^2)=$ the *area* of the quadrant NAG, and
12. *area* of $BDAN=$*area* of sector $BAN+$ *area* of triangle $HBA-$ *area* of sector $BDAH$. Now
13. *area* of sector $BAN=\frac{1}{2}AB\times AB\sin^{-1}(AI\div AB)$
$=(r^2x^2\div 2a^2)\sin^{-1}\left(\frac{x}{2a}\right)$,
14. *area* of triangle $ABH=\frac{1}{2}(AH\times BI)=\frac{1}{2}r\times(rx\div 2a^2)\times\sqrt{(4a^2-x^2)}=(r^2x\div 4a^2)\sqrt{(4a^2-x^2)}$, and
15. *area* of sector $BDAH=\frac{1}{2}[AH\times AH\cos^{-1}(HI\div BH)]$
$=\frac{1}{2}r^2\times\cos^{-1}[1-(x\div 2a)]$.

II. 16. $\therefore$ *Area* of $BDAKGNB=2\left\{\frac{1}{4}\frac{r^2x^2}{a^2}\pi+\frac{r^2x^2}{2a^2}\sin^{-1}\left(\frac{x}{2a}\right)+\frac{r^2x}{4a^2}\sqrt{(4a^2-x^2)}-\frac{1}{2}r^2\cos^{-1}\left(1-\frac{x^2}{2a^2}\right)\right\}$

$=\frac{r^2x^2}{2a^2}\pi+\frac{r^2x^2}{a^2}\sin^{-1}\left(\frac{x}{2a}\right)+\frac{r^2x}{2a^2}\sqrt{(4a^2-x^2)}$

$-r^2\cos_{-1}\left(1-\frac{x^2}{2a^2}\right)$.

17. $\therefore V=\int_0^a\left\{\frac{r^2x^2}{2a^2}\pi+\frac{r^2x^2}{a^2}\sin^{-1}\left(\frac{x}{2a}\right)+\frac{r^2x}{2a^2}\sqrt{(4a^2-x^2)}-r^2\cos^{-1}\left(1-\frac{x^2}{2a^2}\right)\right\}dx=\frac{1}{6}\pi ar^2+\frac{r^2}{a^2}\int_0^a x^2\sin^{-1}\left(\frac{x}{2a}\right)dx+\frac{r^2}{2a^2}\int_0^a x^2\sqrt{(4a^2-x^2)}dx-r^2\int_0^a\cos^{-1}\left(1-\frac{x^2}{2a^2}\right)dx=\frac{1}{6}\pi ar^2+\frac{r^2}{a^2}\left[\frac{1}{3}x^3\sin^{-1}\frac{x}{2a}+\frac{1}{9}(x^2+8a^2)(4a^2-x^2)^{\frac{1}{2}}\right]_0^a-\frac{r^2}{2a^2}\left[\frac{1}{3}(4a^2-x^2)^{\frac{3}{2}}\right]_0^a-r^2\left[x\cos^{-1}\left(1-\frac{x^2}{2r^2}\right)+2(4a^2-x^2)^{\frac{1}{2}}\right]_0^a=\left(\frac{64-27\sqrt{3}-2\pi}{18}\right)ar^2$,

=the volume of the conical part of the excavation.

18. πar^2=the volume of the cylindrical part.

19. $\therefore\ \pi ar^2+\left(\frac{64-27\sqrt{3}-2\pi}{18}\right)ar^2=\left(\frac{64-27\sqrt{3}+16\pi}{18}\right)\times ar^2=337.500554$ cu. ft., the volume of the entire excavation.

III. $\therefore$ The volume of the excavation $=\left(\frac{64-27\sqrt{3}+16\pi}{18}\right)ar^2$,

or 337.50055+cu. ft., correct to the last decimal place.

NOTE.—This problem was proposed in the *School Visitor* by Wayland Dowling, Rome Center, Mich. A solution of the problem, by Henry Gunder, was published in Vol. 9, No. 6, p. 121. The solution there given is by polar coordinates. The editor gives the answers obtained by the contributors; viz., Mr. Dowling, H. A. Wood, R. A. Leisy, and William Hoover. Their answers differ from Mr. Gunder's and from each other. Mr. Gunder's answer is 337.5+cu. ft., the same as above. There is a similar problem in *Todhunter's Integral Calculus, p. 190, prob. 29.*

I. A tree 74 feet high, standing perpendicularly, on a hillside, was broken by the wind but not severed, and the top fell directly down the hill, striking the ground 34 feet from the root of the tree, the horizontal distance from the root to the broken part being 18 feet, find the height of the stub.

Construction.—Let AD be the hill-side, AB the stump, BD

the broken part, and AC the horizontal line from the root of the tree to the broken part. Produce AB to E and draw DE parallel to AC.

II.

1. Let $AB=x$, the height of the stump. Then
2. $BD=74$ ft.$-x=s-x$, the broken part, since $AB+BD=74$ feet.
3. Let $AD=a=34$ ft., the distance from the foot of the tree to where the top struck the ground,
4. $AC=b=18$ ft., the horizontal distance from the foot of the tree to the broken part.
5. $x=AB$, the height of the stump. Then
6. $BC=\sqrt{(AB^2+AC^2)}=\sqrt{(x^2+b^2)}$ . . (1). In the similar triangles BAC and BED,
7. $\sqrt{(x^2+b^2)} : x :: s-x : BE$. Whence,
8. $BE=\frac{x(s-x)}{\sqrt{(x^2+b^2)}}$ . . . .(2). Also,
9. $\sqrt{(x^2+b^2)} : b :: s-x : DE$. Whence
10. $DE=\frac{b(s-x)}{\sqrt{(x^2+b^2)}}$ . . . (3). Now
11. $AE=BE-BA=\frac{x(s-x)}{\sqrt{(x^2+b^2)}}-x$ . . . . . (4).
12. $AE^2+ED^2=AD^2$, or
13. $\left\{\frac{x(s-x)}{\sqrt{(x^2+b^2)}}-x\right\}^2+\left\{\frac{b(s-x)}{\sqrt{(x^2+b^2)}}\right\}^2=a^2$ . . . .(5). Developing (5), we have
14. $4(s^3-a^2+b^2)x^4-4s(s^2-a^2+2b^2)x^3+(s^4+a^4-2a^2s^2+8b^2s^2-4a^2b^2)x^2-4b^2s(s^2-a^2)x=-b^2(s^2-a^2)$ . . . . (6).
15. $1161x^4-91908x^3+1959876x^2-25894080x+377913600=0$ . . . . . (7), by substituting the values of a, b, and s in (6).
16. $\therefore x=24$ feet, the height of the stump, by solving (7) by Horner's method.

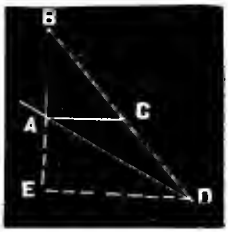


FIG. 84.

III. $\therefore$ The height of the stump is 24 feet.

NOTE.—This problem was taken from the *Mathematical Magazine*, Vol. I., No. 7, prob. 84. In Vol I., p. 184, of the *Mathematical Magazine* is a solution of it, given by C. H. Scharar and Prof. J. F. W. Sheffer The solution there given is different from the one above.

I. What is the longest strip of carpet one yard wide that can be laid diagonally in a room 30 feet long and 20 feet wide?

Construction.—Let $ABCD$ represent the room and $EFGH$ the strip of carpet one yard wide placed diagonally in the room.

II.

1. Let $AB=a=30$ ft., the length of the room,
2. $BC=b=20$ ft., the width, and
3. $HG=c=3$ ft., the width of the carpet. Let
4. $BF=HD=x$. Then
5. $FC=AH=20-x=b-x$.
6. $BE=\sqrt{(EF^2-BF^2)}=\sqrt{(9-x^2)}=\sqrt{(c^2-x^2)}$ . .(1).
7. $AE=GC=AB-EB=a-\sqrt{(c^2-x^2)}$ . . . . (2). By similar triangles,
8. $EF:BF::GF:GC$, or
9. $c:x::GF:a-\sqrt{(c^2-x^2.)}$ Whence,
10. $GF=\dfrac{c[a-\sqrt{(c^2-x^2)}]}{x}$ . . .3

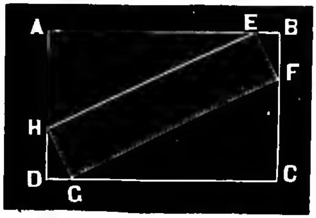


FIG. 85.

Again, we have

11. $EF:BE::GF:FC$, or
12. $c:\sqrt{(c^2-x^2)}::GF:b-x$.
13. $\therefore GF=\dfrac{c(b-x)}{\sqrt{(c^2-x^2)}}$ . . .(4). By equating GF in (3) and (4)
14. $\dfrac{c(b-x)}{\sqrt{(c^2-x^2)}}=\dfrac{c[a-\sqrt{(c^2-x^2)}]}{x}$ . . . (5).
15. $bx-x^2=a\sqrt{(c^2-x^2)}-c^2+x^2$ . . .(6), by dividing (5) by c and clearing of fractions.
16. $c^2-bx-2x^2=a\sqrt{(c^2-x^2)}$ . . .(7), by transposing in (6).
17. $4x^4-4bx^3+(a^2+b^2-4c^2)x^2+2bc^2x=c^2(a^2-c^2)$ . . . . . . (8), by squaring (7) and transposing and combining.
18. $4x^4-80x^3+1264x^2+360x=8019$ . . . , (9), by restoring numbers in (8).
19. $\therefore x=2.5571+$ft., by solving (9) by Horner's method.
20. $\therefore\sqrt{(c^2-x^2)}=\sqrt{(9-x^2)}=1.5689$ ft. Then,
21. $GC=30-\sqrt{(9-x^2)}=28.4311$ ft., and
22. $FC=20-x=17.4429$ ft.
23. $\therefore GF=\sqrt{(FC^2+GC^2)}=\sqrt{[(28.4311)^2+(17.4229)^2]}$ $=33.3554$ ft., the length of the carpet.

III. $\therefore$ The length of the strip of carpet is 33.3554 ft.

I. What length of rope, fastened to a point in the circumference of a circular field whose area is one acre, will allow a horse to graze upon just one acre outside the field?

Construction.—Let $ABPC$ be the circular field and P the point in the circumference to which the horse is fastened. Let BP represent the length of the required rope. Draw the radius BO of the field and the line BC. Then

1. 1 A.=160 sq. rd.=the area of the field $ABPC$, and
2. $BO=OP=R=\sqrt{(160\div\pi)}=4\sqrt{\left(\dfrac{10}{\pi}\right)}$, the radius of

the circular field. Let

3. θ=the angle BPO=the angle OBP. Hence,
4. $\pi-2\theta$=the angle BOP. Now
5. $BP=AP\cos\angle APB=2R\cos\theta$, the length of the required rope. The
6. *area* $BPCD$ over which the horse grazes=*area* $BECDB$—*area*$BECPB$. But
7. *area* of circle $BECD=\pi BP^2=\pi 4R^2\cos^2\theta=4\pi R^2\cos^2\theta$, and the
8. *area* $BECP=2\times($*area* of sector EPB+*area* of segment $BPH)$. Now

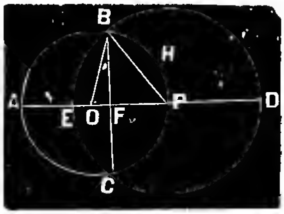


FIG. 86.

9. *area* of sector $EPB=\frac{1}{2}BP\times$ *arc* $BE=\frac{1}{2}\times 2R\cos\theta\times 2R\cos\theta\times\theta=2R^2\theta\cos^2\theta$, and
10. *area* of segment BPH=*area* of sector BOP—*area* of triangle $OBP=\frac{1}{2}BO\times arcBHP-\frac{1}{2}OP\times BF=\frac{1}{2}[R\times R(\pi-2\theta)]-\frac{1}{2}R\times R\sin(\pi-2\theta)$, $=\frac{1}{2}R^2(\pi-2\theta)-\frac{1}{2}R^2\sin 2\theta$.

II.

11. ∴ *Area* $BECP=2[2R^2\theta\cos^2\theta+\frac{1}{2}R^2(\pi-2\theta)-\frac{1}{2}R^2\sin 2\theta]=R^2[4\theta\cos^2\theta+\pi-2\theta-\sin 2\theta]=R^2[4\theta\left(\frac{1+\cos 2\theta}{2}\right)+\pi-2\theta-\sin 2\theta]=R^2[\pi+2\theta\cos 2\theta-\sin^2 2\theta]$.
12. ∴ *Area*$BPCDB=4\pi R^2\cos^2\theta-R^2[\pi+2\theta\cos 2\theta-\sin 2\theta]$. But
13. πR^2=1A.=160 sq. rd.=the area of $BPCDB$, by the conditions of the problem.
14. ∴ $4\pi R^2\cos^2\theta-R^2[\pi+2\theta\cos 2\theta-\sin 2\theta]=\pi R^2$. Whence,
15. $4\pi\left(\frac{1+\cos 2\theta}{2}\right)-[\pi+2\theta\cos 2\theta-\sin 2\theta]=\pi$, or
16. $2\pi+2\pi\cos 2\theta-\pi-2\theta\cos 2\theta+\sin 2\theta=\pi$.
17. ∴ $2\theta\cos 2\theta-\sin 2\theta=2\pi\cos 2\theta$, or
18. $2\theta-\tan 2\theta=2\pi$, by dividing by $\cos 2\theta$. Whence,
19. $\theta=51^\circ\ 16'\ 24''$, by solving the last equation by the method of Double Position.
20. ∴ $BP=2R\cos\theta=8\sqrt{\left(\frac{10}{\pi}\right)}\cos\theta$=8.92926+rods.

III. ∴ The length of the rope is 8.92926+rods.

I. If a 2-inch auger hole be bored diagonally through a 4-inch cube, what will be the volume bored out, the axis of the auger hole coinciding with the diagonal of the cube?

Formula.—$V=r^2\sqrt{3}(\pi e-2r\sqrt{2})$, where e is the edge.

Coustruction.—Let $AFGD$ be the cube and DF the diagonal, which is also the axis of the auger hole. The volume bored out will consist of two equal tetrahedrons $acd—D$ and $efg—F$ *plus* the cylinder $acd—f$, *minus* 6 cylindrical ungulas each equal to $ace—b$. Pass a plane any where between e and b, perpendicular to the axis of the cylinder, and let x be the distance the plane is from D. Now let

II.

1. $AB=e=4$ inches, the edge of the cube;
2. $DF=\sqrt{3}s=4\sqrt{3}$, the diagonal of the cube; and
3. $r=1$ inch, the radius of the auger, or the radius of the circle acd.
4. $ac=ad=dc=r\sqrt{3}=\sqrt{3}$,
5. $Dc=\frac{1}{2}r\sqrt{6}=\frac{1}{2}\sqrt{6}$, by the similar triangles dDc and HDc.
6. $\sqrt{(Dc^2-r^2)}=\sqrt{[(\frac{1}{2}r\sqrt{6})^2-r^2]}=\frac{1}{2}r\sqrt{2}=\frac{1}{2}\sqrt{2}$, the altitude of the tetrahedron $acd—D$.

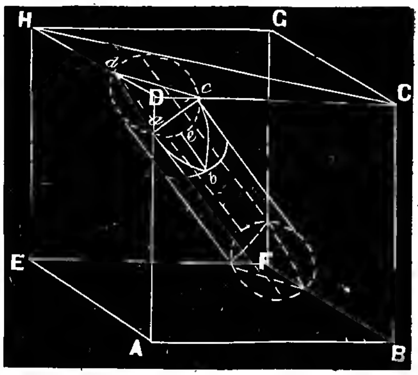


FIG. 87

7. $\therefore\ 2v=\frac{2}{3}(\textit{area}\text{ of base}\times\text{altitude})=2(\frac{1}{4}\sqrt{3}\times ac^2\times\frac{1}{3}\times\frac{1}{2}r\sqrt{2})=\frac{1}{4}\sqrt{6}r^3=\frac{1}{4}\sqrt{6}$, the volume of the two tetrahedrons,
8. $v'=\pi r^2\times(DF\div 2$ times the *altitude* of $acd—D)=\pi r^2(e\sqrt{3}-\frac{1}{2}r\sqrt{2})=\pi(4\sqrt{3}-\frac{1}{2}\sqrt{2})$, the volume of the cylinder $acd—f$.
9. $be=\frac{1}{2}r\sqrt{2}$, by similar triangles, not shown in the figure.
10. $\frac{1}{2}r\sqrt{2}+\frac{1}{2}r\sqrt{2}=r\sqrt{2}=$distance from D to where the auger begins to cut an entire circle.
11. $r-\frac{1}{2}x\sqrt{2}=$*versine* of an arc of the ungulas at a distance x from D.
12. $2r\cos^{-1}\left(\frac{\frac{1}{2}x\sqrt{2}}{r}\right)=$an *arc* of the ungulas at a distance x from D.
13. $r^2\cos^{-1}\left(\frac{\frac{1}{2}x\sqrt{2}}{r}\right)-\frac{1}{2}x\sqrt{2}(r^2-\frac{1}{2}x^2)^{\frac{1}{2}}=$the area of a segment at a distance x from D.
14. $\therefore\ 6v''=6\int_{\frac{1}{2}r\sqrt{2}}^{r\sqrt{2}}\left[r^2\cos^{-1}\left(\frac{\frac{1}{2}x\sqrt{2}}{r}\right)-\frac{1}{2}x\sqrt{2}(r^2-\frac{1}{2}x^2)^{\frac{1}{2}}\right]dx$

$$=6\left[r^2x\cos^{-1}\left(\frac{\frac{1}{2}\sqrt{2}x}{r}\right)-\sqrt{2}r^3\sqrt{\left(1-\frac{x^2}{2r^2}\right)}+\right.$$

$\frac{1}{3}\sqrt{2}(r^2-\frac{1}{2}x^2)^{\frac{3}{2}}\Big]_{\frac{1}{2}\sqrt{2r}}^{\sqrt{2r}}=6r^3(\frac{3}{8}\sqrt{6}-\frac{1}{6}\pi\sqrt{2})=$
$r^3(\frac{9}{4}\sqrt{6}-\pi\sqrt{2})$.

15. $\therefore$ V, the volume bored out, $=2v+v'-6v''=\frac{1}{4}\sqrt{6}r^3+\pi r^2(e\sqrt{3}-\frac{1}{2}r\sqrt{2})-r^3(\frac{9}{4}\sqrt{6}-\pi\sqrt{2})=r^2\sqrt{3}(\pi e-2r\sqrt{2})$ $=16.866105$ cu. in.

III. $\therefore$ The volume bored out is 16.866105 cu. in.

I. A horse is tethered to the outside of a circular corral. The length of the tether is equal to the circumference of the corral. Required the radius of the corral supposing the horse to have the liberty of grazing an acre of grass.

Coustruction.—Let $AEFBK$ be the circular corral, AB the diameter, and A the point where the horse is tethered. Suppose the horse winds the tether around the entire corral; he will then be at A. If he unwinds the tether, keeping it stretched, he will describe an involute, $APGH'$, to the corral. From H' to H, he will describe a semi-circle, radius $AH'=AH=$ to the circumference of the corral. From H through G to A, he will again describe an involute.

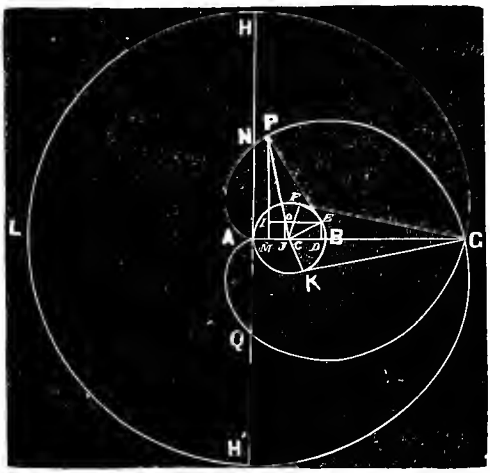


FIG. 88.

Then the area over which he grazes is the semi-circle $HLH'+$ the two equal involute areas $AFGHA$ and $AKGH'A+$the area $BFGKB$.

Let C be the center of the corral and also the origin of co-ordinates, AG the x–axis and P any point in the curve $APGH'$.

1. Let $\theta=$the angle ACE that the radius CE perpendicular to PE, the radius of curvature of the curve $APGH'$, makes with the x-axis,
2. $\theta_0=$the angle $AFEBK$ that the radius CK makes with the x–axis when the radius of curvature PE has moved to the position KG;
3. $R=AC$, the radius of the corral:
4. $\rho=PE=arc\ AFE=R\theta$, the radius of curvature of

the involute;
5. $x=CM$ and
6. $y=PM$, the co-ordinates of the point P; and
7. $x_0=CG$ and
8. $y_0=0$, the co-ordinates of the point G. Then we have
9. $x=CM=IE-CD=PE(=arc\ AFE)\cos\angle IEP$, $=\angle PEC-\angle OEC(=\angle ECD)$, $-CE\cos\angle EGD$ $=R\theta\cos(\angle IEP-\angle ECD)-R\cos(\pi-\theta)=R\theta\cos.$ $[\frac{1}{2}\pi-(\pi-\theta)]-R\cos(\pi-\theta)=R\theta\cos-(\frac{1}{2}\pi-\theta)$ $-R\cos(\pi-\theta)=R\theta\cos\theta+R\sin\theta$ .... (1).
10. $y=PM=PI+IM(=DE)=PE\sin\angle PEI+EC\times\sin\angle ECD=R\theta\sin(\theta-\frac{1}{2}\pi)+R\sin(\pi-\theta)=R\sin\theta-R\theta\sin\theta$ .... (2). When $\theta=\theta_0$=angle $AFEBK$
11. $x_0=CG=R\cos\theta_0+R\theta_0\sin\theta_0$ .... (3), and
12. $y_0=0=R\sin\theta_0-R\theta_0\cos\theta_0$ .... (4). Hence, from (4),
13. $\theta_0=R\sin\theta_0\div R\cos\theta_0=\tan\theta_0$ .... (5). Then, from (3),

II. 14. $x_0=R\cos\theta_0+R\tan\theta_0\sin\theta_0=R\left(\cos\theta_0+\frac{\sin\theta_0}{\cos\theta_0}\sin\theta_0\right)$ $=\frac{R}{\cos\theta_0}=R\sec\theta_0=R\sqrt{(1+\tan^2\theta_0)}=R\sqrt{(1+\theta_0{}^2)}$ .... (6). Now
15. $BFGKB=2[\frac{1}{2}KG\times KC$—sector $BCK]=R^2\theta_0-R^2(\theta_0-\pi)$ .... (7).
16. $AFGHA+AKGH'=2\int dA=2\int\frac{1}{2}\rho^2d\theta=\int_{\theta_0}^{2\pi}R^2\theta^2d\theta$ $=\frac{1}{3}R^2(8\pi^3-\theta_0^3)$ .... (8), and
17. $HH'L=\frac{1}{2}\pi(AH)^2=\frac{1}{2}\pi(2\pi R)^2=2\pi^3R^2$ .... (9). Adding (7), (8), and (9),
18. $R^2\theta_0-R^2(\theta_0-\pi)+\frac{1}{3}R^2(8\pi^3-\theta_0^3)+2\pi^3R^2=$ $R^2(\pi+\frac{14}{3}\pi^3-\frac{1}{3}\theta_0^3)$=area over which the horse grazes.
19. 1 A.=160 sq. rd.=43560 sq. ft.=the area over which the horse grazes.
20. $\therefore R^2(\pi+\frac{14}{3}\pi^3-\frac{1}{3}\theta_0^3)=43560$ sq. ft. Whence,
21. $R=\sqrt{\left(\frac{43560}{\pi+\frac{14}{3}\pi^3-\frac{1}{3}\theta_0^3}\right)}$ .... (10). But
22. $\theta_0=4.494039=264°\ 37'\ 18''.35$ by solving (5) by the method of Double Position.
23. $\therefore R=19.24738$ ft., by substituting the value of θ_0 in (10).

III. $\therefore$ The radius of the corral is 19.24738 ft.

A 20-foot pole stands plumb against a perpendicular wall. A cat starts to climb the pole, but for each foot it ascends the pole slides one foot from the wall; so that when the top of the pole is reached, the pole is on the ground at right angles to the wall. Required the equation to the curve the cat described and the distance through which it traveled.

Construction.=Let AC be the wall, P the position of the cat at any time, and BC the position of the ladder at the same time. Draw AP and to the middle point D of AP draw BD. Then $AB=PB$.

II. A.
1. Let $BC=20$ ft.$=a$, the length of the ladder,
2. $AP=r$, the radius vector of the curve the cat describes, and
3. $\theta=$the angle PAB.
4. $\pi-2\theta=$the angle ABP, because the angle $PAB=$the angle BPA.
5. $AB=BC\cos ABC=a\cos(\pi-2\theta)$ $=-a\cos 2\theta$,
6. $\frac{1}{2}AP=\frac{1}{2}r=AD=AB\cos\angle BAD$ $=-a\cos 2\theta\cos\theta$.
7. $\therefore r=-2a\cos 2\theta\cos\theta$, or
8. $r+2a\cos 2\theta\cos\theta=0$, the equation of the curve described by the cat.

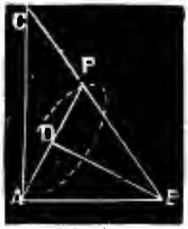


FIG. 89.

B.
1. Let $s=$the distance through which the cat traveled.
2. $s=\int\sqrt{(dr^2+r^2d\theta^2)}=2a\int_0^{\frac{1}{4}\pi}\sqrt{(1-12\cos^2\theta+44\cos^4\theta-32\cos^6\theta)}d\theta$,
3. $=-a\int_0^{\frac{1}{2}\pi}\sqrt{(2-4\cos\phi-\cos^2\phi+4\cos^3\phi)}d\phi$. where $\phi=\pi-2\theta$,
4. $=-\frac{1}{2}a\int_0^{\frac{1}{2}\pi}\sqrt{(6-4\cos\phi-2\cos 2\phi+4\cos 3\phi)}d\phi$ $=1.1193\,a$,
5. $=22.386$ ft., the distance through which the cat travels.

III. $\therefore$ $r+2a\cos 2\theta\cos\theta=0$, is the equation or the curve, and 22.386 ft.=the distance through which cat traveled.

NOTE.—The integration in this problem is performed by Cote's' Method of Approximation.

I. Suppose W. A. Snyder builds a coke oven on a circular bottom 10 feet in diameter. While building it, he keeps one end of a pole 10 feet long, always against the place he is working and the other end in that point of the circumference of the bottom opposite him. Required the capacity of the oven.

Construction.—Let AB be the diameter of the base and CG the altitude. At a distance x from the base pass a plane intersecting the oven in F and E. Draw AE and AC.

II.
1. $AB=2R=10$ feet, the diameter of the base.
2. $AC=AE=2R=10$ feet, by conditions of the problem
3. $CG=\sqrt{(AC^2-AG^2)}=(4R^2-R^2)=R\sqrt{3}$, the altitude.
4. $EH^2=x^2=(3AG+GH)(GB-GH)=3AG^2-2AG\times GH=3R^2-2R\times GH(=EI)$, because EH is the ordinate of a semi-circle whose diameter is $2AB$. From this, we find
5. $EI=\sqrt{(4R^2-x^2)}-R$. Then
6. $\pi EI^2=\pi[\sqrt{(4R^2-x^2)}-R]^2$, the area of the circle whose center is I.

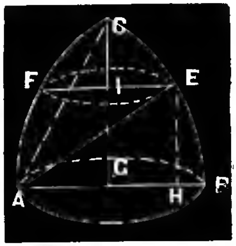


FIG 90.

7. $\therefore V=\int_0^{R\sqrt{3}}\pi[\sqrt{(4R^2-x^2)}-R]^2dx=\int_0^{R\sqrt{3}}[5R^2-x^2-2R\sqrt{(4R^2-x^2)}]dx$,
8. $=\left[5R^2x-\frac{1}{3}x^2-4R^3\sin^{-1}\frac{x}{2R}-Rx\sqrt{(4R^2-x^2)}\right]_0^{R\sqrt{3}}$,
9. $=\frac{1}{3}\pi R^3(9\sqrt{3}-4\pi)=\frac{1}{3}\pi 5^3(9\sqrt{3}-4\pi)$,
10. $=\frac{1}{3}\pi 125(9\sqrt{3}-4\pi)=395.590202+$ cu. ft.

III. $\therefore$ The capacity is 395.590202 cu. ft.

I. At each corner of a square field whose sides are 10 rods, a horse is tied with a rope 10 rods long; what is the area of the part common to the four horses?

Construction.—Let $ABCD$ be the field and $EFGH$ the area common to the four horses. Join EF, FG, GH, and EH. Draw DK perpendicular to EF and draw DE and DF. Since $AF=DE=DF=GB=CE$, the triangles ADF and EDC are equilateral and, consequently, the angle $ADE=\angle ADC-\angle EDC=90°-60°=30°$. Also the angle $FDC=30°$. Hence, $EDF=30°$. Now let

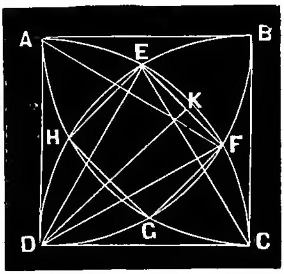


FIG. 91.

II.
1. $AD=ED=a=10$ rods. Then
2. Area of sector $EDF=\frac{1}{2}ED\times$ *arc* $EKF=\frac{1}{2}[a\times(2\pi a)\times\frac{30}{360}]=\frac{1}{12}\pi a^2$.
3. *Area* of triangle $EDF=\frac{1}{2}EF\times DK$. But,
4. $\frac{1}{2}EF=\frac{1}{2}\sqrt{a[2a-\sqrt{(4a^2-AF^2)}]}$, by formula of Prob. XXII., $=\frac{1}{2}a\sqrt{(2-\sqrt{3})}$, and
5. $DK=\sqrt{[DE^2-(\frac{1}{2}EF)^2]}=\frac{1}{2}a\sqrt{(2+\sqrt{3})}$. Hence,
6. *area* of triangle $EDF=\frac{1}{2}a\sqrt{(2+\sqrt{3})}\times\frac{1}{2}a\sqrt{(2-\sqrt{3})}=\frac{1}{4}a^2$.

7. $\therefore$ *Area* of segment $EF=\frac{1}{12}\pi a^2-\frac{1}{4}a^2=\frac{1}{12}a^2(\pi-3)$. The
8. *area* of square $EFGH=EF^2=a^2(2-\sqrt{3})$. Hence,
9. *area* of the figure $EFGH=a^2(2-\sqrt{3})+4\times\frac{1}{12}a^2(\pi-3)$ $=a^2(\frac{1}{3}\pi+1-\sqrt{3})=31.5147$ sq. rd.=the area common to the four horses.

III. $\therefore$ The area of the part common to the four horses is 31.5147 sq. rd.

Note.—This problem is similar to problem 348, *School Visitor*, to which a fine trigonometrical solution is given by Prof. E. B. Seitz.

I. What is the length of the longest straight, inflexible stick of wood that can be thrust up a chimney, the arch being 4 feet high and 2 feet from the arch to the back of the chimney—the back of the chimney being perpendicular?

Construction.—Let $PDEC$ be a vertical section of the chimney, PB the height of the arch, PE the distance from the arch to the back of chimney, and APD the longest stick of wood that can be thrust up the chimney.

FIG. 92.

II.

1. Let $PB=a=4$ feet, the height of the arch,
2. $PE=b=2$ feet, the width of the chimney,
3. x=the length of the longest stick of wood, and
4. θ=the angle DAC. Then
5. $AP=PB\ \text{cosec}\,\theta=a\ \text{cosec}\,\theta$,
6. $PD=PE\sec\theta=b\sec\theta$.
7. $\therefore x=AP+PD=a\ \text{cosec}\,\theta+b\sec\theta$ . . . . . (1). Differentiating (1),
8. $0=-a\cos\theta\div\sin^2\theta+b\sin\theta\div\cos^2\theta$ . . . . (2), or
9. $a\cos^3\theta=b\sin^3\theta$ . . . . (3), by clearing of fractions and transposing in (2).
10. $\therefore \dfrac{\sin^3\theta}{\cos^3\theta}=\tan^3\theta=\dfrac{a}{b}$. Whence,
11. $\tan\theta=\sqrt[3]{\dfrac{a}{b}}$. From (3), we may also have
12. $\cot\theta=\sqrt[3]{\dfrac{b}{a}}$. Now, from trigonometry,
13. $\sqrt{(1+\tan^2\theta)}=\sec\theta$, and
14. $\sqrt{(1+\cot^2\theta)}=\text{cosec}\,\theta$. Hence, by substituting in (1),
15. $x=a\sqrt{(1+\cot^2\theta)}+b\sqrt{(1+\tan^2\theta)}=$

$$a\sqrt{\left[1+\left(\frac{b}{a}\right)^{\frac{2}{3}}\right]}+b\sqrt{\left[1+\left(\frac{a}{b}\right)^{\frac{2}{3}}\right]}=a^{\frac{2}{3}}\sqrt{(a^{\frac{2}{3}}+b^{\frac{2}{3}})}$$
$$+b^{\frac{2}{3}}\sqrt{(a^{\frac{2}{3}}+b^{\frac{2}{3}})}=(a^{\frac{2}{3}}+b^{\frac{2}{3}})(\sqrt{a^{\frac{2}{3}}+b^{\frac{2}{3}}})=(a^{\frac{2}{3}}+b^{\frac{2}{3}})^{\frac{3}{2}},$$
$$=\sqrt{[(a^{\frac{2}{3}}+b^{\frac{2}{3}})^3]}=\sqrt{[(4^{\frac{2}{3}}+2^{\frac{2}{3}})^3]}=8.323876+\text{ft.}$$

III. $\therefore$ The length of the longest stick is 8.323876+ft.

I. A small garden, situated in a level plane is surrounded by a wall having twelve equal sides, in the center of which are twelve gates. Through these and from the center of the garden 12 paths lead off through the plane in a straight direction. From a point in the path leading north and at a distance of 4 furlongs 47$\frac{19}{243}$ yards from the center of the garden, A. and B. start to travel in opposite directions and at the same rate. A. continues in the direction he first takes; B., after arriving at the first road (lying east of him) by a straight line and at right angles with it, turns so as to arrive at the next path by a straight line and at right angles with it and so on in like manner until he arrives at the same road from which he started, having made a complete revolution around the center of the garden. At the moment that B. has performed the revolution, how far will A. and B. be apart?

Let O be the center of the garden, A, the point in the path leading north from which A. and B. start, $C, D, E, F, G, H, I, K, L, M, N, P$, the points at which B. strikes the paths. The triangles OCA, ODC, OED, OFE, &c., are right triangles, OCA, ODC, OED, OEF, &c., being the right angles. Let S in the prolongation of AC denote the position of A., when B., arrives at P. It is required to find the distance AS. Let $OA=a=4$ furlongs, 47$\frac{19}{243}$ yd., $PS=AC+CD+DE+\ldots+NP=x$, $AS=y$, $AP=z$, $n=12$, the number of paths and $\angle AOC=\angle COD=\angle DOE=\ldots\angle NOP=360^\circ\div n=30^\circ$. Then from the right triangles we have $OC=OA\times$

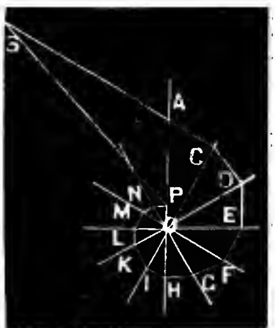


FIG. 93.

$\cos AOC=a\cos\theta$, $OD=OC\cos COD=a\cos^2\theta$, $OE=OD\times\cos DOE=a\cos^3\theta$, $OP=ON\cos NOP=a\cos^n\theta$; $AC=OA\times\sin AOC=a\sin\theta$, $CD=OC\sin COD=a\sin\theta\cos\theta$, $DE=DO\sin DOE=a\sin\theta\cos^2\theta$, $NP=NO\sin NOP=a\sin\theta\cos^{n-1}\theta$. $\therefore z=OA-OP=a(1-\cos^n\theta)$, and $x=a\sin\theta+a\sin\theta\cos\theta+a\sin\theta\cos^2\theta+\ldots\ldots+a\sin\theta\cos^{n-1}\theta=a\sin\theta(1+\cos\theta+\cos^2\theta+\cos^3\theta+\ldots\ldots+\cos^{n-1}\theta)=a\sin\theta(1-\cos^n\theta)\div(1-\cos\theta)=a\cot\frac{1}{2}\theta(1-\cos^n\theta)$. Hence, since $\angle PAS=(90^\circ+\theta)$, we have $y=\sqrt{[x^2+z^2-2xz\times\cos(90^\circ+\theta)]}=a\operatorname{cosec}\frac{1}{2}\theta(1-\cos^n\theta)\times\sqrt{(1+\sin^2\theta)}=\frac{225280}{243}\times\frac{4}{\sqrt{6}-\sqrt{3}}[1-(\frac{3}{4})^6]\times\frac{\sqrt{5}}{2}=3292$ yd., nearly.

NOTE.—This problem was proposed in the *School Visitor*, by Dr. N. R. Oliver, Brampton, Ontario. The above elegant solution was given by Prof. E. B. Seitz, and was published in the *School Visitor, Vol. 3, p. 36.*

I. A fox is 80 rods north of a hound and runs directly east 350 rods before being overtaken. How far will the hound run before catching the fox if he runs towards the fox all the time, and upon a level plain?

Construction.—Let C and A be the position of the hound and fox at the start, P and m corresponding positions of the hound and fox any time during the chase, and P' and n their positions the next instant, B the point where the hound catches the fox and $CPP'B$ the curve described by the hound. Join m and P, and n and P'; they are tangents to the curve at P and P'. Draw Pd and $P'e$ perpendicular to AB, mo perpendicular to $P'n$, and $P'p$ perpendicular to Pd.

II.

1. Let $AC=a=80$ rds.
2. $AB=b=350$ rds.,
3. $Am=x$,
4. $Bd=y$,
5. $Pm=w$,
6. arc $CP=s$,
7. curve $CPB=s_1$, and

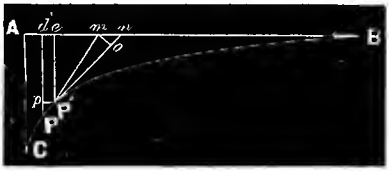


FIG 94.

8. r=ratio of the hound's rate to the fox's. Then we have
9. $mn=dx$,
10. $ed=P'p=dy$,
11. $PP'=ds$,
12. $no-PP'=dw$ .... (1), and
13. $s=rx$ .... (2). From (2), we have, by differentiation,
14. $ds=rdx$. Whence,
15. $\frac{dx}{ds}=\frac{1}{r}$. From the similar right triangles PpP' and mon, we have
16. $PP':mn::pP':mo$, or $ds:dx::dy:mo$. Whence,
17. $mo=\frac{dx\times dy}{ds}=\frac{dy}{r}$, since $\frac{dx}{ds}=\frac{1}{r}$. Substituting in (1),
18. $\frac{dy}{r}-ds=dw$, or
19. $dy-r^2dx=rdw$ .... (3). Integrating (3),
20. $y-r^2x=rw+C$ ....(4). But, since when $x=0$, $y=0$, and $w=a$,
21. $0=ra+C$. Whence, $C=-ra$.
22. $\therefore y-r^2x=rw+C=rw-ra$ .... (5). When $x=b$, $y=b$, and $w=0$, and (5) becomes
23. $b-r^2b=-ar$, or $r^2b-ra=b$. Whence,
24. $r^2-\frac{a}{b}r=1$,
25. $r^2-\frac{a}{b}r+\frac{a^2}{4b^2}=1+\frac{a^2}{4b^2}=\frac{a^2+4b^2}{4b^2}$

26. $r-\frac{a}{2b}=\frac{1}{2b}\sqrt{a^2+4b^2}$,
27. $2rb-a=\sqrt{a^2+4b^2}$,
28. $rb=\frac{1}{2}(a+\sqrt{a^2+4b^2})$. But
29. $rb=s_1$, what (1) becomes when $b=x$.
30. $\therefore s_1=\frac{1}{2}(a+\sqrt{a^2+4b^2})=392.2783$ rods, the distance the hound runs to catch the fox.

NOTE.—This solution is substantially the same as the one given by the Late Professor E. B. Seitz, and published in the *School Visitor*, *Vol. IV*, *p. 207*. The path of the hound is known as the "Curve of Pursuit."

I. A ship starts on the equator and travels due north-east at all times; how far has it traveled when its longitude, for the first time, is the same as that of the point of departure?

Let B be the point of the ship's departure, BPN its course, P its position at any time and N its position at the next instant. Then PN is an element of the curve of the ship, which is known as the *Loxodrome*, or *Rhumb* line. Let $\theta=BF=$the longitude of the point P, $\phi=PF=$the corresponding latitude, (x, y, z) the rectangular co-ordinates of P, and $\varphi=\frac{1}{4}\pi=$the constant angle PNQ.

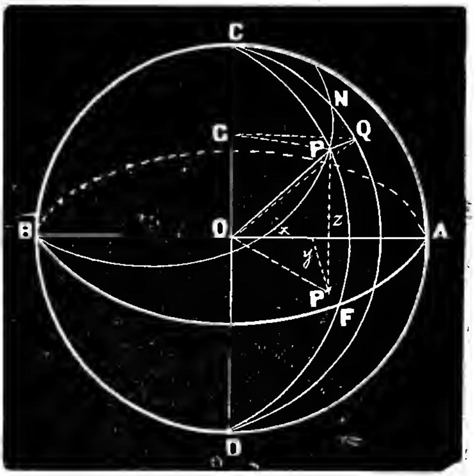


FIG. 95.

Then we have for the equations of the curve, $x=PG\cos\theta$ $=r\cos\phi\cos\theta$ .... (1), $y=PG\sin\theta=r\cos\phi\times\sin\theta$ .... (2), and $z=r\sin\phi$....(3), where r is the radius of the earth. Now an element of a curve of double curvature, referred to rectangular co-ordinates is $\sqrt{(dz^2+dy^2+dx^2)}$. $\therefore PN=ds$ $=\sqrt{(dz^2+dy^2+dx^2)}$ .....(4). Differentiating (1), (2), and (3), $dx=-r(\cos\theta\sin\phi d\phi+\cos\phi\sin\theta d\theta)$, $dy=-r(\sin\theta\sin\phi d\phi-\cos\phi\cos\theta d\theta)$, and $dz=r\cos\phi d\phi$. Substituting these values in (4), $ds=r\sqrt{[\cos^2\phi d\phi^2+(\sin\theta\sin\phi d\phi-\cos\phi\cos\theta d\theta)^2+(\cos\theta\sin\phi d\phi+\cos\phi\sin\theta d\theta)^2]}=r\sqrt{(\cos^2\phi d\phi^2+\sin^2\phi d\phi^2+\cos^2\phi d\theta^2)}=r\sqrt{(d\phi^2+\cos^2\phi d\theta^2)}$ ... (4). Now $PQ=GP\times PQ=r\cos\phi d\theta$ and $NQ=rd\phi$. $\frac{PQ}{NQ}=\tan\angle PNQ$

$=\tan\varphi$. $\therefore \frac{PQ}{NQ}=\frac{\cos\phi d\theta}{d\phi}=\tan\varphi$, or $\cos\phi d\theta=\tan\varphi d\phi$ .... (6). Substituting the value of $\cos\phi d\theta$ in (5), $ds=r\sqrt{(d\phi^2+\tan^2\varphi d\phi^2)}$ $=r\sqrt{(1+\tan^2\varphi)}d\phi=\frac{r}{\cos\phi}d\phi$. $\therefore s=\frac{r}{\cos\varphi}\int_{\phi_2}^{\phi_1}d\phi=$ $\frac{r}{\cos\varphi}(\phi_1-\phi_2)$ ....(7). By integrating (6), $\theta=\tan\varphi\int\frac{d\phi}{\cos\phi}$ $=\tan\varphi\log_e[\tan(\frac{1}{4}\pi+\frac{1}{2}\phi)]$ or $e^{\theta\cot\varphi}=\tan(\frac{1}{4}\pi+\frac{1}{2}\phi)$ ... (8). Whence, $\phi=2\tan^{-1}(e^{\theta\cot\varphi})-\frac{1}{2}\pi$. When $\theta=2\pi$ and $\varphi=\frac{1}{4}\pi$, $\phi=2\tan^{-1}(e^{2\pi})-\frac{1}{2}\pi=89^\circ\ 47'\ 9''.6=.4988\frac{1}{9}\pi$. $\therefore s=\frac{r}{\cos\frac{1}{4}\pi}\times$ $(.4988\frac{1}{9}\pi-0)=r\sqrt{2}(.4988\frac{1}{9}\pi)=2.21615937r=8775.991093$—mi., the distance the ship travels.

The rectangular equations of the Loxodrome are $\sqrt{(x^2+y^2)}\left\{e^{a\tan^{-1}\frac{y}{x}}+e^{-a\tan^{-1}\frac{y}{x}}\right\}=2r$, and $x^2+y^2+z^2=r^2$, where $a=\cot\varphi$.

The last equations are easily obtained from the figure. The **first** is obtained as follows: From (1) and (2), we find $\theta=\tan^{-1}\frac{y}{x}$; also, $x^2+y^2=r^2\cos^2\phi$ or $\cos\phi=\frac{1}{r}\sqrt{(x^2+y^2)}$. From (8), we get $e^{\theta\cot\varphi}=\frac{\cos\phi}{1+\sin\phi}=\frac{\cos\phi}{1+\sqrt{(1-\cos^2\phi)}}$. Whence, $e^{\theta\cot\varphi}+e^{\theta\cot\varphi}\times\sqrt{(1-\cos^2\phi)}=\cos\phi$. Transposing $e^{\theta\cot\varphi}$, squaring, and reducing, we have $\cos\phi(e^{\theta\cot\varphi}+e^{-\theta\cot\varphi})=2$. Substituting the value $\cos\phi$, and θ, we have $\sqrt{(x^2+y^2)}\left\{e^{a\tan^{-1}\frac{y}{x}}+e^{-a\tan^{-1}\frac{y}{x}}\right\}=2r$.

NOTE.—This solution was prepared by the author for problem 1501, *School Visitor*, but it was not published because of its difficult composition.

CHAPTER XXII.

EXAMINATION TESTS WITH ANSWERS.

For the benefit of students preparing for county or state examinations, we write out the answers to the following questions as a specimen of how the *examination paper* ought to be prepared:

ARITHMETIC.

1. How do you divide one fraction by another? Why is the fraction thus divided?

2. Divide four million and four millionths by one ten-thousandth. Write the answer in figures and words.

3. If a liter of air weighs 1.273 gr., what is the weight in kilos., if the air is in a room which contains 78 cu. m.?

4. The base of a cylinder is 12 inches in diameter and its altitude is 25 inches. Required the solid contents.

5. The edge of a cube is 6 inches; what is the length of the diagonal of the cube?

6. A broker bought stock at 4% discount, and sold it at 5% premium, and gained $450. How many shares did he purchase?

7. A ships 500 tons of cheese, to be sold at 9½ cents a lb. He pays his agent 3% for selling; the proceeds are to be invested in sugar, after a commission of 2% is deducted for buying. Required the entire commission.

8. Upon what value are dividends declared? Brokerage estimated? Usual rate of brokerage?

9. What is the face of a note dated July 5, 1881, and payable in 4 months to produce $811, when discounted at 9%?

10. Upon what principle is the United States rule for partial payments based? The Mercantile rule? How does compound interest differ from annual interest? *Ohio State List, 1884.*

ANSWERS.

No.	SUBJECT: *Arithmetic.* *Name of Applicant*	Grade.
1	(*a*) Invert the terms of the divisor and then multiply the numerators of the fractions together for the numerator of the quotient and the denominators together for the denominator of the quotient. (*b*) The fraction is thus divided because inverting the terms of the divisor gives the number of times the divisor is contained in 1, as is shown by analysis. The number of times then it is contained in any other number is obtained by multiplying this number by the number of times the divisor is contained in 1.	
	1. Four million and four millionths=4000000.000004= $\frac{4000000000004}{1000000}$.	

2 II. 2. One ten-thousandth $= .0001 = \frac{1}{10000}$

3. $\frac{40000000000004}{1000000} \div \frac{1}{10000} = \frac{40000000000004}{1000000} \times \frac{10000}{1} = \frac{40000000000004}{100} = 400000000000\frac{4}{100} = 400000000000.04 =$

III. ∴ The quotient is four hundred billion and four hundredths.

3 II.
1. 1 cu. m. = 1000 l.
2. 78 cu. m. = 78 × 1000 l. = 78000 l.
3. 1.273 g. = the weight of 1 l. of air, and
4. 99294 g. = 78000 × 1.273 g. = weight of 78000 l.
5. 1000 g. = 1 kilo.
6. 99294 g. = 99294 g. ÷ 1000 = 99.294 kilos.

III. ∴ The weight of 78 cu. m. of air weighs 99.294 kilos.

4 II.
1. 12 in. = the diameter of the cylinder, and
2. 25 in. = the altitude. Then
3. $\frac{1}{4}\pi 12^2$ sq. in. $= 36\pi$ sq. in., the area of the base of the cylinder.
4. $25 \times 36\pi$ cu. in. $= 900\pi$ cu. in. $= 900 \times 3.141592 \times 1$ cu. in. = 2827.4328 cu. in., the volume of the cylinder.

III. ∴ The volume of the cylinder is 2827.4328 cu. in.

5 II.
1. 6 in. = the length of the edge of the cube.
2. 36 sq. in. + 36 sq. in. = 72 sq. in. = the area of the square described on the diagonal of one of the equal faces, which is the sum of the areas of the squares described on two equal edges.
3. 72 sq. in. + 36 sq. in. = 108 sq. in. = area of square described on the diagonal of the cube, which equals the sum of the areas described on the three edges.
4. $6\sqrt{3}$ in. $= \sqrt{108} \times 1$ in. $= 10.392+$ in., the length of the diagonal of cube.

III. ∴ $6\sqrt{3}$ in. $= 10.392+$ in. = length of diagonal of cube.

6 II.
1. 100% = par value of stock.
2. 4% = discount.
3. 96% = 100% − 4% = market value, or cost of stock.
4. 5% = premium.
5. 105% = 100% + 5% = selling price of stock.
6. 9% = 105% − 96% = gain.
7. $450 = gain.
8. ∴ 9% = $450.
9. 1% = $50, and
10. 100% = $5000 = par value of stock.
11. $100 = par value of one share, usually. Then
12. $5000 = par value of $5000 ÷ $100, or 5 shares.

III. ∴ He purchased 5 shares.

1. $9\frac{1}{2}$ cents = selling price of one lb.
2. $47.50 = 500 × $0.09½ = selling price of 500 lb.
3.
   1. 100% = $47.50.
   2. 1% = $0.475.
   3. 2% = 2 × $0.475 = $0.95 = commission for selling

7

II.

the cheese.

4. $47.50—$0 95=$46.55=proceeds, or the amount to be invested in sugar.
5. 100%=cost of sugar.
6. 3%=commission on sugar.
7. 103%=total cost of sugar.
8. $46.55=total cost of sugar.
9.
 1. ∴ 103%=$46.55.
 2. 1%=$\frac{1}{103}$ of $46.55=$0.45.
 3. 100%=100×$0.45=$45=cost of sugar.
 4. 2%=2×$0.45=$0.90=commission on sugar.
10. ∴ $0.95+$0.90=$1.85=total commission.

III. ∴ $1.85=entire commission.

8

(*a*) Dividends are declared upon the par value.

(*b*) Brokerage is reckoned upon the selling price or purchasing price of bonds in *Commission* and *Brokerage*, but in *Stock Investments* it is reckoned on the par value.

(*c*) The usual rate of brokerage is $\frac{1}{8}$% on the par value of the stock, either for a purchase or a sale.

9

II.

1. 100%=face of note.
2. $3\frac{3}{20}$%=discount for 126 da.
3. $96\frac{17}{20}$%=proceeds.
4. $811=proceeds.
5. ∴ $96\frac{17}{20}$%=$811,
6. 1%=$8.37372, and
7. 100%=$837.372, the face of the note.

1881—	7—	5	when dated.
	4	3	
1881—	11—	5-8	when due.

III. ∴ The face of the note must be $837.372.

10

(*a*) Upon the principle *that payments be applied first to the discharge of interest due, the balance, if any, toward paying the principal and interest. Interest or payment must in no case draw interest.*

(*b*) Upon the principle that partial payments shall draw interest from time of payment until date of settlement.

(*c*) Compound Interest increases in a geometrical ratio, and Annual Interest in an arithmetical ratio.

SECOND LIST.

1. A and B together have $9,500. Two-thirds of A's money equals $\frac{3}{5}$ of B's. How much money has each?

2. A owes a sum equal to $\frac{3}{4}$ of his yearly income. By saving $\frac{6}{19}$ of his income annually for 5 years, he can pay his debt and have $1,200 left. What is his yearly income?

3. Smith and Jones can do a piece of work in 12 days. If Smith can do only $\frac{3}{4}$ as much as Jones, how long will it take each of them to do the work?

4. I am offered 6% stock at 84, and 5% stock at 72. Which investment is preferable, and how much?

5. If in selling cloth $\frac{3}{4}$ of the gain is equal to $\frac{3}{16}$ of the selling price, for how much will $3\frac{1}{8}$ yards sell that cost $5 per yard?

6. The frustum of a cone is 10 feet in diameter at the bottom and 8 feet at the top, with a slant height of 12 feet. What is the height of the cone from which the frustum is cut?

7. A, B and C ate eight pies. If they ate equal shares and A and B furnish the pies, A furnishing 5 and B, 3, and C pays 16 cents for his share, how should A and B divide the money?

8. Which is the heavier, and how much, an ounce of lead or an ounce of gold? *Pickaway County List, 1899.*

ANSWERS.

SUBJECT: *Arithmetic.*

No. | .... (Name of Applicant.) | Grade.

No. 1

II.
1. ∵ $\frac{2}{3}$ of A's money=$\frac{3}{5}$ of B's money,
2. $\frac{1}{3}$ of A's money=$\frac{1}{2}$ of $\frac{3}{5}$ of B's money=$\frac{3}{10}$ of B's money, and
3. $\frac{3}{3}$ of A's money, or A's money,=3 times $\frac{3}{10}$ of B's money =$\frac{9}{10}$ of B's money.
4. Let $\frac{10}{10}$=B's money. Then
5. $\frac{9}{10}$=A's money.
6. $\frac{10}{10}+\frac{9}{10}=\frac{19}{10}$=the sum of A's and B's money.
7. $9500=the sum of A's and B's money.
8. ∴ $\frac{19}{10}$=$9500.
9. $\frac{1}{10}=\frac{1}{19}$ of $9500=$500,
10. $\frac{9}{10}$=9 times $500=$4500=A's money, and
11. $\frac{10}{10}$=10 times $500=$5000=B's money.

III. ∴ { $4500=A's money, and
$5000=B's money.

No. 2

II.
1. Let $\frac{4}{4}$=A's yearly income,
2. $\frac{3}{4}$=his debt.
3. $\frac{6}{19}$=the amount of his income he is to save for 5 years.
4. ∴ $\frac{30}{19}$=the amount he is to save in 5 years.
5. ∴ $\frac{30}{19}$—$1200=his debt.
6. ∴ $\frac{30}{19}$—$1200=$\frac{3}{4}$ of his yearly income.
7. ∴ $\frac{30}{19}-\frac{3}{4}=\frac{63}{76}$=$1200.
8. $\frac{1}{76}$ of his income=$\frac{1}{63}$ of $1200=$19$\frac{1}{21}$.
9. $\frac{76}{76}$, or his annual income,=76×19\frac{1}{21}$=$1447$\frac{13}{21}$.

III. ∴ 1447\frac{13}{21}$=A's annual income.

No. 3

II.
1. Let $\frac{4}{4}$=part of the work Jones does in 1 day. Then
2. $\frac{3}{4}$=part Smith does in 1 day.
3. $\frac{7}{4}$=part they both do in 1 day.
4. $\frac{1}{12}$=part they both do in 1 day, since they can do the work in 12 days.
5. ∴ $\frac{7}{4}=\frac{1}{12}$ of the work,
6. $\frac{1}{4}=\frac{1}{7}$ of $\frac{1}{12}=\frac{1}{84}$ of the work,
7. $\frac{3}{4}=3\times\frac{1}{84}=\frac{1}{28}$=part Smith does in 1 day,
8. $\frac{4}{4}=4\times\frac{1}{84}=\frac{1}{21}$=part Jones does in 1 day.
9. $\frac{28}{28}$=part Smith can do in $\frac{28}{28}\div\frac{1}{28}$, or 28 days.
10. $\frac{21}{21}$=part Jones can do in $\frac{21}{21}\div\frac{1}{21}$, or 21 days.

III. ∴ Smith can do the work in 28 days and Jones in 21 days.

1.
1. 100%=amount invested in the 6% stock.
2. 100%=par value of the 6% stock.
3. 84%=market value of the 6% stock.
4. ∴ 84% of par value=100% of the investment,

Answers — Concluded.

4	II. … 5. 1% of par value $=1\frac{4}{21}$% of the investment, 6. 6% of the par value $=7\frac{1}{7}$% of the investment = the income on the investment. 2. 1. 100% — amount invested in 5% stock. 2. 100% = par value of the 5% stock. 3. 72% = market value of 5% stock. 4. ∴ 72% of par value = 100% of the investment, 5. 1% of par value $=1\frac{7}{18}$% of the investment, 6. 5% of the par value $=6\frac{17}{18}$% of the investment. 3. $7\frac{1}{7}$% — $6\frac{17}{18}$% $=\frac{27}{126}$% = the difference in income on the 6% stock and the 5% stock. III. ∴ The 6% stock at 84 is $\frac{27}{126}$% better than the 5% stock at 72.
5	II. 1. ∵ ¾ of gain $=\frac{9}{16}$ of the selling price, 2. ¼ of gain = ⅓ of $\frac{9}{16}$ of the selling price $=\frac{3}{16}$ of the selling price. 3. $\frac{4}{4}$, or the gain, $=4\times\frac{3}{16}$, or ¾ of the selling price. 4. $\frac{4}{4}$, the selling price, — ¾, the gain, = ¼, the cost. 5. \$5 = the cost of 1 yard. 6. \$16⅔ = 3⅓ × \$5 = cost of 3⅓ yards. 7. ∴ ¼, the cost, = \$16⅔, 8. $\frac{4}{4}$, the selling price, = 4 × \$16⅔ = \$66⅔ = selling price. III. ∴ The cloth would sell for \$66⅔.
6	II. 1. Let *AB*, Fig. 63, = 5 feet, the radius of the lower base of the frustum, 2. *CD* = 4 feet, the radius of the upper base of the frustum, and 3. *AD* = 12 feet, the slant height of the frustum. 4. Then $AK=AB-DC=5$ feet — 4 feet = 1 foot, and 5. $DK=\sqrt{AD^2-AK^2}=\sqrt{144-1}=\sqrt{143}$ feet. 6. ∴ $AK:DK=AB:BE$, by similar triangles; or 7. 1 foot : $\sqrt{143}$ feet = 5 feet : BE; whence, 8. $BE=5\sqrt{143}$ feet, the altitude of the cone from which the frustum is cut. III. ∴ The altitude of the cone from which the frustum is cut is $5\sqrt{143}$ feet.
7	See solution on page 190.
8	II. 1. 1 oz. of gold = 5760 gr. ÷ 12, or 480 gr., since there are 5760 gr. in a pound of gold. 2. 1 oz. of lead = 7000 gr. ÷ 16, or 437½ gr., since there are 7000 gr. in a pound of lead. 3. ∴ 480 gr. — 437½ gr. = 42½ gr., the excess of weight of 1 oz. of gold over 1 oz. of lead. III. ∴ 1 oz. of gold is 42½ gr. heavier than 1 oz. of lead.

EXAMINATION TESTS WITHOUT ANSWERS.

1. Define bonds, coupons, exchange, tariff.

2. A field of 12 acres and 30 perches yields 255 bu. 2 qts of wheat; how much will a field of 15 acres and 10 perches yield at the same rate?

3. Find value of 11% of \$180 + 22% of \$160 + 92% of \$63.

4. A piano was sold for \$297, at a gain of 35%; what would have been the % of gain if it had been sold for \$300?

5. A dealer imported 120 dozen champagne, invoiced at \$23 a dozen, breakage $12\frac{1}{2}$%; what was the duty at 22%?

6. I rent a house for \$300 per year, the rent to be paid monthly in advance; what amount of cash at the beginning of the year will pay one year's rent?

7. The rafters of a house are 20 feet long, the width of the gable is 30 feet, the rafters project two feet; what is the height of the gable?

8. What the convex surface of the frustum of a cone whose slant height is 6 feet, the diameter of its lower base 5 feet, and of its upper 4 feet?

9. To be analyzed: If for every cow a farmer keeps, he allows $\frac{1}{4}$ acre for pasture, and $\frac{2}{5}$ of an acre for corn, how many cows can he keep on 39 acres?

10. How much can I afford to give for 6's of '81 so that I may realize 8% per annum, gold being at a premium of 15? *Hancock County List.*

1. What is the surface of a parallelopiped, 8 feet long, 4 feet wide, and 2 feet high?

2. A starts on a journey at the rate of 3 miles per hour; 6 hours afterwards B starts after him at the rate of 4 miles per hour. How far will B travel before he overtakes A?

3. The time since noon is $\frac{7}{17}$ of the time to 4 o'clock P. M.; what is the time?

4. A man having oranges at 4 cents each, and apples at 2 for 1 cent, gained 20% by selling 5 dozen for \$2.04; how many of each did he sell?

5. The first term of a geometric series is 3, the third term 507; find the ratio.

6. A merchant sold a quantity of goods at a gain of 20%. If, however, he had purchased the goods for \$60 less, his gain would have been 25%. What did the goods cost?

7. There is a park 400 feet square; a walk 3 feet wide is made in it, along the edges, how many square yards would such a walk contain?

8. A man sold wheat, commission 3% and invested the proceeds in corn, commission 2% his whole commission, \$250; for how much did the wheat sell and what was the value of the corn? *Licking County List.*

1. A man had $43\frac{3}{4}$ yards of carpeting, costing \$$26\frac{1}{4}$; he sold $\frac{2}{5}$ of the pieces gaining \$$\frac{1}{4}$ on each yard sold. How much did he receive for it?

2. From the product of $\frac{25}{10}$ and $\frac{75}{100}$ subtract the difference of their squares.

3. How many acres in a field whose length is 40 rods and diagonal 50 rods?

4. How many trees will be required to plant the above by placing them 1 rod apart? By 2 rods apart?

5. Bought a lot of glass; lost 15% by breakage. At what % above cost must I sell the remainder to clear 20% on the whole?

6. After spending 25% of my money, and 25% of the remainder, I had left \$675. How much had I at first?

7. How many fifths in $\frac{7}{25}$? *Ans.* $1\frac{2}{5}$.

8. A box is $3\frac{1}{2}$ inches long, $2\frac{1}{2}$ inches wide, and 2 inches deep will contain how many $\frac{1}{4}$-inch cubes?

9. Change $\frac{3}{8}$ of quart to the decimal of a bushel.

10. A can hoe 16 rows of corn in a day, B 18, C 20, and D 24. What is the smallest number of rows that will keep each employed an exact number of days? *Seneca County List.*

1. (*a*) Define: number, integer, fraction, a common multiple, and the greatest common divisor of two or more numbers.

(*b*) *Prove*, (do not merely *illustrate*) that to divide by a fraction one may multiply by the divisor inverted.

(*c*) Change 74632 from a scale of 8 to a scale of 9.

2. (*a*) The freezing and boiling temperatures of water are 32° and 212°, respectively, when measured by a Fahrenheit thermometer; measured by a centigrade thermometer they are 0° and 100°, respectively; if a Fahrenheit thermometer records a temperature of 74° what would the centigrade record be at the same time?

(b) By what per cent must 8° Fahrenheit be increased so as to equal 8° centigrade?

3. Silver weighs 10.45 times as heavy as water, while gold weighs 19.30 times as heavy as water; find, correct to 3 decimal places, the number of inches in the edge of a cube of gold which is equal in weight to a cube of silver whose edge is 4.3 cm. Also express this weight in (Troy) grains.

4. A 6% bond, which matures in 3 years, with interest payable annually, is selling at 104; a $5\frac{1}{2}$% bond, which matures in $1\frac{1}{2}$ years, with interest payable semi-annually, is selling at 102. Which is the better investment? And how much better is it?

5. A water-tank has connected with it 4 pipes; the first can fill it in 30 min., the second in 40 min., the third can empty it in 50 min., and the fourth can empty it in one hour. If these pipes are so arranged that the third is automatically opened when the tank is precisely $\frac{1}{2}$ filled, and the fourth when the tank is $\frac{3}{4}$ filled, how long will it take to just fill the tank if the second pipe is set running 10 minutes later than the first?

T.

Cornell University — Scholarship Examination, 1899.

1. A and B run a race, their rates of running being as 17 to 18. A runs $2\frac{1}{8}$ miles in 16 minutes, 48 seconds and B the whole distance in 34 minutes. What is the distance run?

2. The surface of the six equal faces of a cube is 1350 sq. inches. What is the length of the diagonal of the cube?

3. A man bought 5% stock at $109\frac{1}{2}$, and $4\frac{1}{2}$% pike stock at $107\frac{1}{2}$, brokerage in each case $\frac{1}{2}$%; the former cost him $200 less than the latter, but yielded the same income. Find the cost of the pike stock.

4. A, B, and C start together and walk around a circle in the same direction. It takes A $\frac{5}{36}$ hours, B $\frac{2}{9}$ hours, C $\frac{35}{96}$ hours to walk once around the circle. How many times will each go around the circle before they will all be together at the starting point?

5. I hold two notes, each due in two years, the aggregate face value of which is $1020. By discounting both at 5%, one by bank, the other by true discount, the proceeds will be $923. Find face of bank note.

6. The hour and minute hands of a watch are together at 12 o'clock; when are they together again?

7. How many cannon balls 12 inches in diameter can be put into a cubical vessel 4 feet on a side; and how many gallons of wine will it contain after it is filled with the balls, allowing the balls to be hollow, the hollow being 6 inches in diameter, and the opening leading to it containing one cubic inch?

8. An agent sold a house at 2% commission. He invested the proceeds in city lots at 3% commission. His commissions amounted to $350. For what was the house sold?

Ohio State List, December, 1898.

1. A, B, and C can do a piece of work in 84 days; A, B, and D in 72 days; A, C, and D in 63 days; B, C, and D in 56 days. In what time can each do it alone?

2. A banker bought U. S. 4's at $128\frac{3}{8}$% and U. S. $4\frac{1}{2}$'s at $106\frac{1}{2}$%, brokerage $\frac{1}{8}$%. The latter cost him $1053.75 more than the former, but yielded him $195 more income. How much was invested in each kind of bonds?

3. $\frac{3}{4}$ of the cost of A's house increased by $\frac{4}{5}$ of the cost of his farm for 2 years at 5%, amounts to $4950. What was the cost of each, if $\frac{3}{5}$ of the cost of the house was only $\frac{2}{7}$ as much as $\frac{4}{5}$ of the cost of the farm?

4. A man desiring to find the height of a tree, places a 12-foot pole upright 54 feet from the base of the tree; he then steps back 6 feet, and looks over the top of the pole at the top of the tree; his eyes are 4 feet above the ground. How high is the tree?

5. I have, as the net proceeds of a consignment of goods sent by me, $3816.48, which the consignor desires me to remit by draft at 2 months. If the rates of exchange are $\frac{3}{4}$% premium, and the rate of interest 6%, what will be the face of the draft?

6. In a certain factory are employed men, women, and boys; the boys receive 3 cents per hour, the women 4 cents, and the men 6 cents; the boys work 8 hours per day, the women 9 hours, the men 12 hours; the boys receive $5 as often as the women receive $10, and for every $10 paid to the women, $24 are paid to the men. How many are there of each, the whole number being 59?

7. Chicago is 87° 35′ west. What is the standard time at Chicago when it is 1 P. M. at Greenwich?

8. From the middle of the side of a square 10-acre field, I run a line cutting off $3\frac{7}{8}$ acres. Find the length of the line.

Ohio State List, June, 1899.

1. How would you present to a class the subject of addition of fractions? Take as an illustrative example, $\frac{2}{5} + \frac{3}{8} + \frac{7}{10}$.

2. A reservoir is 1.50 meters wide, 2.80 meters long, and 1.25 meters deep. Find how many liters it contains when full, and to what height it would be necessary to raise it that it might contain 10 cu. meters.

3. Reduce (*a*) .4685 T. to integers of lower denominations, and (*b*) 1.69408 to a common fraction in its lowest terms.

4. The boundaries of a square and circle are each 40 feet. Which has the greater area and how much?

5. Find the date of a note of $760, at 8% simple interest, which, when it matured December 1, 1891, amounted to $919.60.

6. A gentleman wishes to invest in $4\frac{1}{2}$% bonds, selling at 102, so as to provide for a permanent income of $1620. How much should he invest?

7. From one-tenth take one-thousandth; multiply the remainder by 10000; divide the product by one million, and write the answer in words.

8. Bought 50 gross of buttons for 25, 10, and 5% off, and disposed of the lot for $35.91 at a profit of 12%. What was the list price of the buttons per gross?

9. Had an article cost 10% less, the number of per cent gain would have been 15 more. What was the per cent gain? Give analysis.

10. If the volume of two spheres be 100 cu. in. and 1000 cu. in. respectively. Find the ratio of their diameters to the nearest thousandth of an inch.

Ohio State List, December, 1891.

PROBLEMS.

1. What is the area of a field in the form of a parallelogram, whose length is 160 rods and width 75 rods? *Ans.* 75 A.

2. Find the area of a triangle whose base is 72 rods and altitude 16 rods. *Ans.* 3 A. 2 R. 16 P.

3. Two trees whose heights are 40 and 80 feet respectively, stand on opposite sides of a stream 30 ft. wide. How far does a squirrel leap in jumping from the top of the higher to the top of the lower? *Ans.* 50 feet.

4. How many steps of 3 feet each does a man take in crossing diagonally, a square field that contains 20 acres? *Ans.* 440 steps.

5. Find the cost of paving a court 150 feet square; a walk 10 feet around the whole being paved with flagstones at 54 cents a square yard and the rest at $31\frac{1}{2}$ cents a square yard? *Ans.* $939.40.

6. What is the area of a triangle, the three sides of which are respectively 180 feet, 150 feet, and 80 feet? *Ans.* 5935.85 sq. ft.

7. What is the area of a trapezium, the diagonal of which is 110 feet, and the perpendiculars to the diagonal are 40 feet and 60 feet respectively? *Ans.* 5500 sq. ft.

8. At 30 cents a bushel, find the cost of a box of oats, the box being 8 feet long, 4 feet wide and 4 feet deep. *Ans.* $$30.85\frac{1}{2}$.

9. Two trees stand on opposite sides of a stream 40 feet wide. The height of one tree is to the width of the stream as 8 is to 4; and the width of the stream is to the height of the other as 4 is to 5. What is the distance between their tops? *Ans.* 50 feet.

10. How many miles of furrow 15 in., wide, is turned in plowing a rectangular field whose width is 30 rods and length 10 rods less than its diagoal? *Ans.* $49\frac{1}{2}$ mi.

11. The sides of a certain trapezium measure 10, 12, 14, and 16 rods respectively, and the diagonal, which forms a triangle with the first two sides, is 18 rods; what is the area? *Ans.* 163.796 sq. rds.

12. Three circles, each 40 rods in diameter, touch each other externally; what is the area of the space inclosed between the circles? *Ans.* 64.5 sq. rds.

13. How many square inches in one face of a cube which contains 2571353 cubic inches? *Ans.* 18769 sq. in.

14. Four ladies bought a ball of thread 3 inches in diameter; what portion of the diameter must each wind off to heve equal shares of the thread?

Ans. First, .2743191 in. Second, .3445792 in. Third, .4912292 in. Fourth, 1.8898815 in.

15. A gentleman proposed to plant a vineyard of 10 A. If he places the vines 6 feet apart; how many more can he plant by setting them in the quincunx order than in the square order, allowing the plat to lie in the form of a square, and no vine to be set nearer its edge than 1 foot in either case? *Ans.* 1870.

16. Find the volume generated by the revolution of a circle about a tangent. *Ans.* $2\pi^2R^2$.

17. How many feet in a board 14 feet long and 16 inches wide at one end and 10 inches at the other, and 3 inches thick? *Ans.* $45\frac{1}{2}$ feet.

18. If I saw through $\frac{1}{4}$ of the diameter of a round log, what portion of the cut is made? *Ans.* .196.

19. What is the surface of the largest cube that can be cut from a sphere which contains 14137.2 cu. ft.? *Ans.* 1800 sq. ft.

20. Two boys are flying a kite. The string is 720 feet long. One boy who stood directly under the kite, was 56 feet from the other boy who held the string; how high was the kite? *Ans.* 717.8+feet.

21. How many pounds of wheat in a cylindrical sack whose diameter is 1½ feet, and whose length is 1¾ yards? ($\pi=3.1416$) *Ans.* 447.31 lb.

22. How large a square can be cut from a circle 50 inches in diameter? *Ans.* 35.3553391 in.

23. How many bbl. in a tank in the form of the frustum of a pyramid, 5 feet deep, 10 feet square at the bottom and 9 feet square at the top? *Ans.* 107.26 bbl.

24. From a circular farm of 270 acres, a father gives to his sons equal circular farms, touching each other and the boundary of the farm. He takes for himself a circular portion in the center, equal in area to a son's part, and reserves the vacant tracts around his part for pasture lands and gives each son one of the equal spaces left along the boundary. Required the number of sons and the amount of pasture land each has. *Ans.* 6 sons; 8.46079 A.

25. At each angle of a triangle being on a level plain and having sides respectively 40, 50, and 60 feet, stands a tower whose height equals the sum of the two sides including the angle. Required the length of a ladder to reach the top of each tower without moving at the base. *Ans.* 116.680316+ft.

26. If the door of a room is 4 feet wide, and is opened to the angle of 90 degrees, through what distance has the outer edge of the door passed? *Ans.* 6.2832 feet.

27. A tinner makes two similar rectangular oil cans whose inside dimensions are as 3, 7, and 11. The first hold 8 gallons and the second being larger requires 4 times as much tin as the other. What are the dimensions of the smaller and the contents of the larger?

Ans. { Dimensions of smaller 6, 14, and 22 inches.
Capacity of larger 64 gallons.

28. An 8-inch globe is covered with gilt at 8 cents per square inch; find the cost. *Ans.* $16.08.

29. A hollow cylinder 6 feet long, whose inner diameter is 1 inch and outer diameter two inches, is transformed into a hollow sphere whose outer diameter is twice its inner diameter; find outer diameter. *Ans.* 3.59 in.

30. A circular field is 360 rods in circumference; what is the diagonal of a square field containing the same area? *Ans.* 20.3 rods.

31. What is the volume of a cylinder, whose length is 9 feet and the circumference of whose base is 6 feet? *Ans.* 25.78 cu. ft.

32. How many acres in a square field, the diagonal being 80 rods? *Ans.* 20 acres.

33. How many cubical blocks, each edge of which is ⅓ of a foot, will fill a box 8 feet long, 4 feet wide, and 2 feet thick. *Ans.* 1728 blocks.

34. From one corner of a rectangular pyramid 6 by 8 feet, it is 19 feet to the apex; find the dimentions of a rectangular solid whose dimensions are as 2, 3, and 4, that may be equivalent in volume. *Ans.* 4, 9, and 8 feet.

35.* A solid metal ball, 4 inches radius, weighs 8 lbs.; what is the thickness of spherical shell of the same metal weighing 7⅝ lb., the external diameter of which is 10 inches? *Ans.* 1 inch.

36. What is the difference between 25 feet square and 25 square feet? *Ans.* 600 sq. ft.

37.* Find the greatest number of trees that can be planted on a lot 11 rods square, no two trees being nearer each other than one rod?

Ans. 152 trees.

38.* A straight line 200 feet long, drawn from one point in the outer edge of a circular race track to another point in the same, just touches the inner edge of the track. Find the area of the track and its width.

Ans. Area, $\pi a^2 = 10000\pi$ sq. ft.; width, *indeterminate*.

39. The perimeter of a certain field in the form of an equilateral triangle is 360 rods; what is the area of the field? *Ans*. 543.552 sq. rd.

40. A room is 18 feet long, 16 feet wide, and 10 feet high. What length of rope will reach from one upper corner to the opposite upper corner and touch the floor? *Ans*. 35.3 ft.

41. How many bushels of wheat in a box whose length is twice its width, and whose width is 4 times its height; diagonal being 9 feet?

Ans. 25 bu., *nearly*.

42 Find the area of a circular ring whose breadth is 2 inches and inside diameter 9 inches. *Ans*. 69.1152 sq. in.

43.* A round stick of timber 12 feet long, 8 inches in diameter at one end and 16 inches at the other, is rolled along till the larger end describes a complete circle. Required the circumference of the circle.

Ans. 150.83 feet.

44. A fly traveled by the shortest possible route from the lower corner to the opposite upper corner of a room 18 feet long, 12 feet wide and 10 feet high. Find the distance it traveled. *Ans*. 28.42534 feet.

45.* From the middle of one side and through the axis perpendicularly of a right triangular prism, sides 12 inches, I cut a hole 4 inches square. Find the volume removed. *Ans*. 138.564064 cu. in.

46.* Two isosceles triangles have equal areas and perimeters. The base of one is 24 feet, and one of the equal sides of the other is 29 feet. The area of both is 10 times the area of a triangle whose sides are 13, 14, and 15 feet. Find the perimeters and altitudes.

Ans. Perimeters, 98 feet; altitudes 35 and 21 feet.

47. A grocer at one straight cut took off a segment of a cheese which had ¼ of the circumference, and weighed 3 pounds; what did the whole weigh? *Ans*. 33.023 lb.

48.* A twelve inch ball is in a corner where walls and floor are at right angles; what must be the diameter of another ball which can touch that ball while both touch the same floor and the same walls?

Ans. 3.2154 in. or 44.7846 in.

49. What will it cost to paint a church steeple, the base of which is an octagon, 6 feet on each side, and whose slant height is 80 feet, at 30 cents per square yard? *Ans*. $64.

50. A tree 48 feet high breaks off; the top strikes the level ground 24 feet from the bottom of the tree; find the height of the stump. *Ans*. 18 feet.

51. How many acres in a square field whose diagonal is 5¼ rods longer than one of its sides? *Ans*. 160.6446 sq. rd.

52.* Three poles of equal length are erected on a plane so that their tops meet, while their bases are 90 feet apart, and distance from the point where the poles meet to the center of the triangle below is 65 feet. What is the length of the poles? *Ans*. 83.23 feet.

53. A field contains 200 acres and is 5 times as long as wide. What will it cost to fence it, at a dollar per rod? *Ans*. $960.

54.* What is the greatest number of plants that can be set on a circular piece of ground 100 feet in diameter, no two plants to be nearer each other than 2 feet and none nearer the circumference than 1 foot? *Ans*. 2173.

55. The axes of an ellipse are 100 inches and 60 inches; what is the difference in area between the ellipse and a circle having a diameter equal to the conjugate axis? *Ans.* $600\,\pi = 1884.96$ sq. in.

56. Find the diameter of a circle of which the altitude of its greatest inscribed triangle is 25 feet. *Ans.* $33\frac{1}{3}$ feet.

57. If we cut from a cubical block enough to make each dimension 1 inch shorter, it will lose 1657 cubic inches, what are the dimensions?

58. Show that the area of a rhombus is one-half the rectangle formed by its diagonals. *Noble Co. Ex. Test.*

59. The length and breadth of a rectangular field are in the ratio of 4 to 3. How many acres in the field, if the diagonal is 100 rods?

60. A spherical vessel 30 inches in diameter contains in depth, 1 foot of water, how many gallons will it take to fill it? *Holmes Co. Ex. Test.* *Ans.* 39 gallons.

61. A field is 40 rods by 80 rods. How long a line from the middle of one end will cut off $7\frac{1}{2}$ acres? *Ans.* 80.6 rd., *nearly.*

62. A ladder 20 feet long leans against a perpendicular wall at an angle of 30°. How far is its middle point from the bottom of the wall? *Ans.* 10 feet.

63. Four towers, A 125 feet high, B 75 feet, C 160 feet, and D 65 feet, stand on the same plane. B due south and 40 rods from A; C east of B and D south of C. The distance from A to C plus the distance from C to B is half a mile, and the distance from D to B is $82\frac{1}{2}$ yd. farther than the distance from C to D. What length of line is required to connect the tops of A and D? *Ans.* 240+rds.

64. Find the volume of the largest square pyramid that can be cut from a cone 9 feet in diameter and 20 feet high? *Ans.* 270 cu. ft.

65. A rectangular lawn 60 yd. long and 40 yd. wide has a walk 6 ft. wide around it and paths of the same width through it, joining the points of the opposite sides. Find in square yards the area of one of the four plats inclosed by paths. *Ans.* 459 sq. yd.

66. Which has the greater surface, a cube whose volume is 13.824 cu. ft., or a rectangular solid of equal volume whose length is twice its width, and its width twice its height? *Ans.* Rect. 576 sq. ft., more.

67. The volume of a rectangular tin can is 3 cu. ft. 1053 cu. in.; its dimensions are in the proportion of 11, 7, and 3. Find the area of tin in the can. *Ans.* $16\frac{3}{8}$ sq. ft.

68. A conical well has a bottom diameter of 28 ft. 3 in., top diameter 56 ft. 6 in., and depth 23 ft. 1.2 in. Find its capacity in barrels. *Ans.* 8023 bbl.

69. A cylindrical vessel 1 foot deep and 8 inches in diameter was $\frac{13}{16}$ full of water; after a ball was dropped into the vessel it was full. Find the diameter of the ball. *Ans.* 6 inches.

70. Two logs whose diameters are 6 feet lie side by side. What is the diameter of a third log placed in the crevice on top of the two, if the pile is 9 feet high?. *Ans.* 4 ft.

71. Circles 6 and 10 feet in diameter touch each other; if perpendiculars from the center are let fall to the line tangent to both circles, how far apart will they be? *Ans.* 7.756 ft.

72. What are the linear dimensions of a rectangular box whose capacity is 65910 cubic feet; the length, breadth, and depth being to each other as 5, 3, and 2? *Ans.* 65, 39, and 26 ft.

73. The perimeter of a piece of land in the form of an equilateral triangle is 624 rods; what is the area? *Ans.* 117 A. 13 31 P.

74. Four logs 4 feet in diameter lay side by side and touch each other; on these and in the crevices lay three logs 3 feet in diameter; on these three and in the crevices lay two logs 2 feet in diameter; what is the diameter of a log that will lay on the top of the pile touching each of the logs 2 feet in diameter and the middle one of the logs 3 feet in diameter?

Ans. ————

75. What will it cost to gild a segment of a sphere whose diameter is 6 inches; the altitude of the segment being 2 inches, at 5 ¢ per square inch?

Ans ————.

76. A grocer cut off the segment of a cheese, and found it took $\frac{1}{6}$ of the circumference. What is the weight of the whole cheese, if the segment weighed $1\frac{1}{2}$ lbs? *Ans.* 52.0228+lbs.

77. Two ladders are standing in the street 20 feet apart. They are inclined equally toward each other at the top, forming an angle of 45°. Find, by arithmetic, the length of the ladders? *Ans.* 26.13 ft.

Union Co. Ex. List.

78. Two trees stand on opposite sides of a stream 120 feet wide; the height of one tree is to the width of the stream as 5 is to 4, and the width of the stream is to the height of the other as 5 is to 4; what is the distance between their tops? *Ans.* 131.58—ft.

79. How many gallons of water will fill a circular cistern 6 feet deep and 4 feet in diameter? *Ans.* 564.0162 gal.

80. A cube of silver, whose diagonal is 6 inches, was evenly plated with gold; if 4 cubic inches of gold were used, how thick was the plating?

Ans. $\frac{1}{18}$ in.

81. Required the distance between the lower corner and the opposite upper corner of a room 60 feet long, 32 feet wide, and 51 feet high?

Ans. 85 ft.

82. How deep must be a rectangular box whose base inside is 4 inches by 4 inches to hold a quart, dry measure? *Ans.* 4.2 cu. in..

83. A fly is in the center of the floor of a room 30 feet long, 20 feet wide, and 12 feet high. How far will it travel by the shortest path to one of the upper corners of the ceiling? *Ans.* $\sqrt{709}$+ft.

84. A corn crib 25 feet long holds 125 bushels. How many bushels will one of like shape and 35 feet long hold?

85. Let a cube be inscribed in a sphere, a second sphere in this cube, a second cube in this sphere, and so on; find the diameter of the 7th sphere, if that of the first is 27 inches. (2). What is the volume of all the spheres so inscribed including the first? *Ans.*————.

86. The area of a rectangular building lot is 720 sq. ft.; its sides are as 4 to 5; what will it cost to excavate the earth 7 feet deep at 36¢ per cubic yard? *Ans.* $67.20.

87. A owns $\frac{1}{3}$ and B the remainder of a field 60 rods long and 30 rods wide at one end and 20 rods wide at the other end, both ends being parallel to the same side of the field They propose to lay out through it, parallel with the ends, a road one rod wide, leaving A's $\frac{1}{3}$ of the remainder at the wide end and B's $\frac{2}{3}$ at the narrow end of the field. Required the location and area of the road. *Ans.* ————.

88. The diameter of a circular field is 240 rods. How much grass will be left after 7 horses have eaten all they can reach, the ropes which are allowed them being of equal lengths and attached to posts so located that each can touch his neighbor's territory and none can reach beyond the boundary of the field? *Ans.* 62.831853 A.

89. What is the diameter of a circle inclosing three equal tangent circles, if the area inclosed by the three equal circles is 1 acre? *Ans.* ————.

90. What is the diameter of a circle inclosing four equal tangent circles each being tangent to the the required circle, if the area inclosed by the four equal circles is 1 acre? *Ans.* $R=4\sqrt{[5(4-\pi)]}(\sqrt{2}+1)\div(4-\pi)$.

91. What is the greatest number of stakes that can be driven one foot apart on a rectangular lot whose length is 30 feet and width 20 feet?

Ans.———.

92. What is the greatest number ot inch balls that can be put in a box 15 inches long, 9 inches wide, and 6 inches high? *Ans.*———.

93. A conical vessel 6 inches in diameter and 10 inches deep is full of water. A heavy ball 8 inches in diameter, is put into the vessel; how much water will flow out? *Ans* ———.

94. How far above the surface of the earth would a person have to ascend in order that ⅓ of its surface would be visible? *Ans.* 8000 mi.

95. Where must a frustum of a cone be sawed in two parts, to have equal solidities, if the frustum is 10 feet long, 2 feet in diameter at one end, and 6 feet at the other? *Ans* ———.

96. At the three corners of a rectangular field 50 feet long and 40 feet wide, stands three trees whose heights are 60, 80, and 70 feet. Locate the point where a ladder must be placed so that without moving it at the base it will touch the tops of the three trees, and find the length of the ladder. What must be the height of a tree at the fourth corner so that the same ladder will reach the top, the foot of the ladder not being moved?

Ans. ———.

97. A horse is tied to a corner of a barn 50 feet long and 30 feet wide; what is the area of the surface over which the horse can graze, if the rope is 80 feet long? *Ans.* ———.

98 How many cubic feet in a stone 32 feet high, whose lower base is a rectangle, 10 feet by 4 feet and the upper base 8 feet by 1½ feet?

Ans. 805⅓ cu. ft.

99. To what height above the ground would a platform, 10 feet by 6 feet, have to be elevated so that 720 sq. ft. of surface would be invisible to a man standing at the center of the platform, the man being 5 feet high?.

Ans ———.

100. Required the side of the least equilateral triangle that will circumscribe seven circles, each 20 inches in diameter. *Ans.* 89 28203 in.

101. Required the sides of the least right triangle that will circumscribe seven circles each 20 inches in diameter. *Ans* 123.9320 in. and 107.3205 in.

102. How long a ladder will be required to reach a window 40 feet from the ground, if the distance of the foot of the ladder from the wall is ⅗ of the length of the ladder. *Ans.* 50 ft.

103. A circular park is crossed by a straight path cutting off ¼ of the circumference; the part cut off contains 10 acres Find the diameter of the park. *Ans.* 150 rd., *nearly*

104. Find the length of the minute-hand of a clock, whose extreme point moves 5 ft. 5.9736 in., in 1 da. 18 hr.? *Ans.* ½ in.

105. A, B, and C, own a triangular tract of land. Their houses are located at the vertices of the triangle; where must they locate a well to be used in common so that the distance from the houses to the well, will be the same, the distance from A to B being 120 rods, from B to C 90 rods and A to C 80 rods. *Ans.* ———.

106. A horse is tethered from one corner of an equilateral triangular building whose sides are 100 feet, by a rope 175 feet long. Over what area can he graze? *Ans.* 90021.109181 sq. ft.

107. Find the area of the triangle formed by joining the centers of the squares constructed on the sides of an equilateral triangle, whose sides are 20 feet? *Ans.* ———

GEOMETRY.

I. DEFINITIONS.

1. ***Geometry*** is that branch of mathematics which deduces the properties of figures in space from their defining conditions, by means of assumed properties of space. — *Century Dictionary.*

2. ***Geometry***
   - 1. Metrical Geometry
     - 1_1. Platonic Geometry
       - 1_2. Pure Geometry
         - 1_3 Plane Geometry
         - 2_3. Solid Geometry
       - 2_2. Conic Secions
       - 3_2. Trigon'try
         - 1_3. Plane Trig.
         - 2_3. Analytical Trig.
         - 3_3. Spherical Trig.
     - 2_1. Analytical Geometry
   - 2. Descriptive Geometry, or Projective Geometry

3. ***Metrical Geometry*** is that branch of Geometry which treats of the length of lines and the magnitudes of angles, areas, and solids.

The fundamental operation of metrical relations is MEASUREMENT. The geometry of Euclid and the Ancients is almost entirely metrical. The theorem, *The square described on the hypotenuse of a right-angled triangle is equal to the sum of the squares described on the other two sides*, is a theorem of metrical geometry.

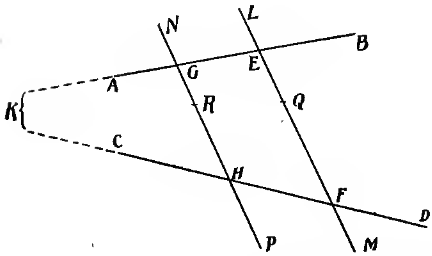
4. ***Descriptive Geometry,*** also called Projective Geometry, Modern Synthetic Geometry, and Geometry of Position, is that branch of Geometry which treats of the positions, the directions, and intersections of lines, the loci of points, and the nature and character of curves and surfaces.

The fundamental operations of Descriptive Geometry are PROJECTION and SECTION. Many of the theorems of Descriptive Geometry are very old, dating as far back as the time of Euclid, but the theories and methods which make of these theorems a homogeneous and harmonious whole is modern having been discovered or perfected by mathematicians of an age nearer our own, such as Monge, Carnot, Brianchon, Poncelet, Moebius, Steiner, Chasles, von Staudt, etc., whose works were published in the earlier half of the present century. Of the synonymous terms I have used to designate this geometry of which I am speaking, the term, Modern Synthetic Geometry is the most comprehensive. Descriptive Geometry was invented by Gaspard Monge (1746–1818) in 1794 and at that time embraced only the theory of making projections of any accurately defined figure such that from these projections can be deduced, not only the projective properties of the figure, but also its metrical properties. Now this term is used to designate the entire theory and development of geometry as embraced in the above definition.

The problem, *To draw a third straight line through the inaccessible point of intersection of two* (converging) *straight lines*, is both metrical and descriptive, that is to say, the required line may be found either by metrical or descriptive geometry, but the method by Descriptive Geometry is far the simpler.

The following are the solutions by both methods:

METRICAL.



I. ***Given*** the two converging lines AB and CD which do not intersect in an accessible point.

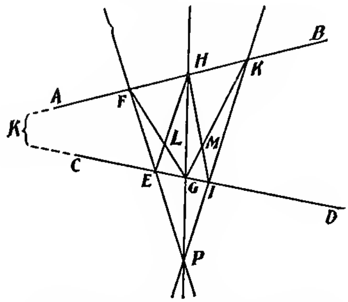
II. ***Required*** to draw a third line through the inaccessible point K.

III. *Construction.*

1. Draw the transversal LM, intersecting AB and CD in E and F respectively.
2. Draw NP parallel to EF and intersecting AB and CD in G and H respectively.
3. Divide EF in any ratio, say $1:2$, and let Q be the point of division.
4. Divide GH in the same ratio and let R be the point of division.
5. The line through QR is the line required.

1. Suppose the line joining the inaccessible point K and the point Q to intersect NP in R', if not in R.

DESCRIPTIVE.



1. Choose some point P outside the two given straight lines AB and CD.
2. Pass through this point any number of transversals, as FP, HP, KP.
3. Draw the diagonals FG, HE, HI, and KG.
4. The points of intersection L and M lie upon the line which passes through the point of intersection of AB and CD.

The proof of this follows from the important harmonic properties of a quadrangle.

IV. *Proof.*

2. Then, from similar triangles, $KR':KQ=R'G:QE$.
3. Also, $KR':KQ=R'H:QF$.
4. $\therefore R'G:QE=R'H:QF$.
5. But, $QE:QF=1:2$. By Hyph.
6. $\therefore R'G:R'H=1:2$.
7. But $RG:RH=1:2$. By Const.
8. $\therefore RG:RH=R'G:R'H$.
9. $\therefore RG=R'G$ and the point R' coincides with R.

Many of the properties of the Conic Sections which are established with great labor and difficulty by Analytical Geometry are easily and elegantly proved by Descriptive Geometry. Descriptive Geometry stands among the first of the branches of pure mathematics in point of interest and simplicity of its methods. The best works on this subject are Luigi Cremona's *Elements of Projective Geometry,* translated by Charles Leudesdorf, and Theodore Reye's *Lectures on Geometry of Position,* Part I., translated by Thomas F. Holgate, and Halsted's *Projective Geometry.*

II. ON GEOMETRICAL REASONING.

5. ***On Geometrical Reasoning.*** We are accustomed to speak of mathematical *reasoning* as being above all other, in accuracy and soundness. This is not correct, if we mean by reasoning the comparing together of different ideas and producing other ideas from the comparison; for, in this view, mathematical reasonings and all other reasonings correspond precisely. The nature of establishing mathematical truths, however, is totally different from that of establishing a truth in history, political economy, or metaphysics, and the difference is this, viz., instead of showing the contrary of the proposition asserted to be only improbable, it proves it at once to be absurd and impossible. For example, suppose one were to ask for the proof of the assassination of Caesar, what would be the method of proof? No one living to-day is *absolutely* certain that Caesar was assassinated, and, in order to establish this truth, we refer to the testimony of historians, men of credit, who lived and wrote their accounts in the very time of which they write; the statements of these historians have been received by succeeding ages as true; and succeeding historians have backed their accounts by a mass of circumstantial evidence which makes it the most improbable thing in the world that the account or any particular part of it is false. In this way we have proved that the truth of the

statement rests on a very high degree of probability, though it does not rise to absolute certainty.

"In mathematics, the case is wholly different. It is true that the facts asserted in these sciences are of a nature totally distinct from those of history; so much so, that a comparison of the evidence of the two may almost excite a smile. But if it be remembered that acute reasoners, in every branch of learning, have acknowledged the use, we might almost say the necessity of a mathematical education, it must be admitted that the points of connection between these pursuits and others are worth attending to. They are the more so, because there is a mistake into which several have fallen, and have deceived others, and perhaps themselves, by clothing some false reasoning in what they called a mathematical dress, imagining that, by the application of mathematical symbols to their subject they secured mathematical argument. This could not have happened if they had possessed a knowledge of the bounds within which the empire of mathematics is contained. That empire is sufficiently wide, and might have been better known, had the time which has been wasted in aggressions upon the domains of others, been spent in exploring the immense tracts which are yet untrodden."* In establishing a mathematical truth, instead of referring to authority, we continually refer our statements to more and more evident statements, until at last we come either to definitions or to statements so evidently true, that to deny them would prove the unsoundness of him who makes the denial.

Geometry must have recourse to the outside world for its first notions and premises, and is, therefore, a natural science.

Yet there is a great difference, between it and the other natural sciences. For example, contrast Geometry and Chemistry. Both derive their constructive materials from sense-perception; but while Geometry is compelled to draw only its first results from observation and is then in a position to move forward deductively to other results without being under the necessity of making fresh observations, Chemistry, on the other hand, is still compelled to make observations and to have recourse to nature.

III. ON THE ADVANTAGES DERIVED FROM THE STUDY OF GEOMETRY, AND MATHEMATICS IN GENERAL.

6. ***On the Advantages derived from the Study of Geometry and Mathematics in General.*** The story is told of Abraham Lincoln that before he began the study of law, he worked through Euclid in order to give his mind that training in logical thinking so necessary to a successful lawyer;

* De Morgan, *Study of Mathematics.*

and his great success as a lawyer and statesman is largely to be attributed to the discipline he thus received.

There should be no conflict between the sciences and the classics. A student taking a college course should give his time to study in both. The study of language enables a person to express his thoughts accurately and clearly while the study of the sciences provides him with thoughts worthy of expression. How far each of these two great departments should be pursued by the student, must be determined by the student himself. But certainly neither should be pursued exclusively. Yet if one were to pursue one or the other of these two great departments of knowledge exclusively, I heartily agree with Professor Earnst Mach who says, "Here I may count upon assent when I say that mathematics and the natural sciences pursued alone as means of instruction yield a richer education in matter and form a more general education, an education better adapted to the needs and spirit of the time, than the philological branches pursued alone would yield."* As to mathematics, "It is admitted by all that a finished or even a competent reasoner is not the work of nature alone; the experience of every day makes it evident that education develops faculties which would otherwise never have manifested their existence. It is, therefore, as necessary to *learn to reason* before we can expect to be able to reason, as it is to learn to swim or fence, in order to attain either of these arts. Now, something must be reasoned upon, it matters not much what it is, provided it can be reasoned upon with certainty. The properties of mind or matter, or the study of languages, mathematics, or natural history, may be chosen for this purpose. Now, of all these, it is desirable to choose the one which admits of the reasoning being verified, that is, in which we can find out by other means, such as measurement and ocular demonstrations of all sorts, whether the results are true or not. . . . Now the mathematics are peculiarly well adapted for this purpose, on the following grounds:

1°. Every term is distinctly explained, and has but one meaning, and it is rarely that two words are employed to mean the same thing.

2°. The first principles are self-evident, and, though derived from observation, do not require more of it than has been made by children in general.

3° The demonstration is strictly logical, taking nothing for granted except the self-evident first principles, resting nothing upon probability, and entirely independent of authority or opinion.

4°. When the conclusion is attained by reasoning, its truth or falsehood can be ascertained, in geometry by actual measure-

*See Professor Mach's *Popular Scientific Lectures*, "On Instruction in the Classics and Sciences." Also Grant Allen's Article in the Oct. No. of the *Cosmopolitan* for 1897.

ment, in algebra by common arithmetical calculation. This gives confidence, and is absolutely necessary, if, as was said before, reason is not to be instructor, but pupil.

5°. There are no words whose meanings are so much alike that the ideas which they stand for may be confounded. Between the meanings of terms there is no distinction, except absolute distinction, and all adjectives and adverbs expressing difference of degree are avoided. Thus it may be necessary to say, "*A* is greater than *B*;" but it is entirely unimportant whether *A* is very little greater than *B* or very much greater than *B*. Any proposition which includes the foregoing assertion will prove its conclusions generally, that is, for all cases in which *A* is greater than *B*, whether the difference be great or little. . . .

"These are the principal grounds on which, in our opinion, the utility of mathematical studies may be shown to rest, as a discipline for the reasoning powers. But the habits of mind which these studies have a tendency to form are valuable in the highest degree. The most important of all is the power of concentrating the ideas which a successful study of them increases where it did exist and creates where it did not. A difficult position, or a new method of passing from one proposition to another, arrests all the attention and forces the united faculties to use their utmost exertions. The habit of mind thus formed soon extends itself to other pursuits, and is beneficially felt in all the business of life.

"As a key to the attainment of other sciences, the use of the mathematics is too well known to make it necessary that we should dwell on this topic. In fact, there is not in this country any disposition to undervalue them as regards the utility of their applications. But though they are now generally considered as a part, and a necessary one, of a liberal education, the views which are still taken of them as a part of education by a large proportion of the community are still very confined."*

The advantages derived from a study of geometry, though very great, are only part of those to be derived from a thorough course of study in mathematics. The eminent mathematician Cayley, "the central luminary, the Darwin of the English School of Mathematicians," as Sylvester calls him, said once that if he had to make a defence of mathematics he would do it in the manner in which Socrates, in Plato's "Republic" defended justice. Justice, according to the Greek sage, was a thing desirable, in itself and for its own sake, quite irrespective of the worldly advantages which might accompany a life of virtue and justice. So just for the sake of learning the beauties and the purest truths which mathematics, the oldest and the noblest, the grandest and the most profound of all sciences, represents,

*De Morgan, *The Study of Mathematics*.

would it be worth while to make ourselves acquainted with its uses as an educational medium and the application it finds in other sciences? Sylvester says, "The world of ideas which mathematics discloses or illuminates, the contemplation of divine beauty and order which it induces, the harmonious connection of its parts, the infinite hierarchy and absolute evidence of truths with which mathematical science is concerned, these, and such like, are the surest grounds of its title to human regard." Sylvester, twenty-five years ago called the attention of the Royal Society to the parallelism between the mathematical and musical ethos: music being the mathematics of the senses, mathematics the music of reason; the soul of each the same. Music the dream, mathematics the working life; each to receive its consummation from the other, when the human intelligence elevated to its perfect type, shall shine forth glorified in some future Beethoven-Gauss.

There is surely something in the beauty of the truths themselves. They enrich us by our mere contemplation of them. What a charm and what a wealth of delight and self-contentment does the finding of mathematical truths afford. In this science, of which geometry is one, out of a few postulates and germinating truths, the mind of man can gradually unfold a system of new and beautiful truths never dreamt of before. Locke says, "The mathematician from very plain and easy beginnings, by gentle degrees, and a continued chain of reasonings, proceed to the discovery and demonstration of truths that appear at first sight beyond human capacity." Because mathematics is a science of pure reason and rigorous logic a mathematician may forget all the preceding propositions of his science and still be able to guide himself with the utmost confidence through the labyrinth of ideas and reach its exit, if he only keeps clearly before him the ends of the threads of thought.

"It is due to the peculiarity of Mathematics, which is a chain of inseparable reasonings, that one part of it can hardly be studied to the exclusion of the others; that in order to understand the whole, only hard and persistent work, the greatest perseverance and the greatest caution, in which all our mental powers and capabilities have to be brought into play, can lead us to the great victory of the mind and enable us to comprehend and see the beauties of pure truths which this magnificent branch of Science represents. To all these peculiarities is due the fact that only a limited number of people are capable of appreciating the beauties of this oldest of all sciences." No fault has ever been found with Mathematics by the *true* student. He who has the courage to study diligently in any line of work, can obtain the same results when studying Mathematics with the same diligence and care. As the drill will not penetrate the granite unless kept to the work hour after hour, so the mind will not penetrate the secrets of Mathematics unless held long and vigorously to the work. As the sun's rays burn only when concentrated, so the mind achieves mastery in Mathematics and indeed in every branch of knowledge only when its possessor hurls all his forces upon it. Mathematics, like all the other sciences, opens its door to those only who knock long and hard. No more damaging evidence can be adduced to prove the weakness of character than for one to have aversion to mathematics; for whether one wishes so or not, it is nevertheless true, that to have aversion for mathematics means to have aversion to accurate, painstaking, and persistent hard study and to have aversion to hard study is to fail to secure a liberal education, and thus fail to compete in that fierce and vigorous struggle for the highest and the truest and the best in life which only the strong can hope to secure.

But we do not judge a painting by the number of its admirers. It is as a rule the lowest kind of art which attracts the largest number of admirers.

In this practical world, in this world of hard struggle for life, where the guiding principle is "swim who can and those who can't may drown," it may not, perhaps, be admissable to judge of the value of a science by its inherent beauty, but rather by the share it contributes to the education of our mental faculties and by the applications it finds in the useful arts and sciences and thus in what measure it contributes to the civilization of the world. He who reads history with some critical judgment cannot fail to notice that the degree of civilization of a country is closely connected with the standard of Mathematics in that country, and this fact is attested by the fierce bidding for the best mathematicians in the world by such countries as France, Russia, and Prussia during the latter part of the last century. Prof. H. J. Stephen Smith, of Oxford, says, "I should not wish to use words which may seem to reach too far, but I often find the conviction forced upon me that the increase of mathematical knowledge is a necessary condition for the advancement of science, and if so, a no less necessary condition for the improvement of mankind. I could not augur well for the enduring intellectual strength of any nation of men, whose education was not based on solid foundation of mathematical learning and whose scientific conception, or in other words, whose notions of the world and of things in it, were not braced and girt together with a strong framework of mathematical reasoning."

Fourier, one of the greatest mathematicians of France, on the completion of his great work on Theory of Heat, says, "Mathematics develops step by step, but its progress is steady and certain amid the continual fluctuations and mistakes of the human mind. Clearness is its attribute, it combines disconnected facts and discovers the secret bond that unites them. When air and light and the vibratory phenomena of electricity and magnetism seem to elude us, when bodies are removed from us into the infinitude of space, when man wishes to behold the drama of the heavens that has been enacted centuries ago, when he wants to investigate the effects of gravity and heat in the deep, impenetrable interior of our earth, then he calls to his aid the help of mathematical analysis. Mathematics renders palpable the most intangible things, it binds the most fleeting phenomena, it calls down the bodies from the infinitude of the heavens and opens up to us the interior of the earth. It seems a power of the human mind conferred upon us for the purpose of recompensing us for the imperfection of our senses and the shortness of our lives. Nay, what is still more wonderful, in the study of the most diverse phenomena it pursues one and the same method, it explains them all in the same language, as if it were to bear witness to the unity and simplicity of the plan of the universe."

Mathematics is the very embodiment of truth. No true devotee of mathematics can be dishonest, untruthful, unjust. Because working ever with that which is true, how can one develop in himself that which is exactly opposite? It would be as though one who was always doing acts of kindness should develop a mean and groveling disposition. Mathematics therefore has ethical value as well as educational value. Its practical value is seen about us every day. To do away with every one of the many conveniences of this present civilization in which some mathematical principle is applied, would be to turn the finger of time back over the dial of the ages to the time when man dwelt in caves and crouched over the bodies of wild beasts.

The practical applications of mathematics have in all ages redounded to the highest happiness of the human race. It rears magnificent temples and edifices, it bridges our streams and rivers; it sends the railroad car with the speed of the wind across the continent; it builds beautiful ships that sail on every sea; it has constructed telegraph and telephone lines and made a messenger of something known to mathematics alone that bears messages of love and peace around the globe; and by these marvellous achievements, it has bound all the nations of the earth in one common brotherhood of man.

IV. AXIOMS.

7. The self-evident first principles of which mention was made in the previous section are called ***axioms.***

Thus, A can not be both B and non-B at the same time; A horse is a horse; Two times two are four; A body in motion will remain in motion, unless acted upon by some external force.

The following are the axioms used in mathematics:

GENERAL AXIOMS.

1. *Things equal to the same thing are equal to each other.*
Thus, if $A=B$ and $B=C$, then $A=C$.

2. *If equals are added to equals, the sums are equal.*
Thus, if $A=B$ and $C=D$, then $A+C=B+D$.

3. *If equals be taken from equals the remainders are equal.*
Thus, if $A=B$ and $C=D$, then $A-C=B-D$.

4. *If equals be added to unequals the sums are unequal in the same order, or sense.*
Thus, if A is greater than B and $C=D$, then $A+C$ is greater than $B+D$.

5. *If equals be taken from unequals the remainders are unequal in the same sense.*
Thus, if A is greater than B and $C=D$, then $A-C$ is greater than $B-D$.

6. *If unequals be taken from equals the remainders are unequal in the opposite sense.*
Thus, if A is greater than B and C is equal to D, then $C-A$ is less than $D-B$.

7. *If equals be multiplied by equals, the products are equal.*
Thus, if $A=B$ and $C=D$, then $AC=BD$.

8. *If unequals be multiplied by equals, the products are unequal in the same sense.*
Thus if A is greater than B and $C=D$, then AC is greater than BD.

9. *If equals be divided by equals, the quotients are equal.*
Thus, if $A=B$ and $C=D$, then $\frac{A}{C}=\frac{B}{D}$.

10. *If unequals be divided by equals, the quotients are unequal in the same sense.*
Thus, if A is greater than B and $C=D$, then $\frac{A}{C}$ is greater than $\frac{B}{D}$.

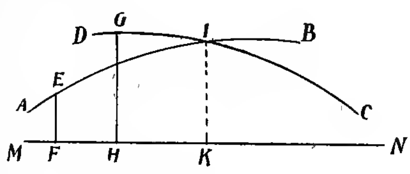
11. *If unequals be added to unequals, the greater to the greater and the lesser to the lesser, the sums will be unequal in the same sense.*
Thus, if A is greater than B and C greater than D, then $A+C$ is greater than $B+D$. If m is less than n and p less than q, then $m+p$ is less than $n+q$.

12. *The whole is greater than any of its parts.*
Thus, if a_1, a_2, a_3, a_4 are parts of A, then A is grerter than any of the a's.

13. *The whole is equal to the sum of all its parts.*
Thus, if a_1, a_2, a_3, a_4, a_5 are the parts of A, then $A=a_1+a_2+a_3+a_4+a_5$.

14. *Magnitudes which coincide with one another are equal to one another.*
Thus, if A coincides with B, then A and B are equal.

15. *If of two unequal quantities, the lesser increases continuously and indefinitely while the other decreases continuously and indefinitely they must become equal once and but once.*
Thus, if, in the figure, the line EF moves parallel to itself, keeping its extremities in AB and MN and the line GH moves parallel to itself keeping its extremities in MN and CD, then the two lines are equal once and only once, viz., when both are equal to the line IK.



16. *If of three quantities the first is greater than the second and the second greater than the third, then the first is greater than the third.*

Thus, if A is greater than B and B greater than C, then A is greater than C.

17. *Two straight lines can not inclose a [finite] space.*

V. ASSUMPTIONS.

8. In addition to the definitions of geometrical magnitudes* and the above axioms the following ***Assumptions,*** or ***Postulates,*** are needed:

(*a.*) ASSUMPTIONS OF THE STRAIGHT LINE.

(1.) *One and only one straight line may be passed through every two points in space; or, briefly, two points determine a straight line.*

(2.) *Two straight lines lying in a plane, determine a point.*

If the two lines are parallel, we still say, for the sake of generality and in harmony with conventions adopted in modern geometry, that the two lines intersect in a point, the point infinity. By taking this view of two parallel lines, many theorems are stated and proved without exceptions to either statement or proof.

(3.) *Through any point in space a line may be drawn and revolved about this point as a center so as to include any assigned point.*

(4.) *A straight line-segment, or a sect, may be produced so as to have any desired length.*

(5.) *A straight line is divided into two parts by any one of its points.*

(*b.*) ASSUMPTIONS OF THE PLANE.

(1.) *Three points not in the same line determine a plane.*

(2.) *A straight line through two points in a plane lies wholly in the plane.*

(3.) *A plane may be passed through a straight line and revolved about it so as to include any assigned point in space.*

(4.) *A portion of a plane may be produced to any desired extent.*

(5.) *A plane is divided into two parts by any of its straight lines.*

(6.) *A plane divides space into two parts.*

(*c.*) ASSUMPTION OF PARALLEL LINES.

(1.) *Through a point without a straight line, only one straight line can be drawn parallel to that line.*

This assumption is a substitute for Euclid's famous eleventh (also called the twelfth) axiom which reads, *If a straight line meet two straight lines so as to make the two interior angles*

*For definitions of geometrical magnitudes, see Mensuration.

on the same side of it taken together less than two right angles, these straight lines being continually produced shall at length meet on that side on which are the angles which are less than two right angles.

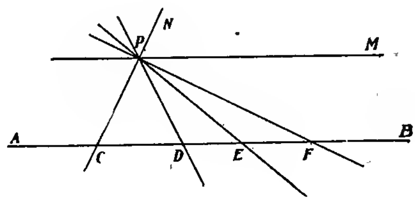
An axiom must possess the following properties: (1) must be self-evident, (2) must be incapable of being proved from other axioms. That the above so-called axiom does not possess the first of these requisites is proved by the fact that there is a dispute among mathematicians as to whether it is an axiom or not. However, it does satisfy the second criterion as, so far, no valid proof of it from other axioms has ever been given. Many proofs have, indeed been given, but it requires very little thought to see that these proofs are all fallacies of *Petitio Principii.*

The many attempts to give a rigorous and valid proof of this assumption, for such it is, has redounded to the eternal glory of geometry in that not only is Euclidean Geometry preserved in all its original purity and integrity but other geometries equally cogent and consistent have been created.

The subject is too abstruse for my present purpose and so I shall do nothing more than show the point of departure of these geometries.

1. Let AB be a given straight line, and P the given point.

2. Through P draw any number of lines.



3. These lines, in relation to the given line, divide themselves into two classes, viz., CUTTING and NON-CUTTING.

Now of the class, *non-cutting,* how many lines are there? On the answer to this question "hang all the law and the prophets." *A priori,* three answers are possible, viz., none, one, many. If we say "none," we have ***Spherical Geometry;*** if we say "one," we have ***Euclidean Geometry;*** if we say "more than one," we have ***Pseudo-Spherical Geometry.***

It is true that the answer, "one," is the answer that is usually insisted upon as being the only possible answer. But this answer is based upon experience and is not, therefore, *a priori.*

In these geometries, the properties of figures are studied, which figures lie in space, or surfaces, possessing the property that the product of the principal radii of curvature at every point of the surfaces shall be constant. If this product is positive, the surface is spherical and the geometry treating of the figures of this surface is Spherical Geometry; if this product is 0, the surface is a plane, and the geometry treating of the properties of figures lying in this surface is the ordinary Euclidean Geometry; if this product is negative, the surface is pseudo-spherical and the geometry treating of the properties of this space is Pseudo-Spherical Geometry.

In the above discussion, it has been assumed tacitly that the measure of a distance remains everywhere the same. Professor Felix Klein has shown that if this be not the case and if the law of measurement of distance be properly chosen, we can obtain three systems of plane geometry analogous to the three systems mentioned above. These are called respectively ***Elliptic, Parabolic, and Hyperbolic Geometries,*** meaning *lacking, equaling,* and *exceeding.* Instead of the above terms given by Klein, we often meet ***Riemannian, Euclidean,*** and ***Lobatschevskian,*** from Riemann, Euclid, and Lobatschevsky, — mathematicians who first set forth clearly the properties of the space-forms. These geometries refer to hyper-space of two dimensions and are called collectively ***non-Euclidean Geometry.***

The notion of hyper-space of two dimensions naturally suggested the question as to whether there are different kinds of hyper-space of three or more dimensions. Riemann showed that there are three kinds of hyper-space of three dimensions having properties analogous to the three kinds of hyper-space of two dimensions already discussed. These hyper-spaces are differentiated by the test whether at every point no geodetical surface, or one geodetical surface, or a fasciculus of geodetical surfaces can be drawn parallel to a given surface, a geodetical surface being defined as such that every geodetic line joining any two points on it lies wholly on the surface. The student who would pursue the subject should read Dr. Halsted's excellent translations of Lobatschevsky and Bolyai, the Lectures and Addresses of Clifford and Helmholtz, Ball's article on *Measurement* in the Encyclopedia Britannica, Professor Schubert's *Essay on the Fourth Dimension,* Russell's *Foundations of Geometry,* and afterwards the monographs of Riemann, Klein, Newcomb, Beltrami, and Killing. For a full bibliography of the literature of the subject up to the time of its publication, see *Bibliography of Non-Euclidean Geometry,* by Dr. Halsted, *American Journal of Mathematics.*

(*d.*) ASSUMPTIONS OF THE CIRCLE.

(1.) *A circle may be constructed with any point as center, and with a radius equal to any given sect.*

(2.) *A circle has but one center.*

(3). *All radii of the same circle are equal, and, hence all diameters of the same circle are equal.*

(4.) *If an unlimited straight line passes through a point within a circle, it must cut the circumference at least twice.*

That it can not cut the circumference more than twice is a theorem.

The region within a circle is defined as that from any point of which no tangents can be drawn to the circle.

(5.) *If one circumference intersects another once, it intersects it again.*

(*e*) ASSUMPTIONS OF THE SPHERE.

(1.) *A sphere may be constructed with any point as center, and with a radius equal to any given sect.*

(2.) *A sphere has but one center.*

(3.) *All radii of the same sphere are equal, and, hence all diameters of the same sphere are equal.*

(4.) *If an unlimited straight line passes through a point within a sphere, it must cut the surface at least twice.*

(5.) *If an unlimited plane or if a spherical surface, intersects a spherical surface, it must intersect it in a closed line.*

(*f*) ASSUMPTIONS OF MOTION.

(1.) *A figure may be moved from one position in three dimensional space to any other position in the same space without altering the size or shape of the figure.*

By this we mean that a figure may be picked up, turned over in any way, and moved to any other position in space without changing the size or shape of the figure. The proof of many theorems in geometry depends upon this assumption.

(2.) *A figure may be moved about in space while one of its points remains fixed.*

Such movement is called "rotation about a center," the center being the fixed point.

(3.) *A figure may be moved about in space while two of its points remain fixed.*

Such movement is called "rotation about an axis," the axis being the line determined by two fixed points.

In the higher mathematics and in Physics and other natural sciences other assumptions are needed.

VI. ON LOGIC.

9. ***On Logic.*** — As a preliminary to the study of geometry a short discussion of the Methods of Reasoning will be of value.

In geometry we are concerned with ***propositions*** about space relations. ***Ideas*** are images of an object formed by the mind. ***Words*** are the spoken or written signs of ideas.

10. ***A judgment*** is an act of the mind affirming a relation between two objects of thought by means of their conceptions.

11. ***A proposition*** is a judgment expressed in words.

For example, take the ideas represented by "all mushrooms" and "things good to eat," posit these ideas in the mind and discern the agreement or disagreement of these two ideas, then express the agreement or disagreement in words. It comes out thus,

"All mushrooms are things good to eat."

Our senses are the instruments by which the qualities of a mushrooms are made known to us. Having found this mushroom good to eat, and this one, and this one, and so on, together with the experience of the race, we arrive at the conclusion, by inductive inference, that "all mushrooms are good to eat." It must be borne in mind that by induction we gain no *certain* knowledge. If the observation of a number of cases shows that alloys of metals fuse at lower temperatures than their constituent metals, we may with more or less probability draw the general inference that

All alloys melt at a lower temperature than their constituent metals.

But this can never rise to the rank of an absolutely certain law until all possible cases have been examined. Not one of the inductive truths which men have established, or think they have established, is really safe from exception or reversal. Lavoisier, when laying the foundations of chemistry, met with so many instances tending to show the existence of oxygen in all acids that he adopted the general conclusion that all acids contain oxygen, yet subsequent experience has shown this to be false. Like remarks may be made concerning all other inductive inferences, the method never leading to absolute certainty.

12. The Powers of the Mind engaged in knowledge are the following three, viz.,

(1). The Power of Discrimination,
(2) The Power of Detecting Identity, and
(3) The Power of Retention.

13. The Laws of Thought are the following three, viz.,

(1) The Law of Identity; as, *That which is, is.*
(2) The Law of Contradiction; as, *A thing cannot both be and not be at the same time.*
(3) The Law of Duality; as, *A thing must either be or not be.*

To these some logicians add a fourth called the "Law of Sufficient reason;" *Every effect has a cause.*

14. *When we join terms together we make propositions; when we join propositions together we make an argument, or piece of reasoning.*

15. ***Terms.*** A concrete term has two meanings, viz., (1) *things to which the term applies,* and (2) *the qualities of those things in consequence of which the term is applied.* The number of different things to which a term is applied is called its ***extension,*** while the number of qualities implied is called its ***intension.***

For example, "table" has a larger "extension" than "round table" for the former term applies to a larger number of objects; the latter has the greater "intension" for it includes all the qualities that the term "table" does and the additional quality "round."

The word "term" comes from the Latin *terminus,* meaning end and is so called because it forms one end of a proposition.

16. ***Propositions.*** Every proposition is composed of a ***subject,*** (Lat., *sub,* under, and *jectum,* laid), a ***copula,*** and a ***predicate*** (Lat. *praedicare,* to assert).

In the proposition, "All mushrooms are things good to eat," "all mushrooms" is the subject, "are" is the copula, and "things good to eat" is the predicate.

Of the kinds of propositions we have

(1) ***Categorical;*** As *A* is *B*. *A* is not *B*; (2) ***Conditional;*** as, If a triangle is equiangular, it is equilateral.

Conditional Propositions are divided into two classes, viz., ***Hypothetical*** and ***Disjunctive.*** The following is a disjunctive proposition:

A is either *B* or *C*.

Of the Categorical Propositions we have,

A. ***The Universal Affirmative;*** as, All horses are animals.
E. ***The Particular Affirmative;*** as, Some animals are horses.
I. ***The Universal Negative;*** as, No horses are cows.
O. ***The Particular Negative;*** as, Some animals are not horses.

Every proposition which expresses accurately a thought, can be reduced to one of the above forms, though the reduction in many cases is not apparent. For example,

Parallel lines never meet, reduces to
Parallel lines are lines which never meet.

The hypothetical proposition, "If gunpowder be damp, it will not explode" reduces to, "Damp gunpowder will not explode."

When we make a statement about all the objects which can be included under a term, we use the term UNIVERSALLY, as logicians say, that is to say, THE TERM IS DISTRIBUTED. In the proposition, "all men are mortal," the term "men" is distributed, because the little word "all" indicates that the statement applies to any and every man. But THE PREDICATE "mortal" IS ONLY TAKEN PARTICULARLY AND IS NOT DISTRIBUTED.

Therefore, we see that a UNIVERSAL AFFIRMATIVE DISTRIBUTES ITS SUBJECT BUT NOT ITS PREDICATE.

As a universal negative proposition take, "No sea-weed is a flowering plant." The subject "sea-weed" is distributed. If there could be found a single flowering plant which is a sea-weed, then the proposition would not be true. Hence the predicate is also distributed.

Hence, THE UNIVERSAL NEGATIVE PROPOSITION DISTRIBUTES ITS SUBJECT AND ITS PREDICATE.

No difficulty is experienced in seeing that *the particular affirmative distributes neither its subject nor its predicate,* and that *the* PARTICULAR NEGATIVE DISTRIBUTES ITS PREDICATE BUT NOT ITS SUBJECT.

In the absence of any knowledge to the contrary, the word "some," in the particular affirmative and particular negative, must be taken to mean "SOME AND IT MAY BE ALL."

17. ***The Law of Converse.*** Two propositions are the converse of each other when the subject of one is the predicate of the other. Thus,

"Equilateral triangles are equiangular, "(direct).
Equiangular triangles are equilateral, (converse).

It does not follow that because a proposition is true its converse will also be true. Thus, "All regular polygons are equilateral (direct); all equilateral (polygons) are regular, (converse). This last is not true. The converse of all definitions are true.

Whenever three theorems have the following relations, their converses are true:

1. If it is known that when $A > B$, then $x > y$, and
2. If it is known that when $A = B$, then $x = y$, and
3. If it is known that when $A < B$, then $x < y$,

then the converse of each of these is true.

For

1_1. If $x > y$, then A cannot equal B and A cannot be less than B without violating 2 or 3; $\therefore A > B$. (Converse of 1.)

2_1. If $x = y$, then A cannot be greater than B and A cannot be less than B without violating 1 or 3; $\therefore A=B$. (Converse of 2.)

3_1. If $x < y$, then A cannot be greater than B and A cannot be equal to B without violating 1 or 2; $\therefore A < B$. (Converse of 3.)

18. The ***opposite*** of a proposition is formed by stating the negative of its hypothesis and conclusions. Thus,

If $A = B$, then $C = D$ (Direct.)
If A is not equal B, then C is not equal D. (Opposite.)

19. *If the direct proposition and its converse are true, the opposite proposition is true; and if a direct proposition and its opposite are true, the converse proposition is true.* Thus,

1. If $A = B$, $C = D$. (Direct.)
   If $C = D$, $A = B$. (Converse.)
   If A is not equal to B, C is not equal to D (Opposite.)
2. If $A = B$, $C = D$. (Direct.)
   If A is not equal to B, C is not equal to D. (Opposite.)
   Then, if $C = D$, $A = B$. (Converse.)

20. ***Methods of Reasoning.*** There are two methods of reasoning, viz., the ***Inductive*** and the ***Deductive.***

The ***Inductive Method*** is used in reaching a general truth or principle by an examination and comparison of particular facts. Thus, This apple is equal to the sum of all its parts, this piece of crayon is equal to the sum of all its parts, this orange is equal to the sum of all its parts, and so with peaches, pears, balls, pebbles, slates, knives, and chairs.

Therefore, the whole of any object is equal to the sum of all its parts, or the whole is equal to the sum of all its parts. This is inductive reasoning.

The ***Deductive Method*** is used in reaching a particular truth or principle from general truths or principles. Thus.

All animals suffer pain.
Flies are animals.
Therefore, flies suffer pain.

21. ***The Syllogism.*** When we compare propositions we reason. Deriving a third proposition from two given propositions is called ***syllogistic reasoning,*** or ***Deductive Reasoning.*** Thus,

1. All English silver coins are coined at Tower Hill.
2. All sixpences are coined at Tower Hill.

Therefore, All sixpences are English silver coins.

The last proposition is called the ***conclusion,*** the other two propositions are called ***premises,*** and the three together the ***syllogism.***

Again,

All electors pay rates. *A.*
No paupers pay rates. *E.*
Therefore, no paupers are electors. *E.*

From the examples given, we see that there are only three terms or classes of things reasoned about; in the first example the three terms are "All English silver coins," "Tower Hill," and "all sixpences." Of these, the class, "English silver coins," does not occur in the conclusion. It is used to enable us to compare together the other two classes of things. It is called the ***middle term.*** (Things) "coined at Tower Hill," is called the ***major term*** for the reason that it has the larger ***extension,*** and "sixpences," the subject of the conclusion, is called the ***minor term*** of the syllogism, for the reason that it has a lesser extension than the subject of the conclusion.

The premise in which the "major term" is found is called the ***major premise,*** and the one in which the minor term is found is called the ***minor premise.***

Hence, ***the middle term is always the term not found in the conclusion; the major term is the predicate of the conclusion; and the minor term is the subject of the conclusion.***

Suppose that the two premises and the conclusion of the last syllogism be varied in every possible way from *affirmative* to *negative,* from *universal* to *particular* and *vice versa.*

Each proposition can be converted into four different propositions and each one of these four may be compounded with any one of the other two. Hence the number of changes (called moods) is $4\times4\times4=64$. These moods may be still further varied, if instead of the middle term being the subject of the first and the predicate of the second, this order may be reversed, or if the middle term the subject of both, or the predicate of both. In this way we see that for each of the sixty-four moods we get four syllogisms called ***figures.***

Of the sixty-four moods, there are altogether ***nineteen moods of the syllogism that are admissible.***

22. ***Rules of the Syllogism.*** To find out whether an argument is valid or not, we must examine it carefully to ascertain whether it agrees with certain rules discovered by Aristotle. Modern logicians have to some extent broken away from these rules. Without going into the matter in detail we state these rules.

I. ***Every syllogism has three terms and only three.*** These terms are called the the *major term*, the *minor term*, and the *middle term*.

II. ***Every syllogism contains three and only three propositions.***

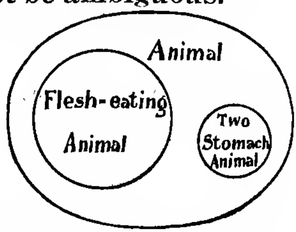
III. ***The middle term must be distributed once at least in the premises and must not be ambiguous.***

Some animals are flesh-eating.
Some animals have two stomachs.

No conclusion can be drawn.

But if we say,

Some animals are flesh-eating,
All animals consume oxygen, we can say

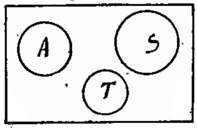


Therefore, some animals consuming oxygen are flesh-eating.

IV. ***If both premises are negative no conclusion can be drawn.***

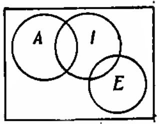
For, from the statements that two things disagree with a third, no proof of agreement or disagreement can be established. Thus the following is inconclusive,

No Americans are slaves.
No Turks are Americans.

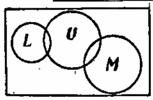


V. ***If both premises are particular no conclusion can be drawn.***

Thus the following are inconclusive:



Some Americans are ignorant.
Some Europeans are ignorant.

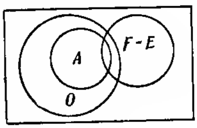


Some laws are unjust.
Some men are unjust.

VI. ***No term must be distributed in the conclusion which was not distributed in the premises.***

From

Some animals eat flesh.
All animals consume oxygen.

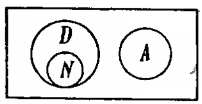


We must conclude that some things that consume oxygen eat flesh.

VII. ***If one premise be negative the conclusion must be negative.***

Thus from

All negroes are dark.
No American is dark.



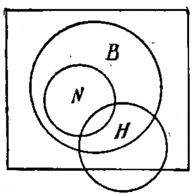
We draw the conclusion

No American is a negro.

VIII. ***If either premise is particular the conclusion must be particular.***

Thus,

All negroes are black.
Some horses are black.



Therefore, some horses are not negroes.

23. ***Logical Fallacies.*** Logical Fallacies result from our neglect to observe the rules of logic. They occur in the mere form of the statement, that is, *in dictione*, as it is known in logic.

There are four *purely logical* fallacies, viz.,

1. Fallacy of four terms (Quaternio Terminorum), — Violation of Rule I.
2. Fallacy of undistributed middle, — Violation of Rule III.
3. Fallacy of illicit process, of the major or minor term. — Violation of Rule VI.
4. Fallacy of negative premises. — Violation of Rule IV.

There are six semi-logical fallacies, viz.,

1. Fallacy of Equivocation.
2. Fallacy of Amphibology.
3. Fallacy of Composition.
4. Fallacy of Division.
5. Fallacy of Accent.
6. Fallacy of Figure of Speech.

In addition to these logical fallicies there are seven Material Fallacies (*extra dictionem*)that is, fallacy in the matter of thought, viz.,

1. Fallacy of Accident.
2. The Converse Fallacy of Accident.
3. The Irrelevant Conclusion.
4. The Petitio Principii.
5. The Fallacy of the Consequent or Non-sequitur.
6. The False Cause.
7. The Fallacy of Many Questions.

We will illustrate some of these fallacies.

Light is contrary to darkness.
Feathers are light.
∴ Feathers are contrary to darkness.

The middle term, "light," has two different meanings in the premises. We have, therefore, four terms instead of three, which violates Rule I. When the middle term is ambiguous, the fallacy is known as the *ambiguous middle.*

Every country under a tyranny is distressed.
This country is distressed.
∴ This country is under a Tyranny. — *Fallacy of Undistributed Middle.*

All moral beings are accountable.
No brute is a moral being.
∴ No brute is accountable. — *Fallacy of the Illicit Process of the Major Term.*

Some men are not just.
No angel is a man.
∴ Some angels are not just.— *Fallacy of Negative Premises.*

EXAMPLES.

Seven is one number.
Two and five are seven.
∴ Two and five are one number. — *Fallacy of Division.*

Three and four are two numbers,
Seven is three and four.
∴ Seven is two numbers. — *Fallacy of Composition.*

The duke yet lives that Henry shall depose. — *Fallacy of Amphibology.*

The conclusion depending upon the interpretation of the meaning of this proposition is doubtful.

A hero is a lion.
A lion is a quadruped.
∴ A hero is a quadruped. — *Fallacy of Figure of Speech.*

Thieves are dishonest;
But thieves are men;
∴ All men are dishonest. — *Fallacy of Accident.*

24. ***How to Prepare a Lesson in Geometry.*** In beginning the study of geometry, great care should be taken to grasp a correct notion of the definitions and illustrations. The definitions, axioms, and assumptions are the foundation on which rests the magnificent structure of geometry. The definitions should be committed to memory, only committing them, however, as they occur in the prosecution of the study. Make haste slowly at first; one proposition per lesson for the first three lessons is quite sufficient: and two propositions may be taken at a lesson for the next seven or eight lessons. After this, if the work is thoroughly in hand three propositions together with several originals should constitute a lesson.

In the preparation of the lesson, the student should carefully read the proposition so as to get its full meaning. After the meaning of the proposition is understood, carefully follow the demonstration in the book, never leaving a statement made in the demonstration until it is thoroughly understood. At first, it may be necessary to repeat this two or three times, perhaps oftener. After the given demonstration is thoroughly understood, close the book, draw a figure on paper or a slate, and write out a demonstration of your own. Compare your demonstration with the one in the book, and make such corrections as are necessary.

By carefully observing this method, it will be a comparatively short time until one reading of the lesson will generally suffice for the necessary preparation. The theorem should always be committed to memory, the demonstration never. It is not a bad practice to commit the proposition exactly as it is stated in the book, for, as a general thing the author has put much time on the statement of each proposition endeavoring to reduce it to its simplest and most elegant form, and upon this work, the student, as a rule, can not improve.

In conducting the recitations, no books should be allowed to be consulted. The propositions should be assigned by stating them in part or in full to the students called upon to recite. The students so called upon, should go to the board and draw as neat and accurate figure as possible, accurate figures often suggesting truths not revealed by carelessly constructed figures. It is generally best not to require any part of the demonstration to be written out, unless, indeed, it includes long and complicated algebraic equations. In reciting, if it is convenient, the student should step to the board and, using a pointer in referring to the various parts of his figure, observe the following order in the discussion of the theorem:

I. ***Statement of the Theorem.*** Here give an accurate statement of the theorem to be demonstrated.

II. ***Given.*** Here state, with reference to the figure constructed whatever is given by the theorem.

III. ***To Prove.*** Here state the exact conclusion to be derived from what is given.

IV. ***Proof.*** Here set forth, in logical order the statements to prove the conclusion just asserted.

The validity, limitations, and general application of the theorem may then be discussed by the class.

Corollaries coming under the various theorems in the lesson may be assigned to students other than those demonstrating the theorems. The proof of a corollary is usually simple, but its proof should be given with the same care and accuracy.

We will now illustrate what we have said by a few propositions. The student should have one of the following excellent texts:

Halsted's *Elements of Geometry.*
Beman and Smith's *Plane and Solid Geometry.*
Phillips and Fisher's *Elements of Geometry.*
Wentworth's *Plane and Solid Geometry.*

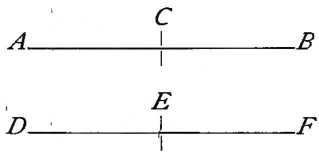
PLANE GEOMETRY.

BOOK I.

ANGLES AND STRAIGHT LINES.

PROPOSITION I.

I. **Theorem.** *All straight angles are equal.*

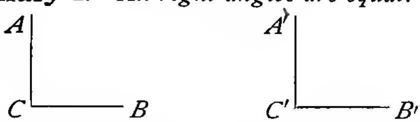


II. **Given** any two straight angles ACB and DEF.

III. **To prove** $\angle ACB = \angle DEF$.

IV. **Proof.**
1. Apply $\angle ACB$ to the $\angle DEF$, so that the vertex C shall fall on the vertex E. (*First assumption of motion.*)
2. Then revolve CB so that it contains the point F. (*Third assumption of the straight line.*)
3. Then CA will coincide with ED. (*First assumption of a straight line and Law of Identity.*)
4. $\therefore \angle ACB = \angle DEF$. Axiom 10.

I. **Corollary** 1. *All right angles are equal.*



II. **Given** any two right angles ACB and $A'C'B'$.

III. **To prove** $\angle ACB = \angle A'C'B'$.

IV. **Proof.**
1. All straight angles are equal. Prop. I.
2. $\angle ACB$ and $\angle A'C'B'$ are each the half of a straight angle. By definition.
3. $\therefore \angle ACB = \angle A'C'B'$. Axiom 7.

I. **Cor.** 2. *The angular units, degree, minute, and second have constant values.*

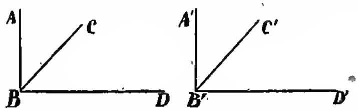
II. **Given** a degree angle.

III. **To prove** that it is a constant magnitude.

IV. **Proof.**
1. A constant magnitude is a magnitude whose value is always the same. By def.
2. A straight angle is a magnitude whose value is always the same. By Prop. I.
3. 1. ∴ A straight angle is a constant magnitude.
4. 2. A degree angle is one hundred eightieth part of a straight angle. By def.
   3. ∴ A degree angle is a constant magnitude. By Aristotle's Dictum,—Whatever may be predicated of a whole may be predicated of a part.

In like manner, we can prove that minute-angles and second-angles are constants.

I. **Cor.** 3. *Complements of equal angles are equal.*



II. **Given** the two equal angles CBD and $C'B'D'$ and their complements ABC and $A'B'C'$, respectively.

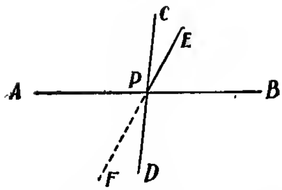
III. **To prove** that $\angle ABC = \angle A'B'C'$.

IV. **Proof.**
1. $\angle ABC$ = the difference between a rt. $\angle$ and $\angle CBD$. By def. of comp.
2. $\angle A'B'C'$ = the difference between a rt. $\angle$ and $\angle C'B'D'$. By def. of comp.
3. But $\angle CBD = \angle C'B'D'$. By hypothesis.
4. ∴ $\angle ABC = \angle A'B'C'$. By Axiom 1.

I. **Cor.** 4. *Supplements of equal angles are equal.*

(Proof same as above.)

I. **Cor.** 5. *At a given point in a given line, one perpendicular, and only one, can be erected in the same plane.*



II. ***Given*** CD perpendicular to AB at P.

III. ***To prove*** that no other perpendicular can be drawn to AB at P in the same plane.

IV. ***Proof.***

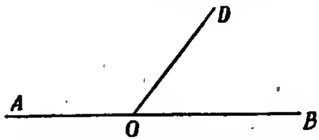
1. Suppose that another perpendicular EF could be drawn.
2. Then $\angle BPE$ would be a rt. $\angle$. By def. of perpendicular.
   (If two lines meet and form a rt. $\angle$, each is said to be perpendicular to the other.)
3. But $\angle BPC$ is a rt. angle.
   (Since CD is perpendicular to AB.)
4. $\therefore \angle BPE$ would equal $\angle BPC$. Prop. I., Cor. 1.
   (All right angles are equal.)
5. But this is impossible. By Axiom 8.
   (The whole is greater than any of its parts.)
6. $\therefore$ The supposition of step 1 is absurd, and a second perpendicular is impossible. *Q. E. D.*

Remark. In this demonstration, we have used what is called the ***Indirect Method,*** or *reductio ad absurdum* which means a reduction to an absurdity, as distinguished from the ***Direct Method*** used in the other proofs. Jevons in his *Principles of Science*, Vol. I, p. 96, says, "Some philosophers, especially those of France, have held that the Indirect Method of Proof has a certain inferiority to a direct method, which should prevent our using it." He goes on to show that the method is not inferior and holds the belief that nearly half our logical conclusions rest upon its employment.

In the above case, by the Law of Duality, a second perpendicular can or can not be drawn. It was shown that by supposing that a second one could be drawn led us to an absurdity. Hence, a second can not be drawn. This method of proof is often used in geometry.

PROPOSITION II.

I. ***Theorem.*** *If two adjacent angles have their exterior sides in a straight line, these angles are supplements of each other.*



II. ***Given*** the exterior sides OA and OB of the adjacent angles AOD and BOD respectively and the straight AB in which these two sides lie.

III. ***To prove*** $\angle AOD = \angle DOB$.

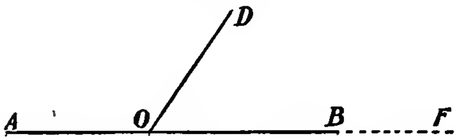
IV. ***Proof.***
1. AOB is a straight line. By hypothesis.
2. $\therefore \angle AOB$ is a st. $\angle$. By def. of a st. $\angle$.
3. But $\angle AOD + \angle DOB = \angle AOB$. By Ax. 9.
4. $\therefore \angle$'s AOD and DOB are supplementary. By def. of supl. angles.

Cor. 1. *The sum of all the angles about a point in a plane is equal to two straight angles.*

Cor. 2. *The sum of all the angles about a point on the same side of a straight line passing through a point, is equal to a straight angle.*

PROPOSITION III.

I. ***Theorem.*** CONVERSELY: *If two adjacent angles are supplements of each other, their exterior angles lie in the same straight line.*



II. ***Given*** that the sum of the adjacent angles AOD and DOB are supplements of each other, that is, equal to a straight angle.

III. ***To prove*** AO and OB in the same straight line.

IV. ***Proof.***
1. Assume OF in the same straight line with OA.
2. Then $\angle AOD + \angle DOF$ is a straight angle. By Prop. II.
3. But $\angle AOD + \angle DOB$ is a straight angle. By hypothesis.
4. $\therefore \angle AOD + \angle DOB = \angle AOD + \angle DOF$. By Ax. I.
5. $\angle AOD = \angle AOD$. By Law of Identity.
6. Subtracting step 5 from step 4, $\angle DOB = \angle DOF$. By Ax. 3.

7. ∴ OB and OF coincide. By converse Ax. 10.
8. ∴ AO and OB are in the same straight line. *Q. E. D.*

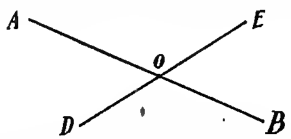
SCHOLIUM. Since Propositions II. and III. are true, their opposites are true, viz.,

If the exterior sides of two adjacent angles are not in a straight line, these angles are not supplements of each other.

If two adjacent angles are not supplements of each other, their exterior sides are not in the same straight line.

PROPOSITION IV.

I. **Theorem.** *If one straight line intersects another straight line, the vertical angles are equal.*



II. ***Given*** the two lines AB and DE intersecting in O.

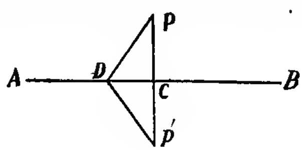
III. ***To prove*** $\angle AOE = \angle DOB$.

IV. ***Proof.***
1. $\angle AOE + \angle AOD$ equals a st. $\angle$. By Prop. I.
2. $\angle AOD + \angle DOB$ equals a st. $\angle$. By Prop. I.
3. ∴ $\angle AOE + \angle AOD = \angle AOD + \angle DOB$. By Ax. 1.
4. Take away from each of these equals the common $\angle AOD$.
5. Then, $\angle AOE = \angle DOB$. By Ax. 3.
6. In like manner we may prove $\angle AOD = \angle EOB$. *Q. E. D.*

I. **Cor.** *If one of the four angles formed by the intersection of two straight lines is a right angle, the other three angles are right angles.*

PROPOSITION V.

I. **Theorem.** *From a point without a straight line one perpendicular, and only one, can be drawn to this line.*



II. ***Given*** the point, P, and the straight line, AB.

III. ***To prove*** that one perpendicular can be drawn from P to AB, and only one.

IV. ***Proof.***

1. Turn the part of the plane above AB about AB as an axis until it falls upon the part below AB and denote the position of P by P'. By Assumption 3 of the Plane.
2. Turn the revolved plane about AB to its original position. By Assumption 3 of the Plane.
3. Draw the straight line PP', cutting AB in C. By Assumptions 1 and 2 of the Straight Line.
4. Take any other point D in AB, and draw PD and $P'D$.
5. Since PCP' is a straight line, PDP' is not a straight line. (Between two points only one straight line can be drawn.)
6. Turn the figure PCD about AB until P falls on P'. By Assumption 3 of the Plane.
7. Then CP will coincide with CP' and DP with DP'
8. $\therefore \angle PCD = \angle P'CD$, and $\angle PDC = \angle P'DC$. Ax. 15.
9. $\therefore \angle PCD$, the half of a st. $\angle PCP'$ is a right $\angle$; and $\angle PDC$, the half of $\angle PDP'$, is not a right angle.
10. $\therefore PC$ is perpendicular to AB, and PD is not perpendicular to AB. By def. of Perpendicular.
11. $\therefore$ one perpendicular, and only one, can be drawn from P to AB. *Q. E. D.*

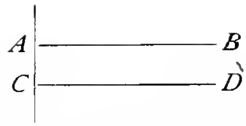
PARALLEL LINES.

Definition. *Parallel lines* are lines lying in the same plane and never meeting however far produced.

On this definition and the assumption of parallel lines rests the whole theory of parallel lines in Euclidean geometry. By convention, we say that parallel lines meet at infinity. Why this convention is adopted will become apparent in studying Higher Modern Geometry.

PROPOSITION VI.

I. ***Theorem.*** *Two straight lines in the same plane perpendicular to the same straight line are parallel.*



II. ***Given*** the two straight lines *AB* and *CD* each perpendicular to the straight line *AC*.

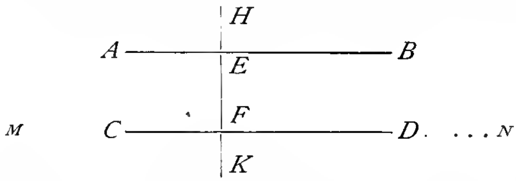
III. ***To prove*** *AB* and *CD* parallel.

IV. ***Proof.***

1. *AB* and *CD*, lying in the same plane, must either meet or not meet. By Law of Duality.
2. If they meet, we shall have two lines from the same point perpendicular to the same line. By hypothesis.
   (The lines *AB* and *CD* being perpendicular to *AC*.)
3. But this is impossible. By Prop. V.
   (From a given point without a straight line, one perpendicular, and only one, can be drawn to a straight line.)
4. ∴ *AB* and *CD* cannot meet, however far produced.
5. ∴ *AB* and *CD* are parallel. By definition of Parallel Lines.

PROPOSITION VII.

I. ***Theorem.*** *If a straight line is perpendicular to one of two parallel lines, it is perpendicular to the other.*



II. ***Given*** the parallel lines *AB* and *CD* and the line *HK* perpendicular to *AB*.

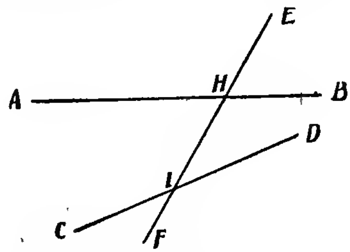
III. ***To prove*** that *HK* is perpendicular to *CD*.

IV. ***Proof.***

1. Suppose *MN* drawn through *F* perpendicular to *HK*.
2. Then *MN* is parallel to *AB*. By Prop. VI.
   (Two lines in the same plane perpendicular to the same line are parallel.)
3. But *CD* is parallel to *AB*. By hypothesis.
4. ∴ *MN* coincides with *CD*. By assumption 1 of parallel lines.
   (Through a point without a straight line only one straight line can be drawn parallel to that line.)
5. *CD* is perpendicular to *HK*; that is,
6. *HK* is perpendicular to *CD*. *Q. E. D.*

TRANSVERSALS.

Definition. A straight line intersecting two or more straight lines is called a *transversal.*



In the figure *EF* is a transversal of the two non-parallel lines *AB* and *CD*.

The angles *AHI, BHI, CIH,* and *DIH* are called *interior angles*, and the angles *AHE, EHB, CIF,* and *FID* are called *exterior angles.*

The angles *AHI* and *HID*, or *BHI* and *HIC* are called *alternate-interior angles.*

The angles *AHE* and *DIF*, or *BHE* and *CIF* are called *alternate-exterior angles.*

The angles *AHE* and *CIH, AHI* and *CIF, EHB* and *HID*, or *BHI* and *DIF* are called *exterior-interior angles.*

PROPOSITION VIII.

I. **Theorem.** *If two parallel lines are cut by a third straight line, the alternate-interior angles are equal; and* **conversely.**

Exercises. I. Find the value of an angle (1) if it is double its complement; (2) if it is one-fourth of its complement.

II. ***Given*** (1) that $\angle A$ is double its complement.

III. ***To find*** the value of $\angle A$.

IV. ***Solution.***

1. rt. $\angle - \angle A$ = complement of $\angle A$. By def. of compl.
2. $\angle A = 2(\text{rt.} \angle - \angle A)$. By hypothesis.
3. $\therefore \angle A = 2$ rt. $\angle$'s $- 2\angle A$. By Distributive Law of Multiplication.
4. Adding $2\angle A$ to these two equals, we have $3\angle A = 2$ rt. $\angle$'s. By Ax. 2.
5. $\therefore \angle A = \frac{2}{3}$ rt. $\angle$. By Ax. 7. *Q. E. F.*

Let the student give the solution of (2).

2. Find the value of an angle (1) if it is three times its supplement; (2) if it is one-third of its supplement.

3. How many degrees in the angle formed by the hands of a clock at 2 o'clock? 3 o'clock? 4 o'clock? 9 o'clock?

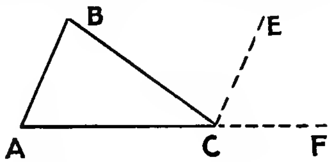
PROPOSITION IX.

I. ***Theorem.*** *If two parallel lines are cut by a third straight line, the exterior angles are equal, and* **conversely.**

Let the student give the demonstration and state and prove the corollaries, if any, coming under the theorem.

PROPOSITION X.

I. ***Theorem.***—*The sum of three interior angles of a triangle is equal to two right angles, or a straight angle.*



II. ***Given*** triangle ABC,

III. ***To prove*** that $\angle A+\angle B+\angle ACB=$st. $\angle$.

IV. ***Proof.***

1. Draw $CE \parallel$ to AC, and prolong AC to F.
2. Then $\angle ACB+\angle BCE+\angle ECF=$st. $\angle$.
   (The sum of all the angles about a point on the same side of a straight line=st. angle.)
3. But $\angle ECF=\angle BAC$,
   (being exterior-interior angles of $\parallel$ lines) and
4. $\angle B=\angle BCF$,
   (being alternate-interior angles of $\parallel$ lines.)
5. Substituting $\angle A$ for $\angle ECF$ and $\angle B$ for $\angle BCF$, in step 2,
6. $\therefore \angle ACB+\angle A+\angle B=$st. $\angle$. *Q. E. D.*

Note.— The truth of this theorem was probably discovered by Thales, 640 B. C.

Attempt to prove this theorem without the use of Euclid's "Eleventh Axiom," or any of its equivalents, and you will see where non-Euclidean Geometry comes into the field of human thought. It is high time that teachers of geometry endeavor to gain a little knowledge of this subject, instead of talking about the "visionary speculations" of the non-Euclidean geometricians. As a help to gain an elementary knowledge of this subject, the reader is recommended to study H. P. Manning's Non-Euclidean Geometry (Ginn & Co.)

BOOK II.

In this book is considered the equality of polygons. We shall consider only one theorem properly belonging to this book.

Problem.—*To bisect a given triangle by a line drawn from a random point in one of its sides.*

Demonstration.—Let ABC be the given triangle, D a random point in the side BC, and E the middle point of BC. Join A and D, A and E. Draw EF parallel to AD. Draw DF. Then DF bisects the triangle ABC. For the triangle ABE is equivalent to the triangle AEC (?). The triangle AFD is equivalent to the triangle ADE (?). Hence, $ABDF$ is equivalent to ABE (?) and, therefore, DF bisects the triangle ABC. *Q. E. D.*

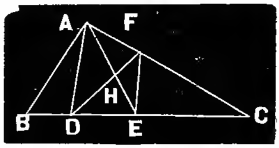


FIG. 1.

Proposition.—*The square described upon the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides.*

I. *Demonstration.*—Let CFD be any right triangle, right angled at F and let AC, CP, and DM be the squares described upon its sides. Then the square AC is equal to the sum of the squares CP and DM. Through F, draw QF perpendicular to AB and produce it to meet OP produced, in G; also produce BC to meet OP in I and AD to meet OP produced, in R. Draw GH parallel to PD, and BT parallel to CF. Draw AE. Now the triangles COI and DFC are equal (?). Hence, $CI=CD=CB$, and therefore the square CP=the parallelogram CG (?)=the parallelogram BF (?)=the rectangle BK (?). In like manner, the square DM can be proved equal to the rectangle AK. Hence, the square AC=the square $CP+$ the square DM. *Q. E. D.*

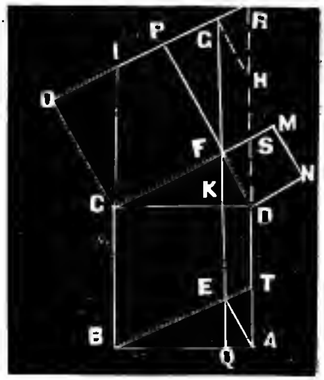


FIG. 2.

Note.—This theorem is known to geometricians as the "Pythagorean Theorem," so called from a Greek geometrician, Pythagoras, (569 B. C.) who was the first to prove it It is also known as the "47th Proposition of Euclid," being the 47th proposition of the first book of Euclid's *Elements*, a mathematical work written by Euclid, a Greek geometer of the 2d century B. C. The above proof is the one essentially given by Euclid. For a great number of different demonstrations the reader is referred to *The American Mathematical Monthly*, Vols. V and VII.

The following demonstration is the one supposed to be given by Pythagoras, and on the discovery of which it is said he sacrificed a hecatomb to the muses that inspired him. This, however, is not authentic.

II. *Demonstration.*—Let EDC be any right triangle, right angled at D. On the sides DE and DC construct the squares $EDHG$ and $DCBM$ respectively. Produce GE and BC until they meet in F, forming the square $FBAG$. On EC, the hypotenuse, construct the square $ECKI$. Then the square $ECKI$ is equal to the sum of the squares $EDHG$ and $DCBM$. For, the square $GFBA$ is equal to $GEDH+DCBM+2\,EDCF$ $(=4ECF)$. The square $GFBA$ is also equal to the square $ECKI+4ECF$. Hence, $ECKI+4ECF=GEDH+DCBM+4ECF$ (?). Whence, $ECKI=GEDH+DCBM$. *Q. E. D.*

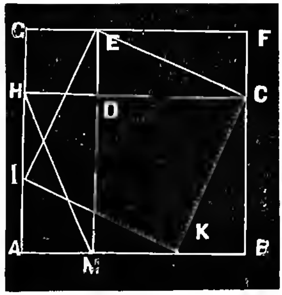


FIG. 3

Proposition.—*In any triangle, each angle formed by joining the feet of the perpendiculars is bisected by the perpendicular from the opposite vertex.*

Demonstration.—Let ABC be any triangle and AD, BE, and CF the three perpendiculars. Join D and E, D and F, and E and F.

In the right triangles AEB and AFC, the angle BAC is common to both. Therefore, they are similar. Hence, $AB:AC=AE:AF$. Now the triangles BAC and FAE have the angle FAE common and the including sides proportional. Therefore, they are similar, and the angle AFE=the angle ACB. In a similar manner we may prove that the angle DFB=the angle ACB; the angle AFE=the angle DFB. From this it follows that the angle CFA—the angle EFA=the angle CFB—the angle DFB. Hence, angle EFC=angle CFD and the angle EFD is bisected by the perpendicular CF. In a similar manner, it can be proved that AD bisects the angle FDE and EB bisects the angle FED. *Q. E. D.*

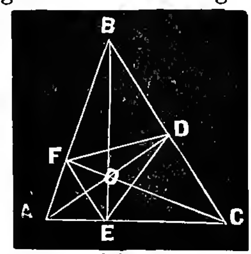


FIG. 4.

Problem.—*From a given point in an arc less than a semicircumference, draw a chord of the circle which will be bisected by the chord of the given arc.*

Demonstration.—Let $ABDC$ be the given circle, AB the given arc, AB the chord of the arc, and P any point of the arc

APC. Draw the diameter POC and on the radius PO as a diameter describe the circle PEO. Then through the points E, and G, of intersection draw the chords PD and PF respectively, and they will be bisected at the points E and G. For draw DC and OE. Then the triangles PEO and PDC are right triangles(?) and are also similar (?). Since PEO and PDC are similar, the line OE is parallel to DC; and since O is the middle point of PC, E is the middle point of PD(?). In like manner, G is the middle point of PF. Q. E. F.

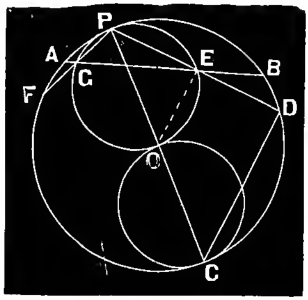


FIG. 5.

Discussion.—There are, in general, two solutions. When arc AB is diminished until B coincides with A, there is no solution. When AB is a semi-circumference, there is one solution and the chord is the diameter POC.

Proposition.—*If two equal straight lines intersect each other anywhere at right angles, the quadrilateral formed by joining their extremities is equivalent to half the square on either straight line.*

Demonstration.—Let AB and CD be two equal straight lines intersecting each other at right angles at E. Join their extremities, forming the quadrilateral $ACBD$. Then $ACBD$ is equivalent to half the square of AB or CD. For the area of the triangle ACB equals $\frac{1}{2}(AB\times CD)$ and the area of the triangle ADB equals $\frac{1}{2}(AB\times ED)$. Hence, the area of $ACBD=\frac{1}{2}(AB\times ED)+\frac{1}{2}(AB\times EC)=\frac{1}{2}AB(ED+EC)=\frac{1}{2}(AB\times CD)$. But CD equals AB, by hypothesis. Hence, $ACBD=\frac{1}{2}AB^2$. Q. E. D.

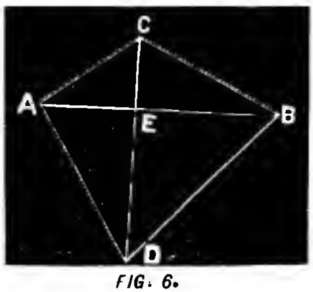


FIG. 6.

A PROBLEM IN MODERN GEOMETRY.

An equilateral hyperbola passes through the middle points D, E, and F of the sides BC, AC, and AB of the triangle ABC, and cutting those sides in order in α, β, and γ. Show that the lines $A\alpha$, $B\beta$, and $C\gamma$ intersect in a point the locus of which is the circumscribing circle of the triangle ABC.

Solution.—The equation to any conic is $u\alpha^2+v\beta^2+w\gamma^2+2u'\beta\gamma+2v'\alpha\gamma+2w'\alpha\beta=0$ .... (1). D is $(0, \frac{1}{2}a\sin C, \frac{1}{2}a\sin B)$; E, $(\frac{1}{2}b\sin C, 0, \frac{1}{2}b\sin A)$; F, $(\frac{1}{2}c\sin B, \frac{1}{2}c\sin A, 0)$. These points being on (1), we should have $c^2v+b^2w+2bcu'=0$ ... (2). $c^2u+a^2w+2acv'=0$ ..... (3), $b^2u+a^2v+2abw'=0$ ....(4).

Whence $u=\frac{a}{bc}(au'-bv'-cw')$ .... (5), $v=\frac{b}{ac}(bv'-cw'-au')$ .. ... (6), $w=\frac{c}{ab}(cw'-au'-bv')$ .... (7). Substituting in the condition $u+v+w-2u'\cos A-2v'\cos B-2w\cos C=0$ .. ... (8) that (1) is an equilateral hyperbola,

$$\frac{a^2(au'-bv'-cw')+b^2(bv'-cw'-au')+c^2(cw'-au'-bv')}{abc}$$

$-2u'\cos A-2v'\cos B-2w'\cos C=0$..... (9). Clearing of fractions and noticing that $2abc\cos A=a(b^2+c^2-a^2)$ ..... (10), $2abc\cos B=b(a^2+c^2-b^2)$ ...(11), $2abc\cos C=c(a^2+b^2-c^2)$ ..... (12), and reducing, $u'\cos A+v'\cos B+w'\cos C=0$....(13). Substituting (5), (6), and (7) in (1) an clearing of fractions, $a^2(au'-bv'-cw')\alpha^2+b^2(bv'-cw'-au')\beta^2+c^2(cw'-au'-bv')$ $+\gamma^2+2u'abc\beta\gamma+2v'abc\alpha\gamma+2w'abc\alpha\beta=0$ . . (14). Where this cuts BC, $\alpha=0$, and (14) gives $b^2(bv'-cw'-au')\frac{\beta^2}{\gamma^2}+2abcu'\frac{\beta}{\gamma}$ $=-c^2(cw'-au'-bv')$ . . (15), whence for the point α; $\alpha_1=0$, $\beta_1=\frac{c}{b}\frac{cu'-bv'-au'}{-cu'+bv'-au'}\gamma_1$ ....(16). By symmetry, for the point β, $\alpha_2=\frac{c}{a}\frac{cw'-au'-bv'}{-cw'+au'-bv'}\gamma_2$, $\beta_2=0$. (17). The equation to $A\alpha$ is found to be $b(-cw'+bv'-aw')\beta-(cw'-bv'-au')\gamma=0$ ... (18); to $B\beta$, $a(-cw'+au'-bv')\alpha-c(cw'-au'-bv')\gamma=0$. ....(19); and to $C\gamma$, $b(-au'+bv'-cw')\beta$ $-a(au'-bv'-cw')\alpha=0$ .... (20). any two of which meet in

$$\left\{\begin{aligned}\alpha'&=\frac{bc(-cw'+bv'-au')(cw'-au'-bv')}{D_1}\\ \beta'&=\frac{ac(cw'-bv'-au')(-cw'+au'-bv')}{D_1},\\ \gamma'&=\frac{ab(-cw'+bv'-au')(-cw'+au'-bv')}{D_1}.\end{aligned}\right. \quad ..(21)$$

The circumscribing circle is $a\beta\gamma+b\alpha\gamma+c\alpha\beta=0$...... (22), which is satisfied by (21) on condition (13), proving the proposition.

NOTE.—This problem was solved by Professor William Hoover, A. M., Ph. D., Professor of Mathematics and Astronomy in the Ohio University, Athens, Ohio, who is one of the leading mathematicians in the United States, and whose biography follows.

BIOGRAPHY.

PROF. WILLIAM HOOVER, A. M., PH. D.

Professor Hoover was born in the village of Smithville, Wayne county, Ohio, October 17, 1850, and is the oldest of a family of seven children. Both parents are living in the village where he was born, still enjoying good health.

Up to the age of fifteen he attended the public schools, and for two or three years after, a local academy. Owing to needy circumstances he was obliged to work for his living quite early, and almost permanently closed attendance at any kind of school at eighteen years of age, sometime before which, going into a store in the county seat, as clerk. Nothing could have been farther from his taste than this work, having been thoroughly in love with study and books long before. After spending two or three years in this way, he went to teaching, about the year 1869, and he has been regularly engaged in his favorite profession to the present day.

He attended Wittenberg College and Oberlin College one term each, a thing having very little bearing on his education. He studied no mathematics at either place, excepting a little descriptive astronomy at the latter.

After teaching three winters of country school, with indifferent success, he was chosen, in 1871, a teacher in the Bellefontaine, Ohio, High School, serving one year, when he was given a place in the public schools of South Bend, Ind. Remaining there two years, he was invited to return to Bellefontaine as superintendent of schools. He afterwards served in the same capacity in Wapakoneta, O., two years, and as principal of the second district school of Dayton, O. In 1883, he was elected professor of mathematics and astronomy in the Ohio University, Athens, Ohio, where he is still in service.

Through all his career of teaching, Professor Hoover has been an incessant student, devoting himself largely to original investigations in mathematics. Although his pretentions in other lines are very modest, he is eminently proficient in literature, language, and history. Before going into college work he had collected a good library. He is indebted to no one for any attainments made in the more advanced of these lines, but by indefatigable energy and perseverance he has made himself the cultured, classic, and renowned scholar he is.

He has always been a thorough teacher, aiming to lead pupils to a mastery of subjects under consideration. His habits of mind and preparation for the work show him specially adapted to his present position, where he has met great success. He studies methods of teaching mathematics, which in the higher parts is supposed to be dry and uninteresting. He sets the example of enthusiasm as a teacher, and rarely fails to impress upon the minds of his students the immense and varied applications of mathematics. He is kind and patient in the class-room and is held in the highest esteem by his students. He is ever ready to aid the patient student inquiring after truth. It seems to be a characteristic of eminent mathematicians that they desire to help others to the same heights to which they themselves have climbed. This was true of Prof. Seitz; it is true of Dr. Martin; and it is true of Prof. Hoover.

In 1879, Wooster University conferred upon Prof. Hoover the degree of Master of Arts, and, in 1886, the degree of Doctor of Philosophy *cum laude*, he submitting a thesis on Cometary Perturbations. In 1889, he was elected a member of the London Mathematical Society and is the only man in his state enjoying this honor. In 1890, he was elected a member of the New York Mathematical Society. He has been a member of the Asso-

Yours Truly
William Hoover

ciation for the Advancement of Science for several years. Papers accepted by the association at the meetings at Cleveland, Ohio, and at Washington, D. C., have been presented on "The Preliminary Orbit of the Ninth Comet of 1886," and "On the Mean Logarithmic Distance of Pairs of Points in Two Intersecting Lines." He is in charge of the correspondence work in mathematics in the Chautauqua College of Liberal Arts and of the mathematical classes in the summer school at Lake Chautauqua, the principal of which is the distinguished Dr. William R. Harper, president of the new Chicago University. The selection of Professor Hoover for this latter position is of the greatest credit, as his work is brought into comparison with some of the best done anywhere.

He is a critical reader and student of the best American and European writers, and besides, is a frequent contributor to various mathematical journals, the principal of which are *School Visitor*, *Mathematical Messenger*, *Mathematical Magazine*, *Mathematical Visitor*, *Analyst*, *Annals of Mathematics*, *American Mathematical Monthly*, and *Educational Times*, of London, England.

His style is concise and his aim is elegance in form of expression of mathematical thought. While greatly interested in the various branches of pure mathematics, he is specially interested in the applications to the advanced departments of Astronomy, Mechanics, and the Physical Sciences —such as Heat, Optics, Electricity, and Magnetism. The "electives" offered in the advanced work for students in his University are among the best mathematics pursued any where in this country.

He is an active member of the Presbyterian church and greatly interested in every branch of church work. He has been an elder for a number of years and was chosen a delegate to the General Assembly meeting at Portland, Oregon, in May, 1892, serving the church in this capacity with fidelity and intelligence. In this biography of Professor Hoover, there is a valuable lesson to be learned. It is this: Energy and perseverance will bring a sure reward to earnest effort. We see how the clerk in a county seat store, in embarrassing circumstances and unknown to the world of thinkers, became the well known Professor of Mathematics and Astronomy in one of the leading Institutions of learning in the State of Ohio. "Not to know him argues yourself unknown."

THE NINE-POINT CIRCLE.

Proposition.—*If a circle be described about the pedal triangle of any triangle, it will pass through the middle points of the lines drawn from the orthocenter to the vertices of the triangle, and through the middle points of the sides of the triangle, in all, through nine points.* (2) *The center of the nine-point circle is the middle point of the line joining the orthocenter and the center of the circumscribing circle of the triangle.* (3) *The radius of the nine-point circle is half the radius of the circumscribing circle of the triangle.* (4) *The centroid of the triangle also lies on the line joining the orthocenter and the center of the circumscribing circle of the triangle, and divides it in the ratio of 2:1.* (5) *The sides of the pedal triangle intersect the sides of the given triangle in the radical axis of the circumscribing and nine-point circles.* (6) *The nine-point circle touches the inscribed and escribed circles of the triangle.*

The Pedal Triangle is a triangle formed by joining the feet of the perpendiculars drawn from the vertices of a triangle to the opposite sides.

The Orthocenter is the point of intersection of these perpendiculars.

Medial Lines, or ***Medians***, are lines drawn from the vertices of a triangle to the middle point of the opposite sides.

The Centroid is the point of intersection of the medians.

The Radical Axis of two circles is the locus of the points whose powers with respect to the two circles are equal.

Demonstration.—Let ABC be any triangle, AD, BF, and CE the perpendiculars from the vertices to the opposite sides of the triangle. O is the orthocenter. Join the points F, E, and D. Then FED is the pedal triangle. About this triangle, describe the circle $FEHDK$. It will then pass through the middle points L, N, and R of the lines, OA, OB, and OC, and the middle points H, G, and K of the sides AB, BC, and AC, in all, through nine points.

Since the angles AFO and AEO of the quadrilateral are both right angles a circle may be described about it. For the same reason circles may be described about the quadrilaterals $EBDO$ and $ODCF$. Draw the lines FR and RG. Now the angles FRE and FDE are equal, being measured by half the same arc FE. But FDE equals $2EDL$, because AD bisects the angle EDF. $\therefore$ FRO equals $2FDL$. Both being measured by the same arc OF, and FRO being two times FDL, FRO is an angle at the center; therefore, since OC is the diameter of the circle circumscribing $FODC$, R is the middle point of OC. In like manner it may be proved that OB and OA are bisected in the points N and L respectively. Draw the line RG. The angles RGC and RGB are equal to two right angles. Also the angles RGB and RED are equal to two right angles, because they are opposite angles of a quadrilateral inscribed in a circle. Therefore RGC is equal to RED. But RED is equal to OBD, because both are measured by half the arc OD. $\therefore$ The angle RGC equals the angle OBD, and consequently the line RG is parallel to the line OB. But, since RG bisects OC in R and is parallel to OB, it bisects BC in G. In like manner, it may be shown that AB and AC are bisected by the nine-point circle in the points H and K respectively. Hence, the circle passes, in all, through nine points. *Q. E. D.*

(2.) Draw the perpendiculars GP, KP, and HP from the middle points of the sides of the triangle. They all meet in a common point P which is the center of the circumscribing circle of the triangle. With P as a center and radius equal to PB, describe the circumscribing circle. Draw the perpendiculars SY SJ, and SZ to the middle points of the chords FK, EH, and

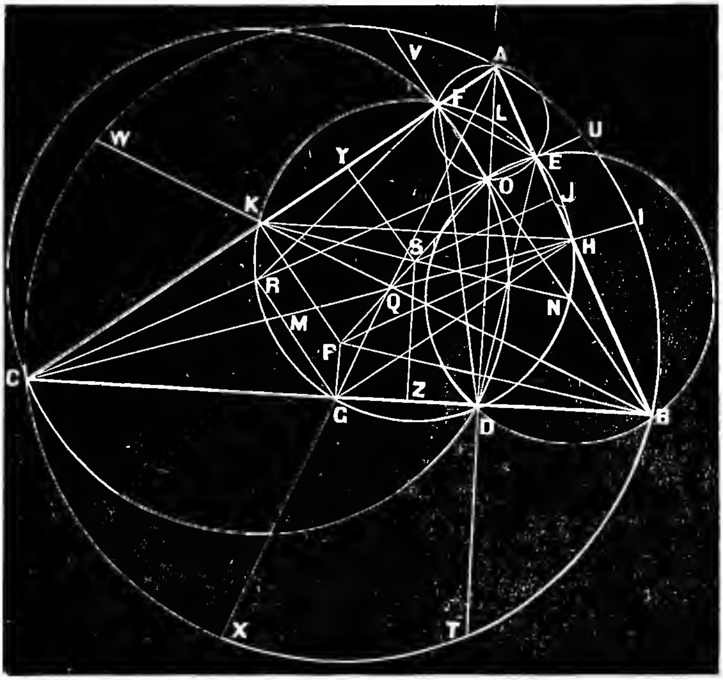


FIG. 6.

DG. These all meet in the same point S, which is the center of the nine-point circle. In the trapezoid $PHEO$, since SJ bisects EH in J and is parallel to PH, it bisects OP in S. Hence, the center of the nine-point circle is the middle point of the line joining the orthocenter and center of the circumscribing circle.
Q. E. D.

(3.) Draw the lines KN and PB. Since the angle KFN is a right angle. the line KN is a diameter of the nine-point circle. $KP=\frac{1}{2}BO=BN$. Since KP and BN are equal and parallel, $KPBN$ is a parallelogram, and consequently $KN=BP$. $\therefore$ $SN=\frac{1}{2}BP$. But SN is the radius of the nine-point circle and BP is the radius of the circumscribing circle of the triangle. Hence, the radius of the nine-point circle is half the radius of the circumscribing circle. *Q. E. D.*

(4.) Draw the medial lines BK, AG, and CH. Draw the line KH. Now the triangles KPH and BOC are similar because the sides of the one are respectively parallel to the sides of the other, and the line HK is half the line BC, because H and K are the middle points of the sides AB and AC. $\therefore BO=2KP$. The triangles KPQ and BOQ are similar, because the angles of one are respectively equal to the angles of the other. Then we have $KP:KQ::BO:BQ$ or $KP:BO::KQ:BQ$. But $BO=2KP$. $\therefore BQ=2KQ$. $\therefore Q$ is the centroid and divides the line joining orthocenter and the center of the circumscribing circle in the ratio of 2:1. *Q.E.D.*

Hence the line joining the centers of the circumscribing and nine-point circles is divided harmonically in the ratio of 2 : 1 by the centroid and orthocenter of the triangles. These two points are therefore centers of similitude of the circumscribing and nine-point circles. $\because$ Any line drawn through either of these points is divided by the circumferences in the ratio of 2 : 1.

(5.) Produce FE till it meets BC in P'. Since two opposite angles of the quadrilateral $BEFC$ are equal to two right angles, a circle may be circumscribed about it. Then we have $P'E.P'F=P'B.P'C$; therefore the tangents from P' to the circles are equal. *Q. E. D.*

(6.) Let O be the orthocenter, and I and Q the centers of the inscribed and circumscribed circles. Produce AI to bisect the arc BC in T. Bisect AO in L, and join GL, cutting AT in S. The nine-point circle passes through G, D, and L, and D is a right angle. Hence, GL is a diameter, and is therefore $=R=QA$. Therefore GL and QA are parallel. But $QA=QT$, therefore $GS=GT=CT\sin\frac{1}{2}A=2R\sin^2\frac{1}{2}A$. Also $ST=2GS\cos\theta$, θ being the angle $GST=GTS$. N being the center of the nine-point circle, its radius $=NG=\frac{1}{2}R$; and r being the radius of the inscribed circle, it is required to show that $NI=NG-r$. $NI^2=SN^2+SI^2-2SN.SI\cos\theta$. Substitute $SN=\frac{1}{2}R-GS$; $SI=TI-ST=2R\sin\frac{1}{2}A-2GS\cos\theta$; and $GS=2R\sin^2\frac{1}{2}A$, to prove the proposition. If J be the center of the escribed circle touching BC, r_1 its radius, it is shown in a similar way that $NJ=NG+r_1$.

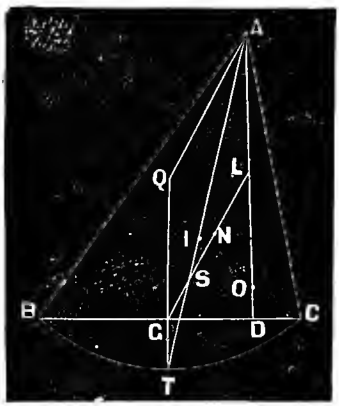


FIG. 7.

THE THREE FAMOUS GEOMETRICAL PROBLEMS OF ANTIQUITY.

The limits of this work forbid our carrying the discussion of elementary geometry further. We have given merely an outline of how the subject may be studied by the student and presented by the teacher and that is our chief aim in this work. But before leaving the subject, it will be of interest to briefly speak of three famous problems in geometry,— problems that have profoundly interested the mathematicians from the time of Plato down to the present time. These problems have been referred to before in this book so that, at this point, we shall only bring them together and speak of them more explicitly. The problems referred to are,

(i) ***The Duplication of the Cube;*** (ii) ***the Trisection of an Angle;*** (iii) ***the Quadrature of the Circle.***

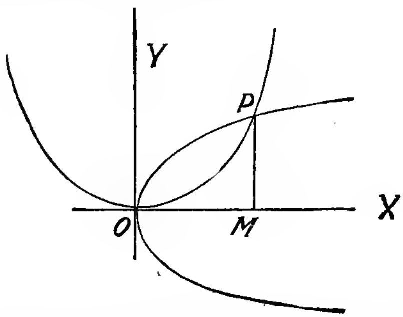
The first of these problems means to find the edge of a cube whose volume shall be twice that of a given cube; the second means to divide any given angle into three equal parts; and the third means to find the side of a square whose area shall be equal to that of a given circle.

As has been said, constructions in pure geometry or Euclidean Geometry admit of the use of an ungraduated ruler and a pair of compasses only. With this restriction, all three problems are insoluble. This is the important point to be observed. The problems are only impossible, because we are limited in the use of our instruments to a straight edge, or ungraduated ruler, and a pair of compasses. In this way many problems may be made impossible. For example, it is impossible, at present, to go across the Atlantic Ocean from Boston to Liverpool on a bicycle, but with a steamship the trip is made very easily. So too, if other instruments are used our three problems are easily solved. The solutions of the first and second problems are implicitly involved in the Galois theory as presented to-day in treatises on higher algebra. The impossibility of the solution of the third was demonstrated in 1882 by Lindemann.

The first two problems may be reduced to one, viz., that of finding two means between two given extremes. In the first problem, if we let a be the edge of given cube and x that of the required cube, then we must have $x^3 = 2a^3$, or $a : x = x : y = y : 2a$. In the second, if a is the sine of the given angle, and x the sine of one-third the angle; then $4x^3 = 3x - a$, or $1 : 4^{\frac{1}{3}}x = 4^{\frac{1}{3}}x : y = y : (3x - a)$, or $1 : x^{\frac{1}{2}} = x^{\frac{1}{2}} : (3 - 4x^2)^{\frac{1}{2}} = (3 - 4x^2)^{\frac{1}{2}} \quad a^{\frac{1}{2}}$.

The problem of the duplication of the cube was known in ancient times as the Delian problem, in consequence of a legend that the Delians had consulted Plato on the subject. It is asserted by Philoponus, that the Athenians in 430 B. C. were suffering from the plague of eruptive typhoid fever and in order to stop it consulted the oracle at Delos as to how it might be done. Apollo replied that they must double the size of the altar of Minerva which was in the form of a cube. This to the unlearned suppliant, was an easy task, and a new altar having each of its edges double that of the old one was constructed, in consequence of which the volume was increased eight-fold. This so enraged the god that he made the pestilence worse than before, and informed a fresh deputation that it was useless to trifle with him as the new altar must be a cube and have a volume exactly double that of the old one. Suspecting a mystery, the Athenians applied to Plato who referred them to the geometricians. In an Arab work, it is related that Plato replied to them, saying, Ye have been neglectful of the science of geometry and, therefore, hath God chastised you, since geometry is the most sublime of all the sciences.

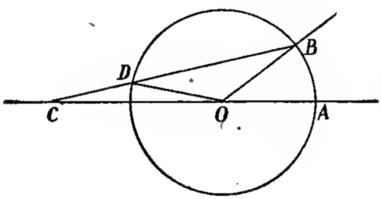
Many solutions of this problem have been given, one of which is given on page 234, by means of the Cissoid. We here give another by means of the parabola.



Let $y^2 = ax$, be the equation of the parabola whose axis coincides with axis of abscissas and $x^2 = 2ay$, the equation of the parabola whose axis coincides with the axis of ordinates. Solving these two equations, we find $y^3 = 2a^3$, that is, $PM^3 = 2a^3$. Hence, if a is the edge of the given cube PM is the edge of the required cube.

To trisect an angle, we proceed as follows:

Let AOB be the given angle. With O as center and any radius, describe a circle, ABD. Draw the secant BDC so that DC shall be equal to the radius OB. (This is impossible unless a graduated ruler is used.) Then draw OD. Then angle $BCA = \frac{1}{3}$ angle AOB. For angle DCO=angle DOC. Why? Angle $BDO = 2\times$angle



DCO. Why? Angle *DCO*+angle *CBO*=angle *BOA*. Why? ∴ Angle *AOB*=3 × angle *BCO*. Why?

The following elegant solution is due Clairaut:

Let *AOB* be the given angle. With *O* as center and any radius describe a circle. Draw *AB* and trisect it in *H* and *K*, so that *AH* = *HK* = *KB*. Bisect the angle *AOB* by *OC*, cutting *AB* in *L*. Then *AH* = 2*HL*. With focus *A*, vertex *H*, and disectrix *OC*, describe a hyperbola. Let the branch of this hyperbola which passes through *H* cut the circle in *P*. Draw *PM* perpendicular to *OC* and produce it to cut the circle in *Q*. Then by the focus and directrix property, we have $AP : PM = AH : HL = 2 : 1$. ∴ $AP = 2\,PM = PQ$. Hence, by symmetry, $AP = PQ = QR$. Hence, $AOP = POQ = QOR$.

The Quadrature of the Circle is effected by the Quadratrix. See page 238.

For a very full treatment of these problems, see Klein's *Famous Problems of Elementary Geometry,* translated from the German by Professors Beman and Smith, also see *Mathematical Recreations and Problems,* by W. W. R. Ball.

PROPOSITIONS.

1. The lines which join the middle points of adjacent sides of any quadrilateral, form a parallelogram.

2. Two medians of a triangle are equal; prove (without assuming that they trisect each other) that the triangle is isosceles.

3. In an indefinite straight line AB find a point equally distant from two given points which are not *both* on AB.

When does this problem not admit of solution?

Construct a right triangle having given:

4. The hypotenuse and the difference of the sides.

5. The perimeter and an acute angle.

6. The difference of the sides and an acute angle.

7. Construct a triangle having given the medians.

8. Construct a triangle, having given the base, the vertical angle, and (1) the sum or (2) the difference of the sides.

9. Describe a circle which shall touch a given circle at a given point, and also touch a given straight line.

10. Describe a circle which shall pass through two given points and be tangent to a given line.

11. Find the point inside a given triangle at which the sides subtend equal angles.

12. Describe a circle which shall be tangent to two intersecting straight lines and passing through a given point.

13. Divide a triangle in two equal parts by a line perpendicular to a side.

14. Inscribe in a triangle, a rectangle similar to a given rectangle.

15. Construct an equilateral triangle equivalent to a given square.

16. Trisect a triangle by straight lines drawn from a given point in one of its sides.

17. Draw through a given point a straight line, so that the part of it intercepted between a given straight line and a given circle may be divided at the given point in a given ratio.

18. Construct a circle equivalent to the sum of three given circles.

19. Find the locus of a point such that the sum of its distances from three given planes is equal to a given straight line.

20. Construct a sphere tangent to three given spheres and passing througn a given point.

21. Draw a circle tangent to three given circles.

NOTE.—This proposition is known as the *Taction Problem*. It was proposed and solved by Apollonius, of Pergæ, A. D. 200. His solution was indirect, reducing the problem to ever simpler and simpler problems. It was lost for centuries, but was restored by Vieta. The first direct solution was given by Gergonne, 1813. An elegant solution of this problem is given by Prof. E. B. Seitz, *School Visitor, Vol. IV, p. 61.*

22. Construct a sphere tangent to four given spheres.

NOTE.—This problem was first solved by Fermat (1601—1665).

23. The perpendicular from the center of gravity of a tetrahedron to any plane without the tetrahedron is one-fourth of the sum of the perpendiculars from the vertices to the same plane.

1. Define: a segment of a circle, four proportional magnitudes, two similar polygons, the projection of a segment of a straight line on another straight line.

2. The sum of all the plane angles about a point is four right angles.

3. The locus of all points equally distant from two fixed points is the straight line that bisects the line joining the two points, at right angles.

4. A straight line that is perpendicular to a radius at its extremity is tangent to the circle; and conversely.

5. Two polygons that are similar to a third polygon are similar to each other.

6. If two triangles have an angle of the one equal to an angle of the other, their areas are to each other as the rectangles of the sides including those angles.

7. The ratio of the circumference of a circle to its diameter is the same for all circles.

8. Find the side of the largest square that can be cut from a tree whose circumference is 14 feet.

Cornell University — Entrance Examination, 1899.

[Proofs by limits are not, in general, satisfactory.]

1. Define: a plane, a straight line perpendicular to a plane, a straight line parallel to a plane, two parallel planes, a diedral angle, the plane angle of a diedral.

2. The sum of the face angles of a convex polyderal angle is less than four right angles.

3. The sections of a prismatic surface made by two parallel planes are equal polygons.

4. The frustum of a triangular pyramid is equal in volume to three pyramids, whose common altitude is the altitude of the frustum, and whose bases are the two bases of the frustum and a mean proportional between them.

5. To draw a plane tangent to a cylindar with circle base.

6. If two angles of a spherical triangle be equal, the opposite sides are equal.

7. The lateral area of a cone of revolution is half the product of the perimeter of its base and its slant height.

8. A cylindrical pail is 6 inches deep and 7 inches in diameter: find how much water it holds, and how much tin it takes to make it.

Cornell University — Entrance Examination, 1899.

1. Define a straight line (preferably without using the ideas of distance or direction). Also define: equal, greater, limit of a variable, length of a curve.

2. If two triangles have two sides of one equal to two sides of the other, and the included angles of the first greater than that of the second, prove that the third side of the first is greater than the third side of the second. Also prove the converse of this theorem.

3. Similar triangles (and similar polygons) are to each other as the squares on homologous sides.

4. Construct a triangle, being given the lengths of the three perdendiculars from the vertices on the opposite sides.

5. Compute the side of a regular pentagon inscribed in a circle whose radius is given.

6. Two straight lines in space have one and but one common perpendicular, and it is the shortest line that can be drawn from one to the other.

7. Compute the volume of a regular octahedron whose edge is two units.

8. Show how to find the radius of a given sphere by means of measurements made on the surface.

9. Prove that the volume of a cone is equal to the area of the base multiplied by one-third of the altitude. Also state without proof the chain of propositions which lead up to this theorem.

10. Find the locus of a point in space the ratio of whose distances from two given points is equal to a given constant.

Cornell University — Scholarship Examination, 1899.
Time, 3 *hours.*

1. State and prove the theorem of Menelaus — and its inverse.

2. *Prove*: Circles described on any three chords from one point of a circle as diameters, have their other three points of intersection co-straight.

3. Explain the Peaucellier Cell.

4. State and prove the *dual theorem* of: The pole of any straight through a point is on the polar of the point.

5. *Prove*: The diagonal triangle of a cyclic quadrangle is self-conjugate (its own reciprocal polar).

6. (*a*) Explain what is meant by a *cross ratio* of a range of four points.

(*b*) If $(PQRS)=3$, what are the other distinct cross ratios of the same range, and what are their magnitudes?

(*c*) Deduce the distinct values of the cross ratios of a harmonic range.

7. Prove Pascal's theorem concerning a cyclic hexagon.

(*b*) State the corollaries as the number of sides is diminished.

8. What is the radical axis of two circles?

(*b*) *Prove*: The difference between the squares of the tangents from any point to two circles equals twice the rectangle of the center sect of the circles and the perpendicular sect from the point to the radical axis.

Examination in Halsted's Modern Geometry. The University of Texas, 1894. *Time,* 3 *hours.*

1. From the common notion "solid" as a starting point, define *surface, line, point, plane, straight line.*

(*b*) Define *angle,* and point out the angles determined by a bi-radial.

(*c*) What is meant by the statement that two magnitudes are *equivalent?* — that one magnitude is *greater* than another?

(*d*) Define the terms, *multiple, submultiple, fraction, ratio.*

(*e*) What is the direct meaning of the statement that two series of magnitudes are *proportional?* State the simplest criteria of proportionality between two series of magnitudes in which to every one of either series there corresponds one of the other series. Apply the test to two such series where it is fulfilled; and again where one criterion fails.

(*f*) When are two figures *perspective?* — when *similar?*

2. Discuss the problem: To describe a circle tangent to three given intersecting straights, not all through the same point. — Also, the analogous problem in a sphere.

3. (*a*) State the conditions of congruence of two plane triangles.
(*b*) State the conditions of similarity of two plane triangles.
(*c*) State the conditions of congruence of two spherical triangles.

4. (*a*) Investigate the form of the quadrilateral made by joining the mid points of consecutive sides of a quadrilateral.
(*b*) Its relative size.

5. Divide a sect internally and externally in a given ratio.

6. (*a*) Prove: If four sects are proportional the rectangle of the extremes is equivalent to the rectangle of the means.
(*b*) State the inverse.

7. (*a*) Prove the Pythagorean theorem without using any other concerning *equivalence* of figures.
(*b*) Prove: The altitude to the hypothenuse is a mean proportional between the segments of the hypothenuse.

8. (a) When is one spherical triangle $A'B'C'$ the *polar* of another, ABC?
(*b*) Prove: The sides of a spherical triangle intersect the corresponding sides of its polar on the polar of their orthocenter.

Examination in Halsted's Elementary Synthetic Geometry. The University of Texas, 1894. *Time, 3 hours.*

ALGEBRA

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1. ***Algebra*** is that branch of mathematics which treats of the general theory of operations with numbers, or quantities.

The operations of ordinary abstract arithmetic are a particular case of algebra. Thus, algebra is sometimes called *generalized arithmetic* and in turn arithmetic is sometimes called *specialized algebra.*

2. ***An Operation,*** in mathematics, is the act of passing from one number to another, the second number having a definite relation to the first.

3. ***An Operator,*** in mathematics, is a letter or symbol designating the operation to be performed.

Thus, $+$, $-$, $\times$, $\div$, $\sqrt{}$, $\frac{d}{dx}$, or D_x, and $\int$ are operators.

4. ***The Fundamental Laws*** of Algebra are the ***Commutative Law;*** the ***Associative Law;*** and the ***Distributive Law.***

The Commutative Law states, that *additions and subtractions may be performed in any order; also the factors of a product may be taken in any order.*

The Associative Law states, that *the terms of an expression may be grouped in any order.* Thus, $a+b-c+d-e=(a+b)-c+(d-e)=a+(b-c)+(d-e)=a+b-(c-d)-e$.
Also *the factors of a product may be grouped in any order.*

The Distributive Law states, that *the product of a compound expression by a single factor is the algebraic sum of the partial products of each term of the compound expression by that factor.*

Thus, $(a+b)c=ac+bc$.

For a very excellent treatment of these laws, the reader is referred to Chrystal's Algebra, Part I.

Note.—The establishment of these three great laws was left for the present century, the chief contributors thereto being De Morgan, Hankel, and Peacock. These men were working at the philosophy of the first principles. Hamilton, Grassmann, and Pierce threw a flood of light on the subject by conceiving algebras whose laws differ from those of ordinary algebra.

The student who would become proficient in mathematics should make himself familiar with ordinary algebra, for it is the basis of all advanced mathematical subjects. For example, in analytical geometry, the subject matter is geometry while the language is algebraic; in the calculus, the subject matter may be physics, astronomy, or political economy while the language is algebraic. We shall solve a few problems in algebra, leaving the student to gain a thorough knowledge of the subject by a study of such works as Chrystal's Algebra, 2 vols.

I. An estate was divided among three persons in such a way that the share of the first was three times that of the second, and the share of the second twice that of the third. The first received $900 more than the third. How much did each receive? [From *Hall and Knight's College Algebra, p. 69, prob 40.*]

II.
1. Let x=the number of dollars in the share of the third person. Then
2. $2x$=the number of dollars in the share of the second, and
3. $6x=3\times2x$=the number of dollars in the share of the first.
4. $6x-x=5x$=the number of dollars the first received more than the second.
5. 900=the number of dollars the first received more than the second.
6. $\therefore 5x=900$,
7. $x=\frac{1}{5}$ of 900=180, the number of dollars in the share of the third,
8. $2x=360$, the number of dollars in the share of the second,
9. $6x=1080$, the number of dollars in the share of the first.

III. ∴ $180=share of the third,
$360= " " " second, and
$1080= " " " first.

I. The length of a room exceeds its breadth by 8 feet; if each had been increased by 2 feet, the area would have been increased by 60 square feet; find the original dimensions of the room. [From *Hall and Knight's College Algebra, p. 69, prob. 33.*]

1. Let x=the number of feet in the breadth of the room. Then
2. $x+8$=the number of feet in the length, and
3. $(x+8)x=x^2+8x$=the number of square feet in the area.
4. $x+2$=the conditional number of feet in the breadth, and
5. $x+2+8=x+10$=the conditional number of feet in the length. Then

II. 6. $(x+2)(x+10)=x^2+12x+20$=the conditional number of square feet in the area.
7. $(x^2+12x+20)-(x^2+8x)=4x+20$=the number of square feet in the increase in the area,
8. 60=the number of square feet in increase in area.
9. $\therefore 4x+20=60$,
10. $4x=40$, by subtracting 20 from both sides.
11. $x=\frac{1}{4}$ of $40=10$, the number of feet in the breadth, and
12. $x+8=18$, the number of feet in the length.

III. $\therefore$ { 10 feet=the breadth, and 18 feet=the length.

I. A takes 3 hours longer than B to walk 30 miles; but if he doubles his pace, he takes 2 hours less time than B; find their rates of walking. [From *Hall and Knight's College Algebra, p. 164, prob. 32.*]

II. 1. Let x=A's rate in miles per hour, and
2. y=B's " " " "
3. Then $\frac{30}{x}$=number of hours it takes A to travel 30 miles.
4. $\frac{30}{y}$=number of hours it takes B to travel 30 miles.
5. $\therefore \frac{30}{x}-\frac{30}{y}=3$, by the first condition of the problem.
6. $2x$=A's conditional rate in miles per hour.
7. Then $\frac{30}{2x}=\frac{15}{x}$=number of hours it takes A to travel 30 miles.
8. $\therefore \frac{30}{y}-\frac{15}{x}=2$, by the second condition of the problem.
9. $\frac{15}{x}-\frac{15}{y}=\frac{3}{2}$, from (5).
10. $\frac{15}{y}=3\frac{1}{2}$, by adding (8) and (9).
11. $\therefore \frac{1}{y}=\frac{7}{30}$.
12. $\therefore y=\frac{30}{7}=4\frac{2}{7}$=number of miles B travels per hour.
13. $\frac{15}{x}-\frac{15}{4\frac{2}{7}}=\frac{3}{2}$, by substituting for y in (9).
14. $\frac{15}{x}-\frac{7}{2}=\frac{3}{2}$,

15. $\frac{15}{x}=\frac{10}{2}=5$,

16. $\frac{1}{x}=\frac{1}{3}$,

17. $\therefore x=3$=number of miles A travels per hour.

III. $\therefore$ { 3 miles=A's rate per hour, and
$4\frac{2}{7}$ miles=B's rate per hour. }

I. In a mile race A gives B a start of 44 yards and beats him 51 seconds. In the second trial A gives B a start of 1 minute and 15 seconds, and is beaten by 88 yards. Find the rate of each in miles per hour. [*Todhunter's Algebra, p. 103, prob. 23. Wentworth's Complete Algebra, p. 179, prob. 55.*]

II.

1. Let x=A's rate in yards per second.
2. y=B's rate in yards per second.
3. 1 mile=1760 yards.
4. $\frac{1760}{x}$=time it takes A to run a mile.
5. $\frac{1760-44}{y}=\frac{1716}{y}$=time B was running in the first trial.
6. $\therefore \frac{1716}{y}-\frac{1760}{x}=51$ . . . . (1).
7. $\frac{1760}{y}$=time it takes B to run 1 mile.
8. $\frac{1760-88}{x}=\frac{1672}{x}$=time A was running in second trial.
9. $\therefore \frac{1760}{y}-\frac{1672}{x}=75$ . . . . (2).
10. $\frac{39}{40y}-\frac{1}{x}=\frac{51}{1760}$ . . . . (3), by dividing (1) by 1760.
11. $\frac{20}{19y}-\frac{1}{x}=\frac{75}{1672}$ . . . . (4), by dividing (2) by 1672.
12. $\frac{59}{760y}=\frac{531}{33440}$, by subtracting (3) from (4); whence
13. $y=\frac{44}{9}$ yards, B's rate per second.
14. $\therefore$ 10 miles$=\frac{3600}{1760}\times\frac{44}{9}$=B's rate in miles per hour.
15. $\frac{1}{x}=\frac{15}{88}$, by substituting the value of y in (3) and changing the signs.
16. $\therefore x=\frac{88}{15}$ yards=A's rate in yards per second.

17. $\therefore$ 12 miles$=\frac{3600}{1760}\times\frac{88}{15}=$A's rate in miles per hour.

III. $\therefore$ $\begin{cases}\text{10 miles=B's rate per hour.}\\ \text{12 miles=A's rate per hour.}\end{cases}$

I. THE QUADRATIC EQUATION.

5. $ax^2+bx+c=0$, is the general quadratic equation.

$$ax^2+bx+c=a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)=a\left(x^2+\frac{b}{a}x+\frac{b^2}{4a^2}+c-\frac{b^2}{4a^2}\right)$$

$$=a\left\{\left(x+\frac{b}{2a}\right)^2-\left(\frac{b^2}{4a^2}-c\right)\right\}=a\left\{\left(x+\frac{b}{2a}\right)^2-\left(\frac{b^2-4ac}{4a^2}\right)\right\}$$

$$=a\left\{x+\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}\right\}\left\{x+\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}\right\}=0.$$

$$\therefore\begin{cases}x+\dfrac{b+\sqrt{b^2-4ac}}{2a}=0\\ x+\dfrac{b-\sqrt{b^2-4ac}}{2a}=0\end{cases}\qquad\therefore\begin{cases}x=-\dfrac{b+\sqrt{b^2-4ac}}{2a}\\ x=-\dfrac{b-\sqrt{b^2-4ac}}{2a}\end{cases}$$

In the solution of exercises involving quadratic equations, students should be thoroughly grounded in the method of completing the square, and this method should not be superseded by the Hindoo Method nor the Method of Factoring, though this latter method should receive due consideration. When the method is thoroughly impressed upon the mind of the student he should be encouraged to solve examples by merely substituting in the general formulæ above.

Thus, find the values of x satisfying the equation
$2x^2+5x-33=0$. Here, $a=2$, $b=5$, and $c=-33$.

$$\text{Then}\begin{cases}x_1=-\dfrac{b+\sqrt{b^2-4ac}}{2a}=-\dfrac{5+\sqrt{25-4\cdot2\cdot-33}}{2\cdot2}=-\dfrac{5+17}{4}=-5\tfrac{1}{2}.\\ x_2=-\dfrac{b-\sqrt{b^2-4ac}}{2a}=-\dfrac{5-\sqrt{25-4\cdot2\cdot-33}}{2\cdot2}=-\dfrac{5-17}{4}=\ 3.\end{cases}$$

I. Find the price of eggs per score when 10 more in $62\frac{1}{2}$ cents' worth lowers the price $31\frac{1}{4}$ cents per hundred. [*Wentworth's Complete Algebra, p. 216, prob. 8.*]

1. Let x=price per score.
2. $\frac{x}{20}$=price per egg.

II.

3. $5x$=price of 100 eggs.

4. $62\frac{1}{2}\div\frac{x}{20}=\frac{1250}{x}$=number of eggs in $62\frac{1}{2}$ cents' worth.

5. $\frac{1250}{x}+10=\frac{1250+10x}{x}$=number if 10 more be added.

6. $62\frac{1}{2}\div\frac{1250+10x}{x}=\frac{25x}{500+4x}$=price of each egg, if 10 more be added to $62\frac{1}{2}$ cents' worth.

7. $\frac{625x}{125+x}$=price of 100 eggs.

8. $\therefore 5x-\frac{625x}{125+x}=31\frac{1}{4}$.

9. $x^2-\frac{25x}{4}=\frac{3125}{4}$, by clearing of fractions, transposing, combining, and dividing through by the coefficient of x^2.

10. $x^2-\frac{25}{4}x+\frac{625}{64}=\frac{50625}{64}$, by completing the square.

11. $x-\frac{25}{8}=\pm\frac{225}{8}$, by extracting the square root.

12. $x=31\frac{1}{4}$, or -25.

III. $\therefore$ $31\frac{1}{4}$c.=price of the eggs per score. The negative value is inadmissible in an arithmetical sense.

$$\left.\begin{aligned} x^2+y &= 11 \quad (1) \\ x+y^2 &= 7 \quad (2) \end{aligned}\right\}$$ Find the values of x and y. [From *Schuyler's Complete Algebra, p. 368, prob. 4.*]

II.

1. $x^2-9=2-y$ (3), by transposing in (1).
2. $x-3=4-y^2$ (4), by transposing in (2).
3. $x-3=\frac{2-y}{x+3}=\frac{2}{x+3}-\frac{y}{x+3}$ (5), by dividing (3) by $x+3$.
4. $\therefore \frac{2}{x+3}-\frac{y}{x+3}=4-y^2$, or
5. $y^2-\frac{y}{x+3}=4-\frac{2}{x+3}$ (6), by transposing.
6. $y^2-\frac{y}{x+3}+\frac{1}{4(x+3)^2}=4-\frac{2}{x+3}+\frac{1}{4(x+3)^2}$ (7), by completing the square in (6).
7. $y-\frac{1}{2(x+3)}=2-\frac{1}{2(x+3)}$ (8), by extracting the square root of (7).

8. $\therefore$ y=2, by transposing $\frac{1}{2(x+3)}$ in (8).
9. $x+4=7$, by substituting value of y in (2).
10. $\therefore x=3$.

III. $\begin{cases} x=3. \\ y=2. \end{cases}$

x and y have three other values in addition to those found. For a number of different solutions giving the four values of x and y, see *The American Mathematical Monthly*.

$$\left.\begin{array}{l} x^2+xy+y^2=a^2 \\ x^2+xz+z^2=b^2 \\ y^2+yz+z^2=c^2 \end{array}\right\} \text{ Find } x, y, \text{ and } z.$$

For a solution of this example, see *The Mathematical Magazine*, edited by Dr. Artemas Martin, Washington, D. C.

I. Find two numbers whose product is equal to the difference of their squares, and the sum of whose squares is equal to the difference of their cubes. [*Ray's Higher Algebra, p. 230, prob. 9.*]

II.
1. Let x=greater number,
2. and y=lesser number.
3. $xy=x^2-y^2$ (1).
4. $x^2+y^2=x^3-y^3$ (2).
5. Let $x=ay$, then
6. $ay^2=a^2y^2+y^2$ (4), by substituting the value of x in (1).
7. $a^2-a=1$ (5), by dividing (4) by y^2, and arranging,
8. whence $a=\frac{1}{2}(1\pm\sqrt{5})$, by completing the square of (5), and extracting the square root, and transposing.
9. $y=\frac{5\pm\sqrt{5}}{2(1\pm\sqrt{5})}=\frac{1}{2}\sqrt{5}$, by substituting the value of a in (2).
10. $x=ay=\frac{1}{2}(1\pm\sqrt{5})\ (\frac{1}{2}\sqrt{5})=\frac{1}{4}(5\pm\sqrt{5})$.

III. $\therefore \begin{cases} x=\frac{1}{4}(5\pm\sqrt{5}), \text{ and} \\ y=\frac{1}{2}\sqrt{5} \end{cases}$

II. INDETERMINATE FORMS.

6. The symbol, 0, is defined by the equation $a-a=0$. It is not used to denote nothing, but is used to denote the absence of quantity.

All operations upon this symbol are impossible. Thus, 0×5,

$5 \div 0$, 5^0, are all impossible operations. Standing apart from the conditions imposed upon the quantities from which 0 arises by certain limitations, the operations above indicated are absolutely meaningless. But such indicated operations, and many others of the same nature, do very frequently occur in mathematical investigations, and when they do thus arise they must be interpreted in such a way as to conform to the fundamental laws of mathematics.

In conformity to these laws, $0 \times a = a \times 0 = 0$; $a+0=0+a=a$; $0 \div b = 0$.

The symbol, ∞, is used to represent a quantity that is larger than any assignable quantity, however large.

What meaning shall be attached to the following indicated operation, $a \div 0$? It is impossible to perform this operation. Suppose we divide a by h. This is possible, provided a and h are real numbers, and is indicated thus, $\frac{a}{h}$. Now what happens to the value of the fraction $\frac{a}{h}$, if we conceive h to diminish indefinitely? We know that as the denominator of a fraction decreases, the value of the fraction increases. Hence, if the value of the denominator becomes very small, the value of the fraction becomes very large. If the denominator becomes less than any assignable quantity, the value of the fraction becomes larger than any assignable quantity. All this is concisely and accurately stated as follows: $\lim_{h \doteq 0} \left[\frac{a}{h}\right] = \infty$, or $\left[\frac{a}{h}\right]_{h \doteq 0} = \infty$.

Hence, the inaccurate though common form, $a \div 0$, must be interpreted in the light of the above explanation and, therefore, $a \div 0 = \infty$, briefly though inaccurately expressed.

$a^0 = 1$, for all finite values of a; but is indeterminate if a is ∞. $0^a = 0$, for all finite values of a; but is infinite if a is infinite. $a \div \infty = 0$, for all finite values of a; but is indeterminate if a is infinite. 0^0 is indeterminate. $\infty - \infty$, $\infty \div \infty$, $0 \div 0$, $0 \times \infty$, and 1^∞ are all indeterminate. But when these forms occur in any mathematical investigation, they usually have a determinate value.

Thus, $\left.\frac{a^2-x^2}{a-x}\right]_{x \doteq a} = \frac{0}{0} = a+x\Big]_{x \doteq a} = 2a$. Here, $\frac{0}{0} = 2a$.

All the other forms may be reduced to the form $\frac{0}{0}$.

Thus, $\infty - \infty = \frac{a}{0} - \frac{a}{0} = \frac{a-a}{0} = \frac{0}{0}$; $0 \times \infty = 0 \times \frac{a}{0} = \frac{0 \times a}{0} = \frac{0}{0}$;

$\infty \div \infty = \frac{a}{0} \div \frac{b}{0} = \frac{a}{0} \times \frac{0}{b} = \frac{0 \times a}{0 \times b} = \frac{0}{0}$; log. $(1^\infty) = \infty$ log. $1 = \infty \times 0 = \frac{0}{0}$.
Since the log. (1^∞) is indeterminate, the quantity 1^∞ is indeterminate.

It is important that the student masters the meaning of these forms, as they occur very frequently in the higher mathematics. For example, the Differential Calculus rests largely upon the proper interpretation of $\frac{0}{0}$.

1. Find the limiting value of $\frac{x^2-5x+6}{x^2-10x+16}$ when $x \doteq 2$.

$$\left.\frac{x^2-5x+6}{x^2-10x+16}\right]_{x\doteq 2} = \frac{0}{0} = \left.\frac{(x-2)(x-3)}{(x-2)(x-8)}\right]_{x\doteq 2} = \left.\frac{x-3}{x-8}\right]_{x\doteq 2} = \frac{-1}{-6} = \tfrac{1}{6}.$$

2. Find the limiting value of $\frac{x^2+2x}{2x^2+3x}$ when $x \doteq 0$ and when $x \doteq \infty$.

(1) $$\left.\frac{x^2+2x}{2x^2+3x}\right]_{x\doteq 0} = \frac{0}{0} = \left.\frac{x(x+2)}{x(2x+3)}\right]_{x\doteq 0} = \left.\frac{x+2}{2x+3}\right]_{x\doteq 0} = \tfrac{2}{3}.$$

(2) $$\left.\frac{x^2+2x}{2x^2+3x}\right]_{x\doteq \infty} = \frac{\infty}{\infty} = \left.\frac{x(x+2)}{x(2x+3)}\right]_{x\doteq \infty} = \left.\frac{x+2}{x+3}\right]_{x\doteq \infty} = \frac{\infty}{\infty} =$$

$$\left.\frac{1+\frac{2}{x}}{1+\frac{3}{x}}\right|_{x\doteq \infty} = \frac{1}{1} = 1.$$

3. Find the limiting value of $\frac{x^2+6x-16}{x^3-12x+16}$ when $x \doteq 2$ and when $x \doteq \infty$.

(1) $$\left.\frac{x^2+6x-16}{x^3-12x+16}\right]_{x\doteq 2} = \frac{0}{0} = \left.\frac{(x+8)(x-2)}{(x+8)(x-2)^2}\right]_{x\doteq 2} = \left.\frac{1}{x-2}\right]_{x\doteq 2} =$$
$$\frac{1}{0} = \infty.$$

(2) $$\left.\frac{x^2+6x-16}{x^3-12x+16}\right]_{x\doteq \infty} = \frac{\infty}{\infty} = \left.\frac{(x+8)(x-2)}{(x+8)(x-2)^2}\right]_{x\doteq \infty} = \left.\frac{1}{x-2}\right]_{x\doteq \infty} =$$
$$\frac{1}{\infty} = 0.$$

4. Find the limiting value of $\frac{\sqrt{a}-\sqrt{x}}{\sqrt[3]{a}-\sqrt[3]{x}}$ when $x \doteq a$.

$$\left.\frac{\sqrt{a}-\sqrt{x}}{\sqrt[3]{a}-\sqrt[3]{x}}\right]_{x\doteq a} = \frac{0}{0}.$$

Let $x=a+h$, where h approaches 0 as a limit. Then

$$\left.\frac{\sqrt{a}-\sqrt{a+h}}{\sqrt[3]{a}-\sqrt[3]{a+h}}\right]_{h\doteq0}=\left.\frac{\sqrt{a}-\left(a^{\frac{1}{2}}+\frac{1}{2}a^{-\frac{1}{2}}h+\frac{\frac{1}{2}(1-\frac{1}{2})}{1\cdot2}a^{-\frac{3}{2}}h^2+\text{etc.}\right.}{\sqrt[3]{a}-\left(a^{\frac{1}{3}}+\frac{1}{3}a^{-\frac{2}{3}}h+\frac{\frac{1}{3}(1-\frac{1}{3})}{1\cdot2}a^{-\frac{5}{3}}h^2+\text{etc}\right.}\right]_{h\doteq0}$$

$$\left.\frac{-\left(\frac{1}{2}a^{-\frac{1}{2}}+\frac{\frac{1}{2}(1-\frac{1}{2})}{1\cdot2}a^{-\frac{3}{2}}h+\text{etc.}\right)h}{-\left(\frac{1}{3}a^{-\frac{2}{3}}+\frac{\frac{1}{3}(1-\frac{1}{3})}{1\cdot2}a^{-\frac{5}{2}}h+\text{etc.}\right)h}\right]_{h\doteq0}=\frac{3}{2}a^{\frac{1}{6}}.$$

EXAMPLES.

1. Find the limiting value of $\left(\frac{1+x}{1-x}\right)^{\frac{1}{x}}$ when $x\doteq0$.

2. Find the limiting value of $\frac{\sqrt{3x-a}-\sqrt{x-a}}{x-a}$ when $x\doteq a$.

3. Find the limiting value of $\frac{1-\sqrt[3]{x}}{1-\sqrt[6]{x}}$ when $x\doteq-1$.

4. Find the limiting value of $\frac{x^3+1}{x^2-1}$ when $x\doteq1$.

5. $\left.\frac{a^x-b^x}{x}\right]_{x\doteq0}=$what?

6. $\left.\frac{e^{mx}-e^{ma}}{x-a}\right]_{x\doteq a}=$what?

7. $\left.\frac{1-x+\log.\ x}{1-\sqrt{2x-x^2}}\right]_{x\doteq1}=$what?

8. $\Big[1+ax\Big]^{\frac{n}{x}}_{x\doteq0}=$what?

$$\Big[1+ax\Big]^{\frac{n}{x}}_{x\doteq0}=\left[\Big[1+ax\Big]^{\frac{1}{x}}_{x\doteq0}\right]^n=\left[\left[1+\frac{1}{x}(ax)+\frac{\frac{1}{x}\left(\frac{1}{x}-1\right)}{1\cdot2}ax^2\right.\right.$$

$$\left.\left.+\frac{\frac{1}{x}\left(\frac{1}{x}-1\right)\left(\frac{1}{x}-2\right)}{1\cdot2\cdot3}(ax)^3+\text{etc.}\right]_{x\doteq0}\right]^n,$$ by the Binomial Theorem,

$$=\left[\left[1+a+\frac{1(1-x)}{1\cdot2}a^2+\frac{1(1-x)(1-2x)}{1\cdot2\cdot3}a^3+\text{etc.}\right]_{x\doteq0}\right]^n=$$

$$\left[1+a+\frac{1}{2!}a^2+\frac{1}{3!}a^3+\frac{1}{4!}a^4+\text{etc.}\right]^n.$$

$$\therefore\ 1^\infty=\left[1+a+\frac{1}{2!}a^2+\frac{1}{3!}a^3+\text{etc.}\right]^n=\left[1+1+\frac{1}{2!}+\frac{1}{3!}+\text{etc.}\right]^n$$

$=[2.71828+]^n=e^n$, when $a=1$. Hence, 1^∞ is indeterminate.

III. PROBABILITY.

7. ***Definition.*** If an event can happen in a ways and fail in b ways, and each of these ways is equally likely, the ***probability,*** or the chance of its happening is $\frac{a}{a+b}$, and the chance of its failing is $\frac{b}{a+b}$.

That is, the probability of an event happening is the number of favorable ways the event can happen divided by the total number of ways it can happen, and the probability of its failing is the number of ways the event can fail divided by the total number of ways it can happen.

For example, if in a lottery there are 6 prizes and 23 blanks, the chance that a person holding 1 ticket will win a prize is $\frac{6}{29}$, and his chance of not winning is $\frac{23}{29}$.

8. The reason for the above definition may be made clear by the following considerations:

If an event can happen in a ways and fail to happen in b ways, and all these ways are equally likely to occur, we can assert that the chance of its happening is to its chance of failing as a to b. Thus if the chance of its happening is represented by ka, where k is an undetermined constant, then the chance of its failing is kb.

$\therefore$ Chance of happening + chance of failing $= k(a+b)$. Now the event is certain to happen or to fail; therefore, the sum of the chances of happening and failing must represent *certainty*. If, therefore, we agree to take certainty as our unit, we have

$$k(a+b)=1, \text{ or } k=\frac{1}{a+b}.$$

$\therefore$ the chance that the event will happen is $\frac{a}{a+b}$, and the chance the event will not happen is $\frac{b}{a+b}$.

9. The subject of probability, from the mathematical point of view, is a very difficult one. That it is a very important subject, no one will deny after having read Jevons's *Principles of Science*, 2 vols., in the first volume of which he has given considerable attention to its treatment. Our space is too limited to give here more than a passing reference to the subject. Those who desire to study the subject thoroughly, should read Tod-

hunter's *History of the Theory of Probability;* LaPlace's *Théorie Analytique des Probabilities*, 1812 (the most exhaustive treatment of the subject ever written); and Whitworth's *Choice and Chance*. The last named book has a large number of problems worked out in full.

EXAMPLES.

1. I. What is the chance of throwing a number greater than 4 with an ordinary die whose faces are numbered from 1 to 6?

II.
1. 6=number of ways the die can fall.
2. 2=number of ways the die can fall so as to give a number *greater* than 4, viz., 5 and 6.
3. ∵ $\frac{2}{6}=\frac{1}{3}$=the required chance, by definition.

III. ∴ the required chance is $\frac{1}{3}$.

2. I. Find the chance of throwing *at least* one ace in a single throw with two dice.

II.
1. 6=number of ways one of the dice may fall.
2. 6×6=36=number of ways the two dice may fall, since with each of the six ways the first may fall, there are six ways in which the second may fall.
3. 5=number of ways one die may be thrown without the ace coming up.
4. 25=5×5=number of ways the two dice can be thrown without either of them being ace.
5. ∴ $\frac{25}{36}$=the chance of not throwing an ace.
6. ∴ $1-\frac{25}{36}=\frac{11}{36}$=the chance of throwing *at least* one ace.

III. ∴ the chance of throwing at least one ace is $\frac{11}{36}$.

3. I. If four coins are tossed find the chance that there should be two heads and two tails.

II.
1. $\frac{1}{2}$=chance of head or tail with one coin.
2. $\frac{1}{16}=(\frac{1}{2})^4$=chance of all heads, tossing 4 coins.
3. $\frac{1}{16}=(\frac{1}{2})^4$=chance of all tails, tossing 4 coins.
4. $\frac{1}{4}=\frac{4}{1}\times\frac{1}{2}(\frac{1}{2})^3$=chance of 1 head and 3 tails.
5. $\frac{1}{4}=\frac{4}{1}\times(\frac{1}{2})^3(\frac{1}{2})$=chance of 1 tail and 3 heads.
6. $\frac{3}{8}=\frac{4\times3}{1\times2}(\frac{1}{2})^2(\frac{1}{2})^2$=chance of 2 heads and 2 tails.

III. ∴ the chance of throwing 2 heads and 2 tails is $\frac{3}{8}$.

4. I. A bag contains 5 white, 7 black, and 4 red balls; find the chance that three balls drawn at random are all white.

II. 1. $\frac{16\cdot15\cdot14}{1\cdot2\cdot3}$=number of ways 3 objects can be selected from 16 objects.

2. $\frac{5\cdot4\cdot3}{1\cdot2\cdot3}$=number of ways 3 objects can be selected from 5 objects, which is the number of ways the 3 white balls may be selected from the 5 white balls.

3. $\therefore \frac{(5\cdot4\cdot3)\div(1\cdot2\cdot3)}{(16\cdot15\cdot14)\div(1\cdot2\cdot3)}=\frac{1}{56}$=the required chance.

III. $\therefore$ the required chance is $\frac{1}{56}$.

I. If three pennies be piled up at random on a horizontal plane, what is the probability that the pile will not fall down?

The pile will stand if the common center of gravity of the second and third coins falls on the surface of the first or bottom coin.

Let r be the radius of a penny; then the center of the second coin may fall anywhere in a circle whose radius is $2r$ and center the center of the surface of the first or bottom coin, and the center of the third coin may fall anywhere in a circle whose radius is $2r$ and the center the center of the surface of the second coin. The number of positions of the center of the second coin is therefore proportional to $4\pi r^2$, and for every one of these positions the center of the third coin can have $4\pi r^2$; hence the total number of positions of the second and third coins is proportional to $16\pi^2 r^4$.

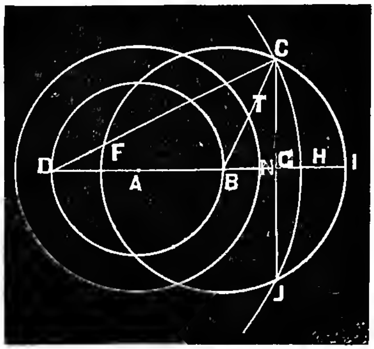


FIG. 8-

We must now determine in how many of these $16\pi^2 r^4$ positions the pile will stand.

Let A be the center of the first or bottom coin, and B the center of the second coin. Take $AD=AB$, and with center D and radius $2r$ describe the arc CHJ. If the center of the third coin is on the surface $CFJH$, the second and third coins will remain on the first, since $BN=NH$, $BT=TC$, and the pile will not fall down.

When AB is not greater than $\frac{1}{2}r$, the circle CHJ will not cut the surface of the second coin, and the pile will stand if the center of the third coin is anywhere on the second.

Let $AB=AD=x, S=$surface $CFJH$, and $p=$the probability required; then $DB=2x$, $BG=\frac{3r^2-4x^2}{4x}$, $DG=\frac{3r^2+4x^2}{4x}$,

$CG=\frac{1}{4x}\left[16r^4-(5r^2-4x^2)^2\right]^{\frac{1}{2}}$, arc $CI=r\cos^{-1}\left(\frac{3r^2-4x^2}{4rx}\right)$,

arc $CH=2r\cos^{-1}\left(\frac{3r^2+4x^2}{8rx}\right)$, $S=\pi r^2+4r^2\cos^{-1}\left(\frac{3r^2+4x^2}{8rx}\right)$

$-r^2\cos^{-1}\left(\frac{3r^2-4x^2}{4rx}\right)-\frac{1}{2}\left[16r^4-(5r^2-4x^2)^2\right]^{\frac{1}{2}}$, and

$$p=\frac{1}{16\pi^2r^4}\int_0^{\frac{1}{2}r}\pi r^2.2\pi x dx+\frac{1}{16\pi^2r^4}\int_{\frac{1}{2}r}^{r}S.2\pi x dx,$$

$=\frac{1}{16}+\frac{1}{8\pi r^4}\int_{\frac{1}{2}r}^{r}(S-\pi r^2)xdx.$ $\quad\int r^2\cos^{-1}\left(\frac{3r^2-4x^2}{4rx}\right)xdx,$

$$=\tfrac{1}{2}r^2x^2\cos^{-1}\left(\frac{3r^2-4x^2}{4rx}\right)-\tfrac{1}{2}r^4\cos^{-1}\left(\frac{5r^2-4x^2}{4r^2}\right)$$

$+\frac{1}{16}r^2\left[16r^4-(5r^2-4x^2)^2\right]^{\frac{1}{2}}$, $\quad\int 4r^2\cos^{-1}\left(\frac{3r^2+4x^2}{8rx}\right)xdx$

$$=2r^2x^2\cos^{-1}\left(\frac{3r^2+4x^2}{8rx}\right)+\tfrac{1}{2}r^4\cos^{-1}\left(\frac{5r^2-4x^2}{4r^2}\right)$$

$-\frac{1}{4}r^2\left[16r^4-(5r^2-4x^2)^2\right]^{\frac{1}{2}}$, $\int\frac{1}{2}\left[16r^4-(5r^2-4x^2)^2\right]^{\frac{1}{2}}xdx$

$=\frac{1}{2}r^4\cos^{-1}\left(\frac{5r^2-4x^2}{4r^2}\right)-\frac{1}{32}(5r^2-4x^2)\left[16r^4-(5r^2-4x^2)^2\right]^{\frac{1}{2}}.$

$$\therefore p=\frac{1}{16}\ \frac{1}{8\pi r^4}\Big[\tfrac{1}{2}r^2x^2\cos^{-1}\left(\frac{3r^2-4x^2}{4rx}\right)-2r^2x^2\times$$

$\cos^{-1}\left(\frac{3r^2+4x^2}{8rx}\right)-\frac{1}{2}r^4\cos^{-1}\left(\frac{5r^2-4x^2}{4r^2}\right)+\frac{1}{32}(5r^2-4x^2)\times$

$\sqrt{16r^4-(5r^2-4x^2)^2}\Big]_{\frac{1}{2}r}^{r}=\frac{1}{16}-\frac{3}{16\pi}\left(\frac{3}{16}\sqrt{15}-2\sin^{-1}\frac{1}{4}\right).$

NOTE.—This solution is due to Artemas Martin, M. A., Ph. D., LL. D., member of the London Mathematical Society, member of the Edinburgh Mathematical Society, member of the Mathematical Society of France, member of the American Mathematical Society, member of the Philosophical Society of Washington and Fellow of the American Association for the Advancement of Science, Washington, D. C. He is one of the peers of mathematical science.

BIOGRAPHY.

ARTEMAS MARTIN, M. A., PH. D., LL. D.

This eminent mathematician was born in Steuben county, N. Y., August 3, 1835. Early, his parents moved to Venango county, Pa., where they lived for many years. Dr. Martin had no schooling in his early boyhood, except a little primary instruction; but by self-application and indefatigable energy which have told the story of many a great man, he has become familiar to every mathematician and lover of science in every civilized country of the world.

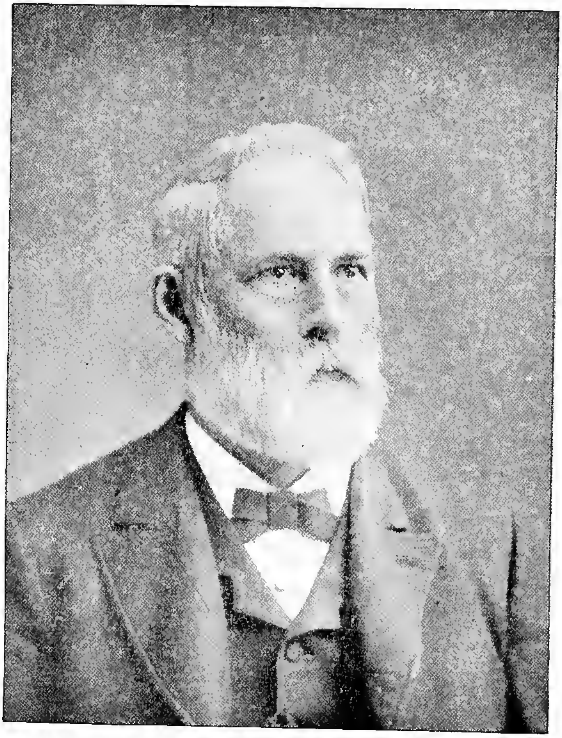
He was never a pupil at school, except when quite small, until in his fourteenth year. He had learned to read and write at home, but knew nothing of Arithmetic. At fourteen he commenced the study of Arithmetic, and after spending two winters in the district school, he commenced the study of Algebra. At seventeen, he studied Algebra, Geometry, Natural Philosophy, and Chemistry in the Franklin Select School, walking two and one-half miles night and morning. Three years after, he spent two and one-half months in the Franklin Academy, studying Algebra and Trigonometry. This finished his schooling. He taught district schools four winters, but not in succession. He was raised on a farm, and worked at farming and gardening in the summer; chopped wood in the winter; and after the discovery of oil in Venango county, worked at drilling oil wells a part of his time, always devoting his "spare moments" to study.

In the spring of 1869, the family moved to Erie county, Pa., where he resided until he entered the U. S. Coast Survey Office in 1885. While in Erie county, after 1871, he was engaged in market-gardening, which he carried on with great care and skill. He began his mathematical career when in his eighteenth year, by contributing solutions to the *Pittsburg Almanac*, soon after contributing problems to the "Riddler Column" of the *Philadelphia Saturday Evening Post*, and was one of the leading contributors for twenty years.

In the summer of 1864 he commenced contributing problems and solutions to *Clark's School Visitor*, afterward the *Schoolday Magazine*, published in Philadelphia. In June, 1870, he took charge of the "Stairway Department" as editor, the mathematical department of which he had conducted for some years before. He continued in charge as mathematical editor till the magazine was sold to Scribner & Co., in the spring of 1875, at which time it was merged into "*St. Nicholas.*"

In September, 1875, he was chosen editor of a department of higher mathematics in the *Normal Monthly*, published by Prof. Edward Brooks, Millersville, Pa., and held that position till the *Monthly* was discontinued in August, 1876. He published in the *Normal Monthly* a series of sixteen articles on the Diophantine Analysis.

In June, 1877, Yale College conferred on him the honorary degree of Master of Arts (M. A.) In April, 1878, he was elected member of the London Mathematical Society. In June, 1882, Rutgers College conferred on him the honorary degree of Doctor of Philosophy (Ph. D.) March 7, 1884, he was elected a member of the Mathematical Society of France. In April, 1885, he was elected a member of the Edinburgh Mathematical Society. June 10, 1885, Hillsdale College conferred on him the honorary degree of Doctor of Laws (LL. D.) February 27, 1886, he was elected a member of the Philosophical Society of Washington. In June, 1881, he was elected Professor of Mathematics of the Normal School at Warrensburg, Mo., but did not accept the position. November 14, 1885, Dr. Martin was appointed



Yours truly
Artemas Martin

Librarian in the office of the U. S. Coast and Geodetic Survey. On August 26, 1890, he was elected a Fellow of the American Association for the Advancement of Science. On April 3, 1891 he was elected a member of the New York Mathematical Society.

All these honors have been worthily bestowed and the Colleges and Societies conferring them have done honor to themselves in recognizing the merits of one who has become such a power in the scientific world through his own efforts.

He has contributed fine problems and solutions to the following journals of the United States: *School Visitor*, *Analyst*, *Annals of Mathematics*, *Mathematical Monthly*, *Illinois Teacher*, *Iowa Instructor*, *National Educator*, *Yates County Chronicle*, *Barnes' Educational Monthly*, *Wittenberger*, *Maine Farmers' Almanac*, *Mathematical Messenger*, and *Educational Notes and Queries*, *American Mathematical Monthly*. Besides other contributions, he contributed thirteen articles on "Average" to the Mathematical Department of the *Wittenberger*, edited by Prof. William Hoover. These are believed to be the first articles published on that subject in America.

Dr. Martin has also contributed to the following English mathematical periodicals: *Lady's*, and *Gentleman's Diary*, *Messenger of Mathematics*, *The Educational Times and Reprint* and the *Mathematical Gazette.*

The *Reprint* contains a large number of his solutions of difficult "Average" and "Probability" problems, which are master-pieces of mathematical thought and skill, and they will be lasting monuments to his memory. His style is direct, clear and elegant. His solutions are neat, accurate and simple. He has that rare faculty of presenting his solution in the simplest mathematical language, so that those who have mastered the elements of the various branches of mathematics are able to understand his reasoning.

Dr. Martin is now (1899) editor of the *Mathematical Magazine*, and *The Mathematical Visitor*, two of the best mathematical periodicals published in America. These are handsomely arranged and profusely illustrated with very beautiful diagrams to the solutions, he doing the typesetting with his own hand. The typographical work of these journals is said to be the finest in America. The best mathematicians from all over the world contribute to these two journals. *The Mathematical Visitor* is devoted to Higher Mathematics, while *The Mathematical Magazine* is devoted to the solutions of problems of a more elementary nature. All solutions sent to Dr. Martin receive due credit, and if it is possible to find room for them the solutions are all published. He has thus encouraged many young aspirants to higher fields of mental activity. He is always ready to aid any one who is laboring to bring success with his work. He is of a kind and noble disposition and his generous nature is in full sympathy with every diligent student who is rising to planes of honor and distinction by self application and against adverse circumstances.

Dr. Martin has a large and valuable mathematical library containing many rare and interesting works. His collection of American arithmetics and algebras is one of the largest private collections of the kind in this country.

I. Find the average or mean distance of every point of a square from one corner.

Taking the corner from which the mean distance is to be found for the origin of orthogonal co-ordinates, and one of the sides of the square for the axis of abscissa, we have for the element of the surface $dx\,dy$, and since this element is at a distance $\sqrt{(x^2+y^2)}$ from the origin, the average distance $=\frac{1}{a^2}\int_0^a\int_0^a dxdy\sqrt{(x^2+y^2)}$

$=\frac{1}{2a^2}\left\{a\int_0^a dx\sqrt{(a^2+x^2)}+\int_0^a x^2dx\log_e\frac{a+\sqrt{(a^2+x^2)}}{x}\right\}$. But

$\int x^2dx\log_e\frac{a+\sqrt{(a^2+x^2)}}{x}=\frac{1}{3}x^3\log_e\frac{a+\sqrt{(a^2+x^2}}{x}+\frac{1}{3}a\int\frac{x^2dx}{\sqrt{(a^2+x^2)}}$

$=\frac{1}{3}x^3\log_e\frac{a+\sqrt{(a^2+x^2)}}{x}+\frac{1}{6}ax\sqrt{(a^2+x^2)}-\frac{1}{6}a^3\log_e\{x+\sqrt{(a^2+x^2)}\}$

$\therefore$ Average distance$=\frac{1}{3}a\,[\sqrt{2}+\log_e(1+\sqrt{2})]$.

NOTE.—This solution is by Prof. J. W. F. Sheffer, Hagerstown, Md., whose name may be found attached to the solutions of many difficult problems proposed in the leading mathematical journals of the United States. The above solution is taken from the *Mathematical Messenger*, published by G. H. Harville, Simsboro, La.

I. All that is known concerning the veracities of two witnesses, A and B, is that B's statements are twice as reliable as A's. What is the probability of the truth of the concurrent testimony of these two witnesses?

Let x=the probability of the truth of any one of A's statements; then $2x$=the probability of any one of B's statements. The event did occur if both witnesses tell the truth, the probability of which is $x\times2x=2x^2$. The event did not occur if both testify falsely, the probability of which is $(1-x)\times(1-2x)=1-3x+2x^2$. Hence, the probability of the occurrence of the event on the supposition that x is known is

$$p'=\frac{2x^2}{2x^2+(1-x)(1-2x)}.$$

Now, as the veracity of B can not exceed unity, the greatest value of x is found by putting $2x=1$, which gives $x=\frac{1}{2}$; hence, x can have any value from 0 to $\frac{1}{2}$.

Therefore, the probability in the problem is

$$p=\int_0^{\frac{1}{2}}p'dx\div\int_0^{\frac{1}{2}}dx=4\int_0^{\frac{1}{2}}\frac{x^2dx}{2x^2+(1-x)(1-2x)}$$

$$=64\int_0^{\frac{1}{2}}\frac{x^2dx}{(8x-3)^2+7}.$$

Let $8x-3=y$. Then $x=\frac{1}{8}(y-3)$, $dx=\frac{1}{8}dy$; the limits of y are 1 and -3, and

$$p=\frac{1}{8}\int_{-3}^{+1}\frac{(y+3)^2dy}{y^2+7}=\frac{1}{8}\int_{-3}^{+1}\left(1+\frac{6y}{y^2+7}+\frac{2}{y^2+7}\right)dy$$

$$=\left[\frac{1}{8}y+\frac{3}{8}\log_e(y^2+7)+\frac{1}{4\sqrt{7}}\tan^{-1}\left(\frac{y}{\sqrt{7}}\right)\right]_{-3}^{+1}$$

$=\frac{1}{2}-\frac{3}{8}\log_e 2+\frac{1}{4\sqrt{7}}\tan^{-1}\sqrt{7}.$

NOTE.—This solution is taken from the *Mathematical Magazine, Vol. II, p. 122.* The solution there given is credited to the author, Prof. William Hoover, and Prof. P. H. Philbrick.

I. A cube is thrown into the air and a random shot fired through it; find the chance that shot passed through opposite faces.

Let AH be the cube. Through P, a point in the face $EFGH$, draw MK parallel to HE, and draw PN perpendicular to HE. Now if PA represents the direction of the shot, it will pass through the face $ABCD$, if it strikes the face $EFGH$ anywhere within $HMPN$.

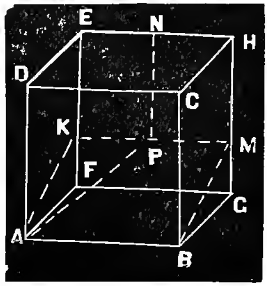


FIG. 9.

Let $AB=1$, $\angle KAF=\theta$, $\angle PAK=\phi$, and area $HMPN=u$. Then $AK=\sec\theta$, $PK=\sec\theta\tan\phi$, $FK=\tan\theta$, $AP=\sec\theta\sec\phi$, $PM=1-\sec\theta\tan\phi$, $PN=1-\tan\theta$, $u=(1-\sec\theta\tan\phi)(1-\tan\theta)$, the area of the projection of $HMPN$ on a plane perpendicular to $PA=u\cos\theta\cos\phi$, and that of $EFGH=\cos\theta\cos\phi$.

Since we are to consider all possible directions of the shot with respect to the cube, the points of intersection of PA with the surface of a sphere whose center is A, and radius unity, must be uniformly distributed. An element of the surface of this sphere is $\cos\phi\, d\theta\, d\phi$. By reason of the symmetry of the cube, the required chance is obtained by finding the number of ways the shot can pass through the opposite faces $EFGH$ and $ABCD$ between the limits $\theta=0$, and $\theta=\frac{1}{4}\pi$, and $\phi=0$ and $\phi=\tan^{-1}(\cos\theta)=\phi'$, and the number of ways it can pass through the face $EFGH$ between the limits $\theta=0$ and $\theta=\frac{1}{2}\pi$, and $\phi=0$ and $\phi=\frac{1}{2}\pi$; and then dividing the former by the latter. Hence, the chance required is

$$p=\frac{\int_0^{\frac{1}{4}\pi}\int_0^{\phi'} u\cos\theta\cos^2\phi\, d\theta\, d\phi}{\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\cos\theta\cos^2\phi\, d\theta\, d\phi}=\frac{4}{\pi}\int_0^{\frac{1}{4}\pi}\int_0^{\phi'} u\cos\theta\cos^2\phi\, d\theta\, d\phi,$$

$$=\frac{2}{\pi}\int_0^{\frac{1}{4}\pi}(\cos\theta-\sin\theta)\tan^{-1}(\cos\theta)\,d\theta,$$

$$=\frac{2}{\pi}\Big[(\sin\theta+\cos\theta)\tan^{-1}(\cos\theta)-\theta+\sqrt{2}\tan^{-1}(\tfrac{1}{2}\sqrt{2}\tan\theta)$$

$$-\tfrac{1}{2}\log_e(1+\cos^2\theta)\Big]_0^{\frac{1}{4}\pi}=\frac{1}{\pi}[4\sqrt{2}\tan^{-1}(\tfrac{1}{2}\sqrt{2})+\log_e(\tfrac{4}{3})-\pi].$$

NOTE.—This solution is due to Professor Enoch Beery Seitz, member of the London Mathematical Society, and late Professor of Mathematics in the North Missouri State Normal School, Kirksville, Mo.

BIOGRAPHY.

PROF. E. B. SEITZ, M. L. M. S.

Professor Seitz, a distinguished mathematician of his day, was born in Fairfield Co., O., Aug. 24, 1846, and died at Kirksville, Mo., Oct. 8, 1883. His father, Daniel Seitz, was born in Rockingham Co., Va., Dec. 17, 1791, and was twice married. His first wife's name was Elizabeth Hite, of Fairfield Co., O., by whom he had eleven children. His second wife's name was Catharine Beery, born in the same county, Apr. 11, 1808, whom he married Apr. 15, 1832. This woman was blessed by four sons and three daughters. Mr. Seitz followed the occupation of a farmer and was an industrious and substantial citizen. He died near Lancaster, O., Oct. 14, 1864, in his seventy-third year; having been a resident of Fairfield Co. for sixty-three years.

Professor Seitz, the third son by his father's second marriage, passed his boyhood on a farm, and like most men who have become noted, had only the advantages of a common school education. Possessing a great thirst for learning, he applied himself diligently to his books in private and became a very fine scholar in the English branches, especially excelling in arithmetic. It was told the author, by his nephew, Mr. Huddle, that when Professor Seitz was in the field with a team, he would solve problems while the horses rested. Often he would go to the house and get in the garret where he had a few algebras upon which he would satiate his intellectual appetite. This was very annoying to his father who did not see the future greatness of his son, and many and severe were the floggings he received for going to his favorite retreat to gain a victory over some difficult problem upon which he had been studying while following the plow. Though the way seemed obstructed, he completed algebra at the age of fifteen, without an instructor. He chose teaching as his profession which he followed with gratifying success until his death. He took a mathematical course in the Ohio Wesleyan University in 1870. In 1872, he was elected one of the teachers in the Greenville High School, which position he held till 1879. On the 24th of June, 1875, he married Miss Anna E. Kerlin, one of Darke county's most refined ladies. In 1879, he was elected to the chair of mathematics in the Missouri State Normal School, Kirksville, Mo., which position he held till death called him from the confines of earth, ere his star of fame had reached the zenith of its glory. He was stricken by that "demon of death," typhoid fever, and passed the mysterious shades, to be numbered with the silent majority, on the 8th of October, 1883. On the 11th of March, 1880, he was elected a member of the London Mathematical Society, being the fifth American so honored.

He began his mathematical career in 1872, by contributing solutions to the problems proposed in the "Stairway" department of the *Schoolday Magazine*, conducted by Artemas Martin. His masterly and original solutions of difficult Average and Probability problems, soon attracted universal attention among mathematicians. Dr. Martin being desirous to know what works he had treating on that difficult subject, was greatly surprised to learn that he had no works upon the subject, but had learned what he knew about that difficult department of mathematical science by studying the problems and solutions in the *Schoolday Magazine*. Afterwards, he contributed to the *Analyst*, the *Mathematical Visitor*, the *Mathematical Magazine*, the *School Visitor*, and the *Educational Times*, of London, Eng.

In all of these journals, Professor Seitz was second to none, as his logical and classical solutions of Average and Probability problems, rising as so many monuments to his untiring patience and indomitable energy and perseverance will attest. His name first appeared as a contributor to the *Educational Times* in Vol. XVIII., of *Reprint*, 1873. From that time until his death the *Reprint* is adorned with some of the finest product of his mighty intellect.

Ever yours,
E. B. Seitz

On page 21, Vol. II., he has given the above solution. This problem had been proposed in 1864 by the great English mathematician, Prof. Woolhouse, who solved it with great labor. It was said by an eminent mathematician of that day that the task of writing out a copy of that solution was worth more than the book in which it was published.

No other mathematician seemed to have the courage to investigate the problem after Prof. Woolhouse gave his solution to the world, till Professor Seitz took it up and demonstrated it so elegantly in half a page of ordinary type, that he fairly astonished the mathematicians of both Europe and America. Prof. Woolhouse was the best English authority on Probabilities even before Professor Seitz was born.

It was the solution of this problem that won for Professor Seitz the acknowledgment of his superior ability to solve difficult Probability problems over any other living man in the world.

In studying his solutions, one is struck with the simplicity to which he has reduced the solutions of some of the most intricate problems. When he had grasped a problem in its entirety, he had mastered all problems of that class. He would so vary the conditions in thinking of one special problem and in effecting a solution that he had generalized all similar cases, so exhaustive was his analysis. Behind the words he saw all the ideas represented. These he translated into symbols, and then he handled the symbols with a facility that has rarely been surpassed.

What he might have accomplished in his maturer years, had he turned his splendid powers to investigations in higher and more fruitful fields of mathematics, no man may say. The solving of problems alone is not a high form of mathematical research. While problem solving is very beneficial and essential at first, yet, if one confines himself to that sort of work exclusively, it becomes a waste of time.

He was a man of the most singularly blameless life; his disposition was amiable; his manner gentle and unobtrusive; and his decision, when circumstances demanded it, was prompt and firm as the rocks. He did nothing from impulse; he carefully considered his course and came to conclusions which his conscience approved; and when his decision was made, it was unalterable.

Professor Seitz was not only a good mathematician, but he was also proficient in other branches of knowledge. His mind was cast in a large mold. "Being devout in heart as well as great in intellect, 'signs and quantities were to him but symbols of God's eternal truth' and 'he looked through nature up to nature's God.' Professor Seitz, in the very appropriate words of Dr. Peabody regarding Benjamin Pierce, Professor of Mathematics and Astronomy in Harvard University, 'saw things precisely as they are seen by the infinite mind. He held the scales and compasses with which eternal wisdom built the earth, and meted out the heavens. As a mathematician, he was adored by his friends with awe. As a man, he was a Christian in the whole aim and tenor of life.' "

Professor Seitz did not gain his knowledge from books, for his library consisted of only a few books and periodicals. He gained such a profound insight into the subtle relations of numbers by close application with which he was particularly gifted. He was not a mathematical genius, that is, as usually understood, one who is born with mathematical power fully developed. But he was a genius in that he was especially gifted with the power to concentrate his mind upon any subject he wished to investigate. This happy faculty of concentrating all his powers of mind upon one topic to the exclusion of all others, and viewing it from all sides, enabled him to proceed with certainty where others would become confused and disheartened. Thread by thread and step by step, he took up and followed out long lines of thought and arrived at correct conclusions. The darker and more subtle the question appeared to the average mind, the more eagerly he investigated it. No conditions were so complicated as to discourage him. His logic was overwhelming.

"THE POND PROBLEM."

I. On a dark night a circular pond is formed in a circular field. A man undertakes to cross the field. Find the chance that he walks into the pond.

Note.—This problem was proposed by Dr. Artemas Martin and published by him in the *Mathematical Visitor* as problem 300. The first published solution of this problem was given by the author and appeared in the Dec. No., of Vol. VII., of the *American Mathematical Monthly*, where also a second solution is given by Professor G. B. M. Zerr. Prof. Zerr solves more difficult problems than any other man in America.

PROBLEMS.

1. State as a theorem the fact implied in the following equations: $9^2-7^2=4\cdot8$; $5^2-3^2=4\cdot4$; $93^2-91^2=4\cdot92$; $3^2-1^2=4\cdot2$. Prove it, and then express the theorem in its most general terms.

2. How find the highest common factor of two polynomials that cannot be readily factored? Prove your answer. Illustrate by finding the H. C. F. of

$$6x^3+7x^2-5x \text{ and } 15x^4+31x^3+10x^2.$$

3. A cistern can be filled in 4 hours by two pipes running together, and in $6\frac{1}{4}$ hours by one of the pipes alone. In how many hours can the other pipe fill the same cistern?

4. If $\sqrt{a}$ and $\sqrt{b}$ are surds, prove that $\sqrt{a}\pm\sqrt{b}$ cannot be a rational number.

5. Simplify $$\frac{\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}+\dfrac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}}{\dfrac{2}{\dfrac{x^2+y^2}{x^2-y^2}+\dfrac{x-y}{\dfrac{1}{x}+\dfrac{1}{y}}}}$$

Check your work by substituting $x=4$ and $y=1$, both in the given expression and in the simplified form, and comparing results.

6. Given the two simultaneous equations

$$x^3+y^3=57 \text{ and } x+y=3;$$

find all the pairs of values of x and y that satisfy them.

Cornell University — Entrance Examination, 1899.

T.

1. Resolve $\dfrac{8-3x-x^2}{x(x+2)^2}$ into partial fractions.

2. At an election there are 4 candidates, and 3 members to

be elected, and an elector may vote for any number of candidates not greater than the number to be elected. In how many ways may an elector vote?

3. Find, by using logarithms, the value of $\sqrt{41.72} \times (.054)^{\frac{2}{3}}$.

4. Show that for any two quantities, the square of their geometric mean is equal to the product of their arithmetic and harmonic means.

5. Draw the graph of the function $x^3 + x^2 + x - 100$; and find the root between 4 and 5, correct to three places of decimals, of the equation $x^3 + x^2 + x - 100 = 0$.

6. If h, k, are the roots of $ax^2 + bx + c = 0$, find the value of $\frac{1}{h^2} + \frac{1}{k^2}$.

7. The square of x varies as the cube of y, and $x = 3$ when $y = 4$. Find the value of y when $x = \frac{1}{\sqrt{3}}$.

Cornell University — Entrance Examination, 1899.

1. (*a*) Simplify the expression

$$\frac{\frac{2bc}{b+c} - b}{\frac{1}{c} + \frac{1}{b-2c}} + \frac{\frac{2bc}{b+c} - c}{\frac{1}{b} + \frac{1}{c-2b}}$$

Show how the symmetry of this expression may be made to serve as a partial check upon the result.

(*b*) As x becomes more and more nearly equal to 2, what value does the fraction $\frac{x^2 - 6x + 8}{x^2 - 5x + 6}$ approach? What is the value of this fraction when $x = 2$?

2. The sum of the three digits of a number is 9; the digit in hundreds' place equals $\frac{1}{8}$ of the number composed of the two other digits, and the digit in units' place equals $\frac{1}{8}$ of the number composed of the two other digits; what is the number?

3. *Prove* that: (*a*) $x^n + y^n$ is exactly divisible by $x + y$ if n is any odd, positive integer whatever.

(*b*) If $a + b$ is constant, then $a \cdot b$ is greatest when $a = b$.

4. (*a*) State what seem to you to be the chief differences between algebra and arithmetic.

(*b*) Define negative number, subtraction, and multiplication, and show how, *from your definition*, the following rules may be deduced: (1) "change the sign of the subtrahend and proceed as in addition"; (2) give the product the positive or the negative sign according as the two factors have like or unlike signs."

5. Given the equation $ax^2+bxy+cy^2=0$ in which a, b, and c are real, — the ratio $x:y$ being unknown. Find the sum and also the product of the roots (i. e. of the values of the ratio $x:y$) of this equation. If a approaches 0 relatively to b and c, what happens to these roots? For what relative values of a, b, and c are the two roots equal? One twice the other? Both imaginary? T.

Cornell University — Scholarship Examination, 1899.

1. Two men, A and B, had a money-box containing $210, from which each drew a certain sum daily; this sum being fixed for each, but different for the two. After six weeks, the box was empty. Find the sum which each man drew daily from the box; knowing that A *alone* would have emptied it five weeks earlier than B *alone*.

Obtain *two solutions,* and interpret the *negative answer.*

2. Solve the equation

$$\frac{x+b}{2a}+\frac{2a}{x-b}=1-\frac{2a}{b}\left(1-\frac{2a-b}{x-b}\right)$$

3. Reduce to their lowest common denominator

$$\frac{1}{12x^3-2x^2-20x-6} \text{ and } \frac{1}{4x^3-6x^2-4x+6}$$

and find, and reduce to their lowest terms, the *difference* and the *quotient* of these two fractions.

4. Write out $(x-y)^9$.

Find the 8th term of $\left(\frac{4\sqrt[4]{b^3}}{3a^5}-\frac{1}{2}a\sqrt{a}.b^{-\frac{1}{5}}\right)^9$; reducing the answer to the simplest possible form, in which it is free from negative and fractional exponents, and has only one radical sign.

Harvard University — Entrance Examination, 1893.

1. Resolve into three factors $(x^2-x)^3-8$.
2. Find the greatest common factor of

$$x^3+5x^2+6x \text{ and } 3x^3+7x^2+3x+2.$$

3. Solve the equation $\sqrt{x-4}+\sqrt{x-11}-\sqrt{2x+9}=0$.

4. Simplyfy

$$\frac{\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}+\dfrac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}}{\dfrac{x^2+y^2}{x^2-y^2}+\dfrac{\frac{2}{x+y}}{\frac{1}{x}+\frac{1}{y}}}$$

5. Solve the simultaneous equations

$$3(x^2 + xy) = 40y,\ x - y = 2.$$

6. What is the geometrical mean between

$$2x - 3 \text{ and } 2x^3 + x^2 - 4x - 3?$$

7. A and B start at the same time from the same point in the same direction. A goes at the uniform rate of 60 miles per day; B goes 14 miles the first day, 16 miles the second day, 18 miles the third day and so on. At the end of 50 days who will be ahead, and by how much?

Massachusetts Institute of Technology — Entrance Examination in Advanced Algebra, 1892.

1. (*a*) Show that

$$\frac{(n+1)\ (n+2)\ (n+3)}{2+3} - 1 = \frac{n}{6}(n^2 + 6\ n + 11).$$

(*b*) Find the algebraic expression which when divided by $x^2 - 2x + 1$ gives a quotient $x^2 + 2x + 1$ and a remainder $x - 1$.

2. (*a*) Reduce to a common denominator (arranging the terms of the numerator according to ascending powers of x)

$$\frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2} + \frac{D}{(x+1)^3}$$

(*b*) Having given that: $A = 2$, $B - 2A = 0$, $C - 2B + 3A = -3$, $D - 2C + 3B = -1$, $E - 2D + 3C = 0$; find the values of B, C, D, and E.

3. Solve for x and y, $2x - 3y, + 14 = 0$, $5y - 4x = 26$.

4. (*a*) Simplify $x^{3p+q}x^{p-4r}(x^2)^{q-2r} \div x^{4p-5r}$.

(*b*) Multiply $x^n + x^{\frac{n}{2}} + 1$ by $x^{-n} + x^{\frac{n}{2}} + 1$.

5. Solve for x (*a*) $x^2 + 2a^2 = 3ax$. Also

$$(b)\quad x^{\frac{1}{3}} + 3x^{-\frac{1}{3}} = 4.$$

6. Solve the simultaneous equations

$$(b)\quad \frac{x}{2} + \frac{y}{5} = 5.$$

$$\frac{2}{x} + \frac{5}{y} = \frac{5}{6}.$$

Also (*b*) $(x + y) = 2a$, $x^2 + y^2 = 2a^2$.

Princeton University — Entrance Examination, 1893.

1. Write the factors of the following expressions:

$$x^4 - (x - 6)^2 \text{ and } m^6 - 64n^6.$$

2. Simplify $\dfrac{x-1-\dfrac{12}{x+4}}{x-5-\dfrac{12}{x+3}}$

3. A and B can do a piece of work in m days; B and C can do it in n days; C and D in p days; and D and A in r days. In how many days can all working together do it?

4. Multiply $x+y\sqrt{-z}$ by $y-z\sqrt{-x}$.

5. Solve the equation $y^2+2(a+6)y=-18a$.

6. Extract the square root of

$$x^3y^{-\frac{2}{3}}-4x^{\frac{3}{2}}y^{-\frac{1}{6}}+6-4x^{-\frac{3}{2}}y^{\frac{1}{3}}+x^{-3}y^{\frac{2}{3}}.$$

7. Simplify. $\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{a}}}$

8. Write the 6th term of $(a-2b)^{\frac{1}{2}}$.

Yale University — Entrance Examination, 1893.

1. (*a*) Solve the equation $ax^2+bx+c=0$.

(*b*) What relation must exist between its coefficients in order (1) that its roots may be imaginary, (2) that they may be real and equal, (3) that they may be real and unequal?

(c) If the coefficient a diminishes without limit, what limits, if any, do the roots respectively approach?

2. Make the first members of the following equations perfect squares, without introducing fractions:

$$2x^2-3x=2,\ 3x^2-8x=-4.$$

3. Solve completely the simultaneous equations $x^2+xy+y^2=19$, $x^2-xy+y^2=7$, and group distinctly the corresponding values of x and y.

4. Convert 3.14159 into a continued fraction, and obtain four convergents. What is the limit of the error in taking the fourth convergent for the value of the decimal?

5. (*a*) Derive the formula for the number of permutations of m things taken n at a time.

(*b*) From 10 different things in how many ways can a selection of 4 be made?

6. (*a*) Write equivalents for the following expressions: $\log_a 1$; $\log_a a$; $\log_a 0$, if $a>1$; $\log a^2 - \log b^2$;

$$\log \sqrt{\left(\frac{a^2-b^2}{c^3}\right)^5}$$

(b) Given the mantissa of $\log_{10} 257 = 0.40993$, to find $\log_{10} \sqrt{0.257}$.

7. Given $\log_a N$ and $\log_a b$, to find $\log_b N$.

Sheffield Scientific School of Yale University—Entrance Examination in Advanced Algebra, 1892.

BIOGRAPHY.

RENE′ DESCARTES.

René Descartes, the first of the modern school of mathematicians, was born at La Haye, a small town on the right bank of the Creuse and about midway between Tours and Poitiers, on March 31st, 1596, and died at Stockholm, on February 11th, 1650. "The house is still shown where he was born, and a *metairie* about three miles off still retains the name of Les Cartes. His family on both sides was of Poitevin descent and had its headquarters in the neighboring town of Châtterault, where his grandfather had been a physician. His father, Joachim Descartes, purchased a commission as counsellor in the Parlement Rennes and thus introduced the family into that demi-noblesse of the robe of which, in stately isolation between the bourgeoisie and the high nobility, maintained a lofty rank in the hierarchy of France. For one-half of each year required for residence the elder Descartes removed, with his wife, Jeanne Brochard, to Rennes. Three children, all of whom first saw the light at La Haye, sprang from the union, — a son, who afterwards succeeded to his father in the Parlement, a daughter who married a M. du Crevis, and a second son, René. His mother, who had been ailing beforehand, never recovered from her third confinement; and the motherless infant was intrusted to a nurse, whose care Descartes in after years remembered by a small pension."*

Descartes, who early showed an inquisitive mind, was called by his father, "my philosopher." At the age of eight, Descartes was sent to the school of La Flèche, which Henry IV had lately founded and endowed for the Jesuits, and here he continued from 1604 to 1612. Of the education here given, of the equality maintained among the pupils, and of their free intercourse, he spoke at a later period in terms of high praise. Descartes himself enjoyed exceptional privileges. His feeble health excused him from the morning duties, and thus early he acquired the habit of matutinal reflection in bed, which clung to him throughout life. When he visited Pascal in 1647, he told him that the only way to do good work in mathematics and to preserve his health was never to allow any one to make him get up in the morning before he felt inclined to do so. Even at this period he had begun to distrust the authority of tradition and his teachers.

Two years before leaving school (1610) he was selected as one of twenty-four gentlemen who went forth to receive the heart of the murdered king as it was borne to its resting place at La Flèche. During the winter of 1612, he completed his preparations for the world by lessons in horsemanship and fencing; and then in the spring of 1613 he started for Paris to be introduced to the world of fashion. Fortunately the spirit of dissipation did not carry him very far, the worst being a passion for gaming. Here through the medium of the Jesuits he made the acquaintance of Mydorge, one of the foremost mathematicians of France, and renewed his schoolboy friendship with Father Mersenne, and together with them he devoted the two years of 1615 and 1616 to the study of mathematics.

"The withdrawal of Mersenne in 1614 to a post in the provinces was the signal for Descartes to abandon social life and shut himself up for nearly two years in a secluded house of the Faubourg St. Germain. Accident, however, betrayed the secret of his retirement; he was compelled to leave his mathematical investigations and to take a part in entertainments, where the only thing that chimed in with his theorizing

***Britannica Encyclopedia*, Ninth Edition.

RENE' DESCARTES.

reveries was the music. The scenes of horror and intrigue which marked the struggle for supremacy between the various leaders who aimed at guiding the politics of France made France no fit place for a student and held out little honorable prospect for a soldier. Accordingly, in May, 1617, Descartes, now twenty-one years of age, set out for the Netherlands, and took service in the army of Prince Maurice of Orange, one of the greatest generals of the age, who had been engaged for some time in a war with the Spanish forces in Belgium. At Breda, he enlisted as a volunteer, and the first and only pay which he accepted he kept as a curiosity through life. There was a lull in the war; and the Netherlands were distracted by the quarrels of Gomarists and Arminians. During the leisure thus arising, Descartes one day, as he roved through Breda, had his attention drawn to a placard in the Dutch tongue; and as the language of which he never became perfectly master, was then strange to him, he asked a bystander to interpret it in either French or Latin. The stranger, who happened to be Isaac Beeckman, principal of the College of Dort, offered with some surprise to do so into Latin, if the inquirer would bring him a solution of the problem — for the advertisement was one of those challenges which the mathematicians of the age, in the spirit of the tournament of chivalry, were accustomed to throw down to all comers, daring them to discover a geometrical mystery known as they fancied to themselves alone. Descartes promised and fulfilled; and a friendship grew up between him and Beeckman — broken only by the literary dishonesty of the latter, who in later years took credit for the novelty contained in a small essay on music (Compendium Musicae) which Descartes wrote at this period and intrusted to Beeckman."*

The unexpected test of his mathematical attainments afforded by the solution of the problem referred to, its solution costing him only a few hours study, made the uncongenial army life distasteful to him, but under family influence and tradition, he remained a soldier, and was pursuaded at the commencement of the thirty years' war to volunteer under Count de Bucquoy in the army of Bavaria. The winter of 1619, spent in quarters at Neuburg on the Danube, was the critical period in his life. Here, in his warm room (*dans un poêle*), he indulged in those meditations which afterwards led to the *Discours de la Methode* (Discourse of Method.) It was here that, on the eve of St. Martin's Day, November 10, 1619, he "was filled with enthusiasm, and discovered the foundations of a marvelous science."

He retired to rest with anxious thoughts of his future career, which haunted him through the night in three dreams, that left deep impressions on his mind. "Next day," he says, "I began to understand the first principles of my marvelous discovery." Thus the date of his philosophical conversion is fixed to a day. This day marks the birth of modern mathematics. His discovery, viz., the coöperation of ancient geometry and algebra, is epoch-making in the history of mathematics.

It is frequently stated that Descartes was the first to apply algebra to geometry. This statement is not true, for Vieta and others had done this before him, and even the Arabs sometimes used algebra in connection with geometry. "The new step that Descartes did take was the introduction into geometry of an analytical method based on the notion of variables and constants, which enabled him to represent curves by algebraic equations. In the Greek geometry, the idea of motion was wanting, but with Descartes it became a very fruitful conception. By him a point was determined in position by its distances from two fixed lines or axes. These distances varied with every change of position in the point. This geometric idea of *co-ordinate representation* together with the algebraic idea of *two variables in one equation* having an indefinite number of simultaneous values, furnished a method for the study of loci,

***Encyclopedia Britannica*, Ninth Edition.

which is admirable for the generality of its solutions. Thus the entire conic sections of Apollonius is wrapped up and contained in a single equation of the second degree."†

"Descartes found in mathematics, as did Kant and Comte, the type of all faultless thought; and he proved his appreciation of his insight by the invention of a new symbolic mechanism and artifice for the applications of algebra to geometry (*Analytic Geometry,* as it is now called, which, in a growing sense, let it be said, existed before him), and by his discoveries in the theory of equations, which were fundamental in their importance."*

After a short sojourn in Paris, Descartes moved to Holland, then at the height of its power. There for twenty years he lived, giving up all his time to philosophy and mathematics. Science, he says, may be compared to a tree, metaphysics is the root, physics is the trunk, and the three chief branches are mechanics, medicine, and morals, these forming the three applications of our knowledge, namely, to the external world, to the human body, and to the conduct of life; and with these subjects alone his writings are concerned.

He spent the time from 1629 to 1633 writing *Le Monde,* a work embodying an attempt to give a physical theory of the universe; but finding its publication likely to bring on him the hostility of the Church, and having no desire to pose as a martyr, he abandoned it. The incomplete manuscript was published in 1664.

He then devoted himself to composing a treatise on universal science; this was published at Leyden in 1637 under the title Discours de la méthode pour bien conduire sa raison et chercher la verité dans les sciences. and was accompanied with three appendices entitled La Dioptrique, Les Météores, and La Géométrie. It is from the last of these that the invention of analytical geometry dates. In 1641, he published a work called Meditations, in which he explained at some length his views of philosophy as sketched out in the *Discourse.* In 1644, he issued the Principia Philosophiae, the greater part of which was devoted to physical science especially the laws of motion and the theory of vortices. In his theory of vortices, he commences with a discussion of motion; and then lays down ten laws of nature, of which the first two are almost identical with the first two as laid down by Newton. The remaining eight are inaccurate. He next proceeds to a discussion of the nature of matter which he regards uniform in kind though there are three forms of it. He assumes that the matter of the universe is in motion, that this motion is constant in amount, and that the motion results in a number of vortices. He states that the sun is the center of an immense whirlpool of this matter, in which the planets float and are swept round like straws in a whirlpool of water.

Each planet is supposed to be the center of a secondary whirlpool by which its satellites are carried, and so on. All of these assumptions are arbitrary and unsupported by any investigation. It is a little strange that a man who began his philosophical reasonings by doubting all things and finally coming to *cogito, ergo sum* should have made assumptions so groundless.

While Descartes was a philosopher of a very high type, yet his fame will ever rest on his researches in mathematics. The first important problem solved by Descartes in his geometry is the problem of Pappus, viz.: "Given several straight lines in a plane, to find the locus of a point such that perpendiculars, or, more generally, straight lines at given angles, drawn from the point to the given lines, shall satisfy that the product of certain of them shall be in given ratio to the product of the rest." "The most important case of this problem is to find the locus of a point such that the product of the perpendiculars on m given lines be in a constant

†*Cajori's History of Mathematics.*
**The Open Court,* August, 1898.

ratio to the product of the perpendiculars on n other given straight lines. The ancients had solved this geometrically for the case $m = 1$, $n = 1$, and the case $m = 1$, $n = 2$. Pappus had further stated that if $m = n = 2$, the locus was a conic, but he gave no proof. Descartes also failed to prove this by pure geometry, but he showed that the curve was represented by an equation of the second degree, that is, was a conic; subsequently Newton gave an elegant solution of the problem by pure geometry."*

In algebra, Descartes expounded and illustrated the general methods of solving equations up to those of the fourth degree (and believed that his method could go beyond), stated the law which connects the positive and negative roots of an equation with the change of signs in the consecutive terms, known as Descartes' Law of Signs, and introduced the method of indeterminate coefficients for the solution of equations.

In appearance, Descartes was a small man with large head, projecting brow, prominent nose, and black hair coming down to his eyebrows. His voice was feeble. Considering the range of his studies he was by no means widely read, had no use for Greek, as is shown by his disgust when he found that Queen Christina devoted some time each day to its study, and despised both learning and art unless something tangible could be extracted therefrom. In philosophy, he did not read much of the writings of others. In disposition, he was cold and selfish. He never married, and left no descendants, though he had one illegitimate daughter, Francine, who died in 1640, at the age of five.

In 1649, through the instigation of his close personal friend, Chanut, he received an invitation to the Swedish court, and in September of that year he left Egmond for the north. Here, on the 11th of February, 1650, he died of inflammation of the lungs brought about by too close devotion to the sick-room of his friend Chanut, who was dangerously ill with the same disease. — By B. F. Finkel. From *the American Mathematical Monthly*.

**Ball's Short Account of the History of Mathematics.*

BIOGRAPHY.

SIR ISAAC NEWTON.

Isaac Newton was born in Lincolnshire, England, December 25, 1642, and died at Kensington, London, March 20, 1727.

His father, who was a yeoman farmer, died before Newton was born, and it was intended that Newton should carry on the paternal farm. He was sent to school at Grantham. At first he showed no signs of extraordinary powers. One day while in a quarrel with one of his schoolmates, who was older and stronger, he received a severe kick from his antagonist, which provoked in Newton an ambition to defeat his assailant by excelling in his classes. It was not long until his learning and mechanical proficiency excited attention. Among his inventions, was a clock worked by water which kept fair time.

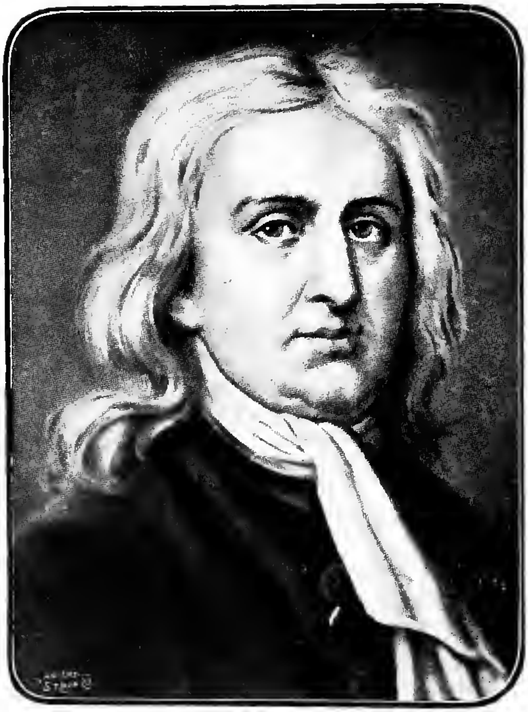
In 1856, Newton returned home to learn the business of farming but instead of applying himself in that way, he spent most of his time solving problems, making experiments, and devising mechanical models. His mother noticing this, determined to find more congenial occupation for him, and it was accordingly decided that he be sent to Trinity College, Cambridge.

Newton entered Trinity College, in 1661, as a subsizar, and for the first time found himself in the midst of surroundings likely to develop his powers. He cared little for general society or for any pursuits except science and mathematics. He had read no mathematics before entering Trinity, but was familiar with Sanderson's *Logic*, which was then frequently read as a preliminery to mathematics. At the beginning of his first term, in a stroll down to Stourbridge Fair, he picked up a book on astrology, but could not understand it on account of the geometry and trigonometry. He therefore bought a Euclid and was surprised to find how obvious the propositions seemed. He thereupon read Oughtred's Clavis and Descarte's Géométrie, the latter of which with some difficulty he mastered by himself. The interest he felt in mathematics led him to take it up as a serious study rather than chemistry.

His subsequent mathematical reading as an undergraduate was founded on Kepler's *Optics*, the the works of Vieta, Van Schooten's *Miscellanies*, and Wallis's *Arithmetica Infinitorum.* At a later time, on reading Euclid more carefully, he formed a high opinion of it as an instrument of education, and he used to express regret that he had not applied himself to geometry before proceeding to algebraic analysis.

On account of the plague, the college was closed in the summer of 1665, and for the larger part of the next year and a half Newton lived at home. This period was crowned with brilliant discoveries. He thought out the fundamental principles of his theory of gravitation, viz., that every particle of matter attracts every other particle, directly as the product of their masses and inversely as the square of the distance between them. At this time he also worked out the fluctional calculus, the greatest contribution to mathematics ever made. It was also at this time that he devised instruments for grinding lenses to particular forms other than spherical, and perhaps he decomposed solar light into different colors.

On his return to Cambridge in 1667, Newton was elected to a fellowship at his college and permanently took up his residence there. In 1669, Barrow resigned the Lucasian Chair of Mathematics in Newton's favor. When first appointed, Newton chose optics for the subject of his lectures and researches, and before the end of 1669, he had worked out the details of his discovery of the decomposition of a ray of white



Sir Isaac Newton.

light into rays of different colors by means of prisms. The complete explanation of the theory of the rainbow followed this discovery. By the end of 1675 he had worked out his corpuscular or emission theory of light.

The *Universal Arithmetic*, which is on algebra, theory of equations, and miscellaneous problems, contains the substance of Newton's lectures during the years 1673 and 1683. Among several new theorems on various points in algebra and the theory of equations, Newton here enunciates the important result that the equation whose roots are the solution of a given problem will have as many roots as there are different possible cases. He also considered how it happened that the equation to which a problem led might contain roots which did not satisfy the original question. He extended Descarte's rule of signs to give limits to the number of imaginary roots. He used the principle of continuity to explain how two real and unequal roots might become imaginary in passing through equality; thence he showed that imaginary roots must occur in pairs. He also gave rules to find the superior limit to the positive roots of a numerical equation, enunciated the theorem known by his name for finding the sum of the *n*th powers of the roots of an equation, and laid the foundation of the theory of symmetrical functions of the roots of an equation.

His chief works, taking them in their order of publication, are the *Principia*, published in 1687; the *Optics*, published in 1704; the *Universal Arithmetic*, published in 1707; the *Analysis per Series, Fluxiones*, etc., published in 1711; the *Lectiones Opticae*, published in 1729; the *Method of Fluxions*, etc., translated by J. Colson and published in 1736, and the *Methodus Differentialis*, published in 1736.

In appearance, Newton was short, and towards the close of his life rather stout, but well set, with a square jaw, brown eyes, a very broad forehead, and rather sharp features. His hair turned grey before he was thirty, and remained thick and white as silver till his death.

As to his manners, he dressed slovenly, was rather languid, and was often so absorbed in his own thoughts as to be anything but a lively companion. Many anecdotes of his extreme absence of mind when engaged in any investigation have been preserved.

Thus once when riding home from Grantham he dismounted to lead his horse up a steep hill, when he returned at the top to remount he found that he had the bridle in his hand, while his horse had slipped it and gone away. Again on a few occasions when he sacrificed his time to entertain his friends, if he left them to get more wine or for any similar reason, he would as often as not be found after the lapse of some time working out a problem, oblivious alike of his expectant guests and of his errand. He took no exercise, indulged in no amusements, and worked incessantly, often spending eighteen or nineteen hours out of the twenty-four in writing.

In character he was religious and conscientious, with an exceptionally high standard of morality, having as Bishop Burnet said, "the whitest soul" he ever knew. He modestly attributed his discoveries largely to the admirable work done by his predecessors; and once explained that, if he had seen farther than other men, it was only because he had stood on the shoulders of giants. He summed up his own estimate of his work in the sentence, "I do not know what I may appear to the world; but to myself I seem to have been only like a boy, playing on the seashore, and diverting myself in now and them finding a smoother pebble, or prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me."

In intellect he has never been surpassed and probably never been equaled. Of this his extant works are the only proper test.

Lagrange describes the *Principia* as the greatest production of the human mind and said he felt dazed at such an illustration of what man's intellect might be capable. In describing the effect of his own writings and those of Laplace, it was a favorite remark of his that Newton was not only the greatest genius that had ever existed but he was also the most fortunate, for as there is but one universe it can happen but to one man in the world's history to be the interpreter of its laws. Laplace, who is in general very sparing of his praise, makes of Newton the one exception, and the words in which he enumerates the causes which "will always assure to the *Principia* a pre-eminence above all other productions of the human intellect" have been often quoted. Not less remarkable is the homage rendered by Guass: for other great mathematicians or philosophers, he used the epithets magnus, or clarus, or clarissimus; for Newton alone he kept the prefix summus.

Abridged from the biography of Newton in Ball's *A Short History of Mathematics.*

Those readers who are interested in the controversy between Newton and Leibnitz as to the discovery of the Differential Calculus should read the entire biography in the above mentioned work, also see the *Britannica Encyclopedia*, subject *Infinitesimal Calculus.*

LEONHARD EULER.

BIOGRAPHY.

LEONHARD EULER.

Leonhard Euler (oi'ler), one of the greatest and most prolific mathematicians that the world has produced, was born at Basel, Switzerland, on the 15th day of April, 1707, and died at St. Petersburg, Russia. November the 18th (N. S.), 1783. Euler received his preliminary instruction in mathematics from his father who had considerable attainments as a mathematician, and who was a Calvinistic* pastor of the village of Riechen, which is not far from Basel. He was then sent to the University of Basel where he studied mathematics under the direction of John Bernoulli, with whose two sons, Daniel and Nicholas, he formed a lifelong friendship. Geometry soon became his favorite study. His genius for analytical science soon gained for him a high place in the esteem of his instructor, John Bernoulli, who was at the time one of the first mathematicians of Europe. Having taken his degree as Master of Arts in 1723, Euler afterwards applied himself, at his father's desire, to the study of theology and the Oriental languages, with the view of entering the ministry, but, with his father's consent, he returned to his favorite pursuit, the study of mathematics. At the same time, by the advice of the younger Bernouillis, who had removed to St. Petersburg in 1725, he applied himself to the study of physiology, to which he made useful applications of his mathematical knowledge; he also attended the lectures of the most eminent professors of Basel. While he was eagerly engaged in physiological researches, he composed a dissertation on the nature and propagation of sound. In his nineteenth year he also composed a dissertation in answer to a prize-question concerning the masting of ships, for which he received the second prize from the French Academy of Sciences.

When his two close friends, Daniel and Nicholas Bernoulli, went to Russia, they induced Catherine I, in 1727, to invite Euler to St. Petersburg, where Daniel, in 1733, was assigned to the chair of mathematics. Euler took up his residence in St. Petersburg, and was made an associate of the Academy of Sciences. In 1730 he became professor of physics, and in 1733 he succeeded his friend Daniel Bernoulli, who resigned on a plea of ill health.

At the commencement of his astonishing career, he enriched the Academical collection with many memoirs, which excited a noble emulation between him and the Bernouillis, though this did not in any way affect their friendship. It was at this time that he carried the integral calculus to a higher degree of perfection, invented the calculation of sines, reduced analytical operations to greater simplicity, and threw new light on nearly all parts of pure or abstract mathematics. In 1735, an astronomical problem proposed by the Academy, for the solution of which several eminent mathematicians had demanded several months' time,

The Encyclopedia Britannica says Euler's father was a Calvinistic minister, while W. W. R. Ball, in his *History of Mathematics*, says he was a Lutheran minister. Euler himself was a Calvinist in doctrine, as the following, which is his apology for prayer, indicates: "I remark, first, that when God established the course of the universe, and arranged all the events which must come to pass in it, he paid attention to all the circumstances which should accompany each event; and particularly to the dispositions, to the desires, and prayers of every intelligent being; and that the arrangement of all events was disposed in perfect harmony with all these circumstances. When, therefore, a man addresses God a prayer worthy of being heard it must not be imagined that such a prayer came not to the knowledge of God till the moment it was formed. That prayer was already heard from all eternity; and if the Father of Mercies deemed it worthy of being answered, he arranged the world expressly in favor of that prayer, so that the accomplishment should be a consequence of the natural course of events. It is thus that God answers the prayers of men without working a miracle."

was solved by Euler in three days with the aid of improved methods of his own, but the effort threw him into a fever which endangered his life and deprived him of his right eye, his eyesight having been impaired by the severity of the climate. With still superior methods, this same problem was solved later by the illustrious German mathematician, Gauss.

In 1741, at the request, or rather command, of Frederick the Great, he moved to Berlin, where he was made a member of the Academy of Sciences, and Professor of Mathematics. He enriched the last volume of the *Mélanges* or Miscellanies of Berlin, with five memoirs, and these were followed, with astonishing rapidity, by a great number of important researches, which were scattered throughout the annual memoirs of the Prussian Academy. At the same time, he continued his philosophical contributions to the Academy of St. Petersburg, which granted him a pension in 1742.

The respect in which he was held by the Russians was strikingly shown in 1760, when a farm he occupied near Charlottenburg happened to be pillaged by the invading Russian army. On its being ascertained that the farm belonged to Euler, the general immediately ordered compensation to be paid, and the Empress Elizabeth sent an additional sum of four thousand crowns. The despotism of Anne I caused Euler, who was a very timid man, to shrink from public affairs, and to devote all his time to science. After his call to Berlin, the Queen of Prussia who received him kindly, wondered how so distinguished a scholar should be so timid and reticent. Euler replied, "Madam, it is because I come from a country where, when one speaks, one is hanged."

In 1766, Euler, with difficulty, obtained permission from the King of Prussia to return to St. Petersburg, to which he had been originally called by Catherine II. Soon after returning to St. Petersburg a cataract formed in his left eye, which ultimately deprived him of sight, but this did not stop his wonderful literary productiveness, which continued for seventeen years — until the day of his death. It was under these circumstances that he dictated to his amanuensis, a tailor's apprentice who was absolutely devoid of mathematical knowledge, his *Anleitung zur Algebra,* or *Elements of Algebra,* 1770, a work which, though purely elementary, displays the mathematical genius of its author, and is still considered one of the best works of its class. Euler was one of the very few great mathematicians who did not deem it beneath the dignity of genius to give some attention to the recasting of elementary processes and the perfecting of elementary text-books, and it is not improbable that modern mathematics is as greatly indebted to him for his work along this line as for his original creative work.

Another task to which he set himself soon after returning to St. Petersburg was the preparation of his *Lettres à une Princesse d' Allemagne sur quelques sujects de Physique,* (3 vols. 1768–72). These letters were written at the request of the princess of Anhalt-Dessau, and contain an admirably clear exposition of the principal facts of mechanics, optics, acoustics, and physical astronomy. Theory, however, is frequently unsoundly applied in it, and it is to be observed generally that Euler's strength lay rather in pure than in applied mathematics. In 1755, Euler had been elected a foreign member of the Academy of Sciences at Paris, and sometime afterwards the academical prize was adjudged to three of his memoirs *Concerning the Inequalities in the Motions of the Planets.* The two prize-problems proposed by the same Academy in 1770 and 1772 were designed to obtain a more perfect theory of the moon's motion. Euler, assisted by his eldest son, Johann Albert, was a competitor for these prizes and obtained both. In his second memoir, he reserved for further consideration the several inequalities of the moon's motion, which he could not determine in his first theory on account of the complicated calculations in which the method he then employed had engaged him. He afterward reviewed his whole theory with the assistance of his son and Krafft and Lexell, and pursued his researches until he had constructed the new tables, which appeared with the great work in 1772. Instead of confining himself, as before, to the fruitless integration of three differential equations of the second degree, which are furnished by mathematical principles, he reduced them to three ordinates which determine the place of the moon; and he divides into classes all the inequalities of that planet, as far as they depend either on the elongation of the sun and moon, or upon the eccentricity, or the parallax, or the inclination of the lunar orbit. The inherent difficulties of this task were immensely enhanced by the fact that Euler was virtually blind, and had to carry all the elaborate computations involved in his memory. A further difficulty arose from the burning of his house and the destruction of a greater part of his property in 1771. His manuscripts were fortunately preserved. His own life only was saved by the courage of a native of Basel, Peter Grimmon, who carried him out of the burning house.

Some time after this, the celebrated Wenzell, by couching the cataract, restored his sight; but a too harsh use of the recovered faculty, together with some carelessness on the part of the surgeons, brought about a relapse. With the assistance of his sons, and of Krafft and Lexell, however, he continued his labors, neither the loss of his sight nor the infirmities of an advanced age being sufficient to check his activity. Having engaged to furnish the Academy of St. Petersburg with as many memoirs as would be sufficient to complete its acts for twenty years after his death, he in seven years transmitted to the Academy above seventy memoirs, and left above two hundred more, which were revised and completed by another hand.

Euler's knowledge was more general than might have been expected in one who had pursued with such unremitting ardor, mathematics and astronomy, as his favorite studies. He had made considerable progress in medicine, botany, and chemistry, and he was an excellent classical scholar and extensively read in general literature. He could repeat the *Aeneid* of Virgil from the beginning to the end without hesitation, and indicate the first and last line of every page of the edition which he used. But such lines from Virgil as, "The anchor drops, the rushing keel is staid," always suggested to him a problem and he could not help enquiring what would be the ship's motion in such a case.

Euler's constitution was uncommonly vigorous and his general health was always good. He was enabled to continue his labors to the very close of his life so that it was said of him, that he ceased to calculate and to breathe at nearly the same moment. His last subject of investigation was the motions of balloons, and the last subject on which he conversed was the newly discovered planet Herschel.

On the 18th of September, 1783, while he was amusing himself at tea with one of his grandchildren, he was struck with apoplexy, which terminated the illustrious career of this wonderful genius, at the age of seventy-six. His works, if printed in their completeness, would occupy from 60 to 80 quarto volumes. However, no complete edition of Euler's writings has been published, though the work has been begun twice.

He was simple, upright, affectionate, and had a strong religious faith. His single and unselfish devotion to the truth and his joy at the discoveries of science whether made by himself or others, were striking attributes of his character. He was twice married, his second wife being a half-sister of his first, and he had a numerous family, several of whom attained to distinction. His *éloge* was written for the French Academy by Condorcet, and an account of his life, with a list of his works, was written by Von Fuss, the secretary of the Imperial Academy of St. Petersburg.

As has been said, Euler wrote an immense number of works, chief of which are the following: *Introductio in Analysin Infinitorum,* 1748, which was intended to serve as an introduction to pure analytical mathematics. This work produced a revolution in analytical mathematics, as the subject of which it treated had hitherto never been presented in so general and systematic a manner. The first part of the Analysis Infinitorum contains the bulk of the matter which is to be found in modern text-books on algebra, theory of equations, and trigonometry. In the algebra, he paid particular attention to the expansion of various functions in series, and to the summation of given series, and pointed out explicitly that an infinite series can not be safely employed in mathematical investigations unless it is convergent. In trigonometry, he introduced (simultaneously with Thomas Simpson in England) the now current abbreviations for trigonometric functions, and simplified formulæ by the simple expedient of designating the angles of a triangle by A, B, C, and the opposite sides by a, b, c. He also showed that the trigonometrical and exponential functions are connected by the relation $\cos\theta+i\sin\theta=e^{i\theta}$. Here too we meet the symbol e used to denote the base of the Naperian logarithms, namely the incommensurable number 2.7182818 . . . and the symbol π used to denote the incommensurable number 3.14159265 . . The use of a single symbol to denote the number 2.7182818 . . . seems to be due to Cotes, who denoted it by M. Newton was probably the first to employ the literal exponential notation, and Euler using the form a^z, had taken a as the base of any system of logarithms. It is probable that the choice of e for a particular base was determined by its being the vowel consecutive to a, or, still more probable because e is the initial of the word *exponent.*

The use of a single symbol to denote 3.14159265 . . appears to have been introduced by John Bernouilli, who represented it by c. Euler in 1734 denoted it by p, and in a letter of 1736 in which he enunciated the theorem that the sum of the square of the reciprocals of the natural numbers is $\frac{1}{6}\pi^2$, he uses the letter c. The symbol π was first used to represent 3.141592 . . . by William Jones in his "Synopsis Palmariorum Matheseos", London, 1706, and after the publication of Euler's Analysis, the symbol π was generally employed, the choice of π being determined by the initial of the word, *περιφέρεια=peripheria.*

The second part of the *Analysis Infinitorum* is on analytical geometry. Euler begins this part by dividing curves into algebraic and transcendental, and establishes a number of propositions which are true for all algebraic curves. He then applied these to the general equation of the second degree in two dimensions, showed that it represents the various conic sections, and deduces most of their properties from the general equation. He also considered the classification of cubic, quartic, and other algebraic curves. He next discussed the question as to what surfaces are represented by the general equation of the second degree in three dimensions, and how they may be discriminated one from the other. Some of these surfaces had not been previously investigated. In this work he also laid down the rules for the transformation of coördinates in space. Here also we find the first attempt to bring the curvature of surfaces within the domain of mathematics, and the first complete discussion of tortuous curves.

In 1755 appeared *Institutiones Calculi Differentialis,* to which the *Analysis Infinitorum* was intended as an introduction. This is the first text-book on the differential calculus which has any claim to be regarded as complete, and it may be said that most modern treatises on the subject are based upon it.

At the same time, the exposition of the principles of the subject is often prolix and obscure, and sometimes not quite accurate

This series of works was completed by the publication in three volumes in 1768 to 1770 of the *Institutiones Calculi Integralis,* in which the results of several of Euler's earlier memoirs on the same subjects and on differential equations are included. In this treatise as in the one on the differential calculus was summed up all that was at that time known on the subject. The beta and gamma functions were invented by Euler, and are discussed here, but only as methods of reduction and integration. His treatment of elliptic integrals is superficial. The classic problems on isoperimetrical curves, the brachistochrone in a resisting medium, and theory of geodesics had engaged Euler's attention at an early date, and the solving of which led him to the calculus of variations. The general idea of this was laid down in his *Curvarum Maximi Minimive Praprietate Gaudentium Inventia Nova ac Facilis*, published in 1744, but the complete development of the new calculus was first effected by Lagrange in 1759. The method used by Lagrange is described in Euler's integral calcnlus, and is the same as that given in most modern text-books on the subject.

In 1770, Euler published the *Anleitung zur Algebra* in two volumes. The first volume treats of determinate algebra. This work includes the proof of the binomial theorem for any index, which is still known by Euler's name. The proof, which is not accurate according to the modern views of infinite series, depends upon the principle of the permanence of equivalent forms, and may be seen in C. Smith's *Treatise on Algebra,* pages 336–7. Euler's proof with important additions due to Cauchy, may be seen in G. Chrystal's Algebra, Part II.

It is a fact worthy of note that Euler made no attempt to investigate the convergency of the series, though he clearly recognized the necessity of considering the convergency of infinite series. While Euler recognized the convergency of series, his conclusions in reference to infinite series are not always sound. In his time no clear notion as to what constitutes a convergent series existéd, and the rigid treatment to which infinite series are now subjected was undreamed of. Euler concluded that the sum of the oscillating series $1-1+1-1+1-1+\ldots\ldots=\frac{1}{2}$, for the reason, that by stopping with an even number of terms the sum is 0, and by stopping with an odd number of terms the sum is 1. Hence, the sum of the series is $\frac{1}{2}(0+1)=\frac{1}{2}$. Guido Grandi went so far as to conclude that $\frac{1}{2}=0+0+0+0$ The paper in which Euler cautions against divergent series contains the proof that $\ldots\frac{1}{n^2}+\frac{1}{n}+1+n+n^2+n^3\ldots=0.$

His proof is as follows, $n+n^2+n^3+\ldots=\frac{n}{1-n}$, $1+\frac{1}{n}+\frac{1}{n^2}+\ldots=\frac{n}{n-1}$, $\frac{n}{n-1}+\frac{n}{1-n}=0$. Euler had no hesitation in writing $1-3+5-7+9-\ldots=0$, and he confidently believed that $\sin\varphi-2\sin2\varphi+3\sin3\varphi-\ldots=0$.

A remarkable development, due to Euler, is what he named the hypergeometrical series, the summation of which he observed to be dependent upon the integration of linear differential equations of the second order, but it remained for Gauss to point out that for special values of the letters, this series represented nearly all the functions then known. By giving the factors 641×6700417 of the number $2^{2^n}+1=4294967297$ when $n=5$, he pointed out the fact that this expression did not always represent primes, as was supposed by Fermat. — By B. F. Finkel. From the *American Mathematical Monthly,* Vol. IV, No. 12.

BIOGRAPHY.

KARL FRIEDRICH GAUSS.

This versitile and prolific mathematician, Karl Friedrich Gauss, was born at Brunswick, Germany, April 30*, 1777, and died at Göttingen on February 23, 1855. His father was a brick-layer, and was desirous of profiting by the wages of his son as a laborer, but young Gauss's talents attracted the attention of Bartels, afterwards professor of mathematics at Dorpat, who brought him to the notice of Charles William, Duke of Brunswick. The duke undertook to educate the boy and sent him to Caroline college, in 1792. By 1795 it was admitted alike by professors and pupils that he knew all that the professors could teach him. It was while at this school that he investigated the method of least squares, and proved by induction the Law of Quadratic Reciprocity. He gave the first rigorous proof of this law and succeeded in discovering eight different demonstrations of it.† While at Caroline college, Gauss manifested as great an aptitude for language as for mathematics, a very general characteristic of eminent mathematicians.

In 1795 Gauss went to Göttingen, as yet undecided whether to pursue philology or mathematics. While at Göttingen he studied mathematics under Abraham Gotthelf Kaestner, who was not a very inspiring teacher and who is now chiefly remembered for his *History of Mathematics*, 1796, and by the fact that he was a teacher of the illustrious Gauss. In 1796 he discovered a method of inscribing in a circle a regular polygon of seventeen sides, and it was this discovery that encouraged him to pursue mathematics rather than philology — a rather insignificant incident to be fraught with such stupendous consequences — consequences materially affecting our present progress in mental and material development.

A detailed construction of this problem by elementary geometry was first made by Pauker and Erchinger.

Gauss worked quite independently of his teachers at Göttingen, and it was while he was there as a student that he made many of his greatest discoveries in the theory of numbers, his favorite subject of investigation. Among his small circle of intimate friends was Wolfgan Bolyai, the discoverer of non-Euclidean geometry.

In 1798 Gauss returned to Brunswick, where he earned a livelihood by private tuition. Later in the year he repaired to the University of Helmstadt to consult the library, and it was while here that he made the acquaintance of Pfaff, a mathematician of great power. Laplace, when asked who was the greatest mathematician in Germany, replied, Pfaff. When the questioner said he should have thought Gauss was, Laplace replied: "Pfaff is the greatest mathematician in Germany; but Gauss is the greatest in all Europe."‡

In 1799 Gauss published his demonstration that every algebraical equation with integral coefficients has a root of the form $a+bi$, a theorem of which he gave three distinct proofs. In 1801, he published *Disquisitiones Arithmeticae*, a work which revolutionized the whole theory of numbers. "The greater part of this most important work was sent to the French Academy the preceding year, and had been rejected with a sneer which, even if the work had been as worthless as the referees believed, would have been unjustifiable."** Gauss had written

*Cf. *Britannica Encyclopedia* and *Century Dictionary*.

†For Gauss's third proof, as modified by Dirichlet, see Mathews's *Theory of Numbers*, pages 38-41.

‡Cajori's *A History of Mathematics*.

**Ball's *A Short History of Mathematics*.

Karl Fredrick Gauss.

far in advance of the judges of his work, and so the recognition of its merits had to wait until the mathematical world came in sight of this splendid creation. Gauss was deeply hurt because of this unfortunate incident, and it was partly due to it that he was so reluctant to publish his subsequent investigations.

The next important discovery of Gauss was in a totally different department of mathematics. The absence of a planet between Mars and Jupiter, where Bode's Law would have led observers to expect one, had long been remarked, but not until 1801 was any of the numerour groups of minor planets which occupy that space observed. On the first of January, 1801, Piazzi of Palermo discovered the first of these planets, which he called Ceres, after the tutelary goddess of Sicily.§ While the announcement of this discovery created no great surprise, yet it was very interesting, since it occurred simultaneously with a publication by the philosopher Hegel, in which he severely criticised astronomers for not paying more attention to philosophy, a science, said he, which would have shown them at once that there could not possibly be more than seven planets, and a study of which would have prevented, therefore, an absurd waste of time in looking for what in the nature of things could not be found. This is only one instance of the many refutations of dogmatic statements of philosophers who presage nature's laws without confirming them by actual observations.

However, the new planet was seen under conditions so unfavorable as to render it almost impossible to forecast its orbit. Fortunately the observations of the planet were communicated to Gauss. Gauss made use of the fact that six quantities known as elements completely determine the motion of a planet unaffected by perturbations. Since each observation of a planet gives two of these, e. g., the right ascension and declination, therefore, three observations are sufficient to determine the six quantities and therefore to completely determine the planet's motion. Gauss applied this method and that of least squares and his anlysis proved a complete success,‡ the planet being rediscovered at the end of the year in nearly the position indicated by his calculations. This success proved him to be the greatest of astronomers as well as the greatest of "arithmeticians."

The attention excited by these investigations procured for him in 1807 the offer, from the Emperor of Russia, of a chair in the Academy of St. Petersburg. But Gauss, having a marked objection to a mathematical chair, by the advice of the astronomer Olbers, who desired to secure him as director of a proposed new observatory at Göttingen, declined the offer of the emperor and accepted the position at Göttingen. He preferred this position because it afforded him an opportunity to devote all his time to science. He spent his life in Göttingen in the midst of continuous work, and after his appointment never slept away from his observatory except on one occasion when he accepted an invitation from Humboldt¶ and attended a scientific congress at Berlin, in 1828. The only other time that he was absent from Göttingen was in 1854, when a railroad was opened between Göttingen and Hanover.†*

For some years after 1807 his time was almost wholly occupied by work connected with his observatory. In 1809 he published at Hamburg his *Theoria Motus Corporum Coelestium*, a treatise which contributed largely to the improvement of practical astronomy, and introduced the principle of curvilinear triangulation. In this treatise are found four formulæ in spherical trigonometry, commonly called "Gauss's Analogies," but which were published somewhat earlier by

§ Young's *General Astronomy*, edition of 1898, page 360.
‡ Berry's *A Short History of Astronomy*.
¶ *Britannica Encyclopedia*, 9th edition, Vol. X, page 104.
†* Cajori's *A History of Mathematics*.

Karl Brandon Mollweide of Leipzig, 1774–1825, and still earlier by Jean Baptiste Joseph Delambre (1749–1822.)* On observations in general (1812–1826) we have his memoir, *Theoria Combinationes Observationum Erroribus Minimis Obnoxia*, with a second part and supplement.

A little later he took up the subject of geodesy and from 1821 to 1848 acted as scientific adviser to the Danish and Hanoverian governments for the survey then in progress. His papers of 1843 and 1866, *Ueber Gegenstaende der hoeheren Geodaesie*, contain his researches on the subject.

Gauss's researches on *Electricity and Magnetism* date from about the year 1830. In 1833 he published hin first memoir on the theory of magnetism, the title of which is *Intensitas vis Magneticae Terrestris ad Mensuram Absolutam Revocata.* A few months afterward he, together with Weber, invented the declination instrument and bifilar magnetometer. The same year they erected at Göttingen a magnetic observatory free from iron (as Humbolt and Arago had previously done on a smaller scale), where they made megnetic observations and showed in particular that it was possible and practical to send telegraphic signals, having sent telegraphic signals to neighboring towns. At this observatory he founded an association called the *Magnetische Verein*, composed at first almost entirely of Germans, whose continuous observations at fixed times extended from Holland to Sicily. The volumes of their publications, *Resultate aus den Beobachtungen des Magnetischen Vereins*, extend from 1833 to 1839. In these volumes for 1838 and 1839 are contained two important memoirs by Gauss, one on the general theory of earth-magnetism, the other on the theory of forces attracting according to the inverse squares of the distance. Like Poisson, he treated the phenomena in electrostatics as due to attractions and repulsions between imponderable particles. In electro-dynamics he arrived, in 1835, at a result equivalent to that given by W. E. Weber in 1846, viz: that the attraction between two electrified particles, e and e', whose distance apart is r, depends on their relative motion and position according to the formula

$$ee'r^{-2}[1+(rd^2r-\tfrac{1}{2}dr^2)^2c^{-2}].$$

Gauss, however, held that no hypothesis was satisfactory which rested on a formula and was not a consequence of physical conjecture, and as he could not form a plausible physical conjecture he abandoned the subject. Such conjectures were proposed by Riemann in 1858, and by C. Neumann and E. Betti in 1868, but Helmholtz in 1870, 1873, and 1874 showed that these conjectures were untenable.

In 1833, in a memoir on capillary attraction, he solved a problem in the Calculus of Variation, involving the variation of a certain double integral, the limits of integration also being variable; it is the earliest example of the solution of such a problem.

In 1846 was published his *Dioptrische Untersuchungen*, researches on optics, including systems of lenses.

As has already been observed, Gauss's most celebrated work in pure mathematics is the *Disquisitiones Arithmeticae*, and a new epoch in the theory of numbers dates from the time of its publication. This treatise, Legendre's *Theorie des nombres* and Dirichlet's *Vorlesungen ueber Zahlentheorie* are the standards on the Number Theory.

In this work Gauss has discussed the solution of binominal equations of the form $x^n=1$, which involves the celebrated theorem that the only regular polygons which can be constructed by elementary geometry are those of which the number of sides is $2^m(2^n+1)$, where m and n are integers and 2^n+1 is a prime. These equations are called *cyclotomic*

*Cajori's *A History of Mathematics.*

equations, when n is prime and when they are satisfied by a primitive nth root of unity.

Gauss developed the theory of ternary quadratic forms involving two indeterminates, and also investigated the theory of determinants on whose results Jacobi based his researches on this subject.

The theory of Functions of Double Periodicity had its origin in the discoveries of Abel and Jacobi. Both arrived at the Theta Functions which play so large a part in the Theory of Double Periodic Functions. But Gauss had independently and at a far earlier date discovered these functions and their chief properties, having been led to them by certain integrals which occurred in the *Determinatio Attractionis*, to evaluate which he invented the transformation now associated with the name of Jacobi. In the memoir, *Determinatio Attractionis*, it is shown that the secular variations, which the elements of the orbit of a planet experience from the attraction of another planet which disturbs it, are the same as if the mass of the disturbing planet were distributed over its orbit into an elleptic ring in such a manner that equal masses of the ring would correspond to arcs of the orbit described in equal times.

Gauss's collected works have been published by the Royal Society of Göttingen, in seven 4-to volumes, 1863–1871, under the editorship of E. J. Schering. They are as follows: (1) The *Disquisitiones Arithmeticae*, (2) *Theory of Numbers*, (3) *Analysis*, (4) *Geometry and Method of Least Squares*, (5) *Mathematical Physics*, (6) *Astronomy*, and (7) *Theoria Motus Corporium Coelestium.* These include besides his various works and memoirs, notices by him of many of these, and of works of other authors in the *Goettingen gelehrte Anzeigen*, and a considerable amount of previously unpublished matter, *Nachlass.* Of the memoirs in pure mathematics, comprised for the most part in volumes ii, iii, and iv (but to these must be added those on *Attraction* in volume v), there is not one which has not signally contributed to the branch of mathematics to which it belongs, or which would not require to be carefully analyzed in a history of the subject.

His collected works show that this wonderful mind had touched hidden laws in Mathematics, Physics, and Astronomy, and every one of the subjects which he investigated was greatly extended and enriched thereby. He was also well versed in general literature and the chief languages of modern Europe, and was a member of nearly all the leading scientific societies in Europe.

He was the last of the great mathematicians whose interests were nearly universal. Since his time, the literature of most branches of mathematics has grown so rapidly that mathematicians have been forced to specialize in some particular department or departments.

Gauss was a contemporary of Lagrange and Laplace, and these three, of which he was the youngest, were the great masters of modern Analysis. In Gauss that abundant fertility of invention which was marvelously displayed by the mathematicians of the preceding period, is combined with an absolute rigorousness in demonstration which is too often wanting in their writings. Lagrange was almost faultless both in form and matter, he was careful to explain his procedure, and, though his arguments are general, they are easy to follow. Laplace, on the other hand, explained nothing, was absolutely indifferent to style, and, if satisfied that his results were correct, was content to leave them either without a proof or even a faulty one. Many long and abstruse arguments were passed by with the remark, "it is obvious." This led Dr. Bowditch, of Harvard university, while translating Laplace's *Mechanique Celeste*, to say that whenever he came to Laplace's "it is obvious," he expected to put in about three weeks of hard work in order to see the obviousness. Gauss, in his writings, was as exact and elegant as Lagrange, but even more difficult to follow than Laplace, for he removed every trace of the analysis by which he reached his results, and even studied to give a proof which, while rigorous, should be as concise and synthetical as possible. He said: "Mathematics is the queen of the sciences, and arithmetic is the queen of mathematics," and his *Disquisitiones* confirms the statement.

Gauss had a strong will, and his character showed a curious mixture of self-conscious dignity and child-like simplicity. He was little communicative, and at times morose.

He possessed a remarkable power of attention and concentration, and in this power lies the secret of his wonderful achievements. As a proof of this power of attention we quote from Carpenter's *Mental Physiology*. Gauss, while engaged in one of his most profound investigations, was interrupted by a servant who told him that his wife (to whom he was known to be deeply attached, and who was suffering from a severe illness) was worse. "He seemed to *hear* what was said, but either did not comprehend it or immediately forgot it, and went on with his work. After some little time, the servant came again to say that his mistress was much worse, and to beg that he would come to her at once; to which he replied: 'I will come presently.' Again he lapsed into his previous train of thought, entirely forgetting the intention he had expressed, most probably without having distinctly realized to himself the import either of the communication or of his answer to it. For not long afterwards when the servant came again and assured him that his mistress was dying and that if he did not come immediately he would probably not find her alive, he lifted up his head and calmly replied, 'Tell her to wait until I come,' — a message he had doubtless often before sent when pressed by his wife's request for his presence while he was similarly engaged."

In bringing this imperfect sketch to a close, we wish to call attention to the fact that it has been conclusively shown that Gauss was not the first to give a satisfactory representation of complex numbers in a plane, this having been first satisfactorily done by Casper Wessel in 1797, though Wallis had made some use of graphic representation of complex numbers as early as 1785. Gauss needs no undue credit to make him famous — the writing alone of any one of the seven of his collected works being sufficient to rank him among the great mathematicians of his day. However, it was Gauss who in 1831, "by means of his great reputation, made the representation of imaginary quantities in the 'Gaussian plane' the common property of all mathematicians." He brought also into general use the sign i for $\sqrt{-1}$, though it was first suggested by Euler. He called $a+ib$ a *complex number* and called a^2+b^2 the *norm.* — By B. F. Finkel. From the *American Mathematical Monthly*.

SOPHUS LIE.

BIOGRAPHY.

SOPHUS LIE.*

Sophus Lie was born on the 17th of December, 1842, at Nordfjordeid (near Florö) where his father, John Herman Lie, was pastor. The studies of his childhood and youth did not reveal in him that exceptional aptitude for mathematics which is signalized so early in the lives of the great geometers: Gauss, Abel, and many others. Even on leaving the University of Christiania in 1865, he still hesitated between philology and mathematics. It was the works of Plücker on modern geometry which first made him fully conscious of his mathematical abilities and awakened within him an ardent desire to consecrate himself to mathematical research. Surmounting all difficulties and working with indomitable energy he published his first work in 1869, and we can say that from 1870 on he was in possession of the ideas which were to direct his whole career.

At this time I frequently had the pleasure of meeting and conversing with him in Paris where he had come with his friend F. Klein. A course of lectures by Sylow revealed to Lie all the importance of the theory of substitution groups; the two friends studied this theory in the great treatise of our colleague Jordan; they saw fully the essential rôle which it would be called upon to play in all the branches of mathematics to which it had then not been applied. They have both had the good fortune to contribute by their works to impressing upon mathematical studies the direction which appeared to them to be the best.

A short note of Lie "Sur une transformation géométrique," presented to our Academy in October, 1870, contains an extremely original discovery. Nothing resembles a sphere less than a straight line and yet, by using the ideas of Plücker, Lie found a singular transformation which makes a sphere correspond to a straight line, and which consequently makes possible the derivation of a theorem relative to an ensemble of spheres from every theorem relative to an aggregate of straight lines, and vice versa. It is true that if the lines are real, the corresponding spheres are imaginary. But such difficulties are not sufficient to deter geometers. In this curious method of transformation, each property relative to asymptotic lines of a surface is transformed into a property relative to lines of curvature. The name of Lie will remain attached to these concealed relations which connect the two essential and fundamental elements of geometric investigation, the straight line and sphere. He has developed them in detail in a memoir full of new ideas which appeared in 1872 in the *Mathematische Annalen.*

The works following this brilliant beginning fully confirmed all the hopes to which it gave birth. Since the year 1872 Lie has put forth a series of memoirs upon the most difficult and most advanced parts of the integral calculus. He commences by a profound study of the works of Jacobi on the partial differential equations of the first order and at first coöperates with Mayer in perfecting this theory in an essential point. Then, by continuing the study of this beautiful subject, he is led to construct progressively that masterful theory of continuous transformation groups which constitutes his most important work and in which, at least at the start, he was aided by no one. The detailed analysis of this vast theory would require too much space here. It is proper, however, to point out particularly two elements wholly essential to these researches: first, the use of contact transformations which throws such

*From the Bulletin of the American Mathematical Society. Translated by Edgar Odell Lovett from *Comptes Rendus.*

a vivid and unexpected light upon the most difficult and obscure parts of the theories relative to the integration of partial differential equations; second, the use of infinitesimal transformations. The introduction of these transformations is due entirely to Lie; their use, like that of Lagrange's variation, naturally greatly extends both the notion of differential and the applications of the infinitesimal calculus.

The construction of so extended a theory did not satisfy Lie's activity. In order to show its importance he has applied it to a great number of particular subjects, and each time he has had the good fortune of meeting with new and elegant properties. I find my preference in the researches which he has published since 1876 on minimal surfaces. The theory of these surfaces, the most attractive perhaps that presents itself in geometry, still awaits, and may await a long time, the complete solution of the first problem to be proposed in it, namely, the determination of a minimal surface passing through a given contour. But, in return, it has been enriched by a great number of interesting propositions due to a multitude of geometers. In 1866 Weierstrass made known a very precise and simple system of formulæ which has called forth a whole series of new studies on these surfaces. In his works Lie returns simply to the formulæ of Monge; he gives their geometric interpretation and shows how their use can lead to the most satisfactory theory of minimal surfaces. He makes known methods which permit of determining all algebraic minimal surfaces of given class and order. Finally, he studies the following problem: to determine all algebraic minimal surfaces inscribed in a given algebraical developable surface. He gives the complete solution for the case where only one of these surfaces inscribed in the developable is known.

Of great interest also are the researches which we owe to him on the surfaces of constant curvature, in the study of which he makes use of a theorem of Bianchi on geodesic lines and circles, likewise those on surfaces of translation, on the surfaces of Weingarten, on the equations of the second order having two independent variables, et cetera. I should reproach myself for forgetting, even in so rapid a résumé, the applications which Lie has made of his theory of groups to the non-Euclidean geometry and to the profound study of the axioms which lie at the basis of our geometric knowledge.

These extensive works quickly attracted to the great geometer the attention of all those who cultivate science or are interested in its progress. In 1877 a new chair of mathematics was created for him at the University of Christiania, and the foundation of a Norwegian review enabled him to pursue his work and publish it in full. In 1886 he accepted the honor of a call to the University of Leipzig; he taught in this university with the rank of ordinary professor from 1886 to 1898. To this period of his life is to be referred the publication of his didactic works, in which he has coördinated all his researches. Six months ago he returned to his native land to assume at Christiania the chair which had been especially reserved for him by the Norwegian parliament, with the exceptional salary of ten thousand crowns. Unfortunately, excess of work had exhausted his strength and he died of cerebal anæmia at the age of fifty-six years.

Nowhere is his loss felt more keenly than in our country, where he had so many friends. True, in 1870 a misadventure befell him, whose consequences I was instrumental in averting. Surprised at Paris by the declaration of war, he took refuge at Fontainebleau. Occupied incessantly by the ideas fermenting in his brain, he would go every day into the forest, loitering in places most remote from the beaten path, taking notes and drawing figures. It took little at this time to awaken suspicion. Arrested and imprisoned at Fontainebleau, under conditions otherwise very comfortable, he called for the aid of Chasles, Bertrand, and others; I made the trip to Fontainebleau and had no trouble in convincing the procureur impérial; all the notes which had been seized and in which figured complexes, orthogonal systems, and names of geometers, bore in no way upon the national defenses. Lie was released; his high and generous spirit bore no grudge against our country. Not only did he return voluntarily to visit it but he received with great kindness French students, scholars of our Ecole Normale who would go to Leipzig to follow his lectures. It is to the Ecole Normale that he dedicated his great work on the theory of transformation groups. A number of our thesis at the Sorbonne have been inspired by his teaching and dedicated to him.

The admirable works of Sophus Lie enjoy the distinction, to-day quite rare, of commanding the common admiration of geometers as well as analysts. He has discovered fundamental propositions which will preserve his name from oblivion, he has created methods and theories which, for a long time to come, will exercise their fruitful influence on the development of mathematics. The land where he was born and which has known how to honor him can place with pride the name of Lie beside that of Abel, of whom he was a worthy rival and whose approaching centenary he would have been so happy in celebrating. — By Professor Gaston Darboux.

BIOGRAPHY.

SIMON NEWCOMB, PH. D., LL. D.

Simon Newcomb was born in Wallace, Nova Scotia, in 1835. After being educated by his father he engaged for some time in teaching. He came to the United States in 1853, and was engaged for two years as a teacher in Maryland. There he became acquainted with Joseph Henry and Julius E. Hilgard, who recognizing his aptitude for mathematics, secured his appointment in 1857 as computer on the "Nautical Almanac," which was then published in Cambridge, Mass. In Cambridge he came under the influence of Professor Benjamin Peirce. He entered the Lawrence Scientific School and was graduated in 1858, continuing thereafter for three years as a graduate student.

In 1861 he was appointed professor of mathematics in the U. S. Navy and assigned to duty at the U. S. Naval Observatory in Washington. There he negotiated the contract for the 26-inch equatorial telescope authorized by congress, supervised its construction and planned the tower and dome in which it is mounted.

He was chief director of the commission created by congress to observe the transit of Venus on December 8, 1874. He visited the Saskatchewan region in 1860 to observe an eclipse of the Sun, and in 1870–1 was sent to Gibralter for a similar purpose. In 1882 he commanded an expedition to observe the transit of Venus at the Cape of Good Hope. Meanwhile in 1887 he became senior professor of mathematics in the U. S. Navy, and since that time has been in charge of the office of the "American Ephemeris and Nautical Almanac." Professor Newcomb has a large corps of assistants in Washington.

In addition to these duties, in 1884 he became professor of mathematics and astronomy in Johns Hopkins, (succeeding the distinguished Sylvester, upon the departure of the latter to accept a professorship at Oxford), where he has had charge of the *American Journal of Mathematics*. However he is not now editor of that Journal, having recently severed his immediate active connection with the Johns Hopkins University for the next two or three years.

Professor Newcomb has been intimately associated with the equipment of the Lick observatory of California, and examined the glass of the great telescope and its mounting before its acceptance by the trustees.

The results of his scientific work have been given to the world in more than one hundred papers and memoirs. Concerning these, Arthur Cayley, president of the Royal Astronomical Society of Great Britain, said: "Professor Newcomb's writings exhibit, all of them, a combination on the one hand of mathematical skill and power and on the other of good hard work, devoted to the furtherance of astronomical science."

His work has been principally in the mathematical astronomy of the solar system, particularly Neptune, Uranus, and the Moon, but the whole plan includes the most exact possible tables of the motions of all the planets. Amongst the most important of his papers are: "On the Secular Variations and Mutual Relations of the Orbits of the Asteroids" (1860); "An Investigation of the Orbit of Neptune, with general tables of its motion" (1874); "Researches on the Motion of the Moon" (1876); "Measure of the Velocity of Light" (1884); and "Development of the Purturbative Function and its Derivative in the Sines and Cosines of the Eccentric Anomaly, and in Powers of the Eccentricities and Inclinations" (1884).

In 1874 Columbian University of Washington conferred on him the degree of LL. D., and in 1875 he received the same degree from Yale,

SIMON NEWCOMB, PH D., LL. D.

also from Harvard in 1874, and from Columbia College in 1887, while on the 300th anniversary of the founding of the University of Leyden in 1875, that institution gave him him the degree of Master of Mathematics and Doctor of Natural Philosophy, and on the 500th anniversary of the University of Heidelberg in 1886 he received the degree of Ph. D. Besides the degrees just mentioned he received one from Edinburgh in 1891, one on the occasion of the tercentenary of the University of Dublin in 1892, and one from Paris on the tercentenary of Galileo's connection with the University in 1893.

He was awarded the gold medal of the Royal Astronomical Society in 1874 and in 1878 received the great gold Huyghens medal of the University of Leyden, which is given to astronomers once in 20 years for the most important work accomplished in that science between its awards. Besides the two gold medals mentioned Professor Newcomb received a third in 1890, the Copley medal, given by the Royal Society of England.

In 1887 the Russian Government ordered the portrait of Professor Newcomb to be painted for the collection of famous astronomers at the Russian observatory at Pulkowa, and also ordered to be presented to him a vase of jasper with marble pedestal seven feet high. The University of Tōkyō has also presented him with two vases of bronze.

He was elected an associate member of the Royal Astronomical Society in 1872, corresponding member of the Institute of France in 1874, and foreign member of the Royal Society 1877; and he also holds honorary or corresponding relations to nearly all the European academies of Science. In 1877 he was elected one of the eight members of the council of the Astronomische Gesellschaft, an international astronomical society that meets once in two years. He was elected to the National Academy of Sciences in 1869 and since 1883 has been its vice president. In 1876 he was elected president of the American Association for the Advancement of Science, and delivered his retiring address at the St. Louis meeting in 1878. He also held the presidency of the American Society for Physical Research.

He was elected member of the New York Mathematical Society in 1891, and delivered an address, entitled "Modern Mathematical Thought" before the annual meeting of the Society, December 28, 1893, which was published in the *Bulletin* of the Society for January 1894, and in *Nature* of February 1, 1894.

Professor Newcomb's book on Popular Astronomy (1877) has been republished in England and translated into German, while "School Astronomy" by Newcomb and Holden (1879), and their "Briefer Course" (1883), are used as text books in most of our colleges.

Professor Newcomb has also carried on important investigations on subjects purely mathematical. An important contribution by him on "Elementary Theorems Relating to the Geometry of a Space of Three Dimensions and of Uniform Positive Curvature in the Fourth Dimension," was published in Borchardt's Journal, Berlin, 1877. Full extracts of this important contribution to non-Euclidean geometry are given in the Encyclopedia Britannica, article "Measurement." In Vol. I. of the American Journal of Mathematics he has a note "On a Class of Transformations which Surfaces may Undergo in Space of more than Three Dimensions," in which he shows, for instance, that if a fourth dimension were added to space, a closed material surface (or shell) could be turned inside out by simple flexure without either stretching or tearing. Later articles have been on the theory of errors in observations. In former years he also contributed to the *Mathematical Monthly* and the *Analyst.*

He has also written a series of mathematical text-books, comprising Algebra (1881); Geometry (1881); Trigonometry and Logarithms

(1882); School Algebra (1882); Analytic Geometry (1884); Essentials of Trigonometry (1884); and Calculus (1887). These works have been favorably received and are everywhere regarded as text-books of decided merits.

Professor Newcomb refers to astronomy as his profession and to political economy as his recreation, and in the latter branch has written several books and a number of magazine articles. — By J. M. Colaw. From the *American Mathematical Monthly,* Vol. I, No. 8.

GEORGE BRUCE HALSTED.

BIOGRAPHY.

GEORGE BRUCE HALSTED.

Dr. Halsted, a direct descendant of Abram Clark, signer of the declaration of independence, was born in Newark, N. J., November 25, 1853. The Halsteds have been for four generations graduates of Princeton. Winning the Mathematical Fellowship at Princeton, Halsted went for applied mathematics to Columbia School of Mines, won while there an intercollegiate prize, and was made one of the first Fellows of the new Johns Hopkins University. Sylvester was particularly partial to him, and Halsted's life of Sylvester is recognized as the authority on the American part of his career. Says Dr. Fabian Franklin: "Professor Halsted, in his account of Sylvester's work already referred to, points out how the vicissitudes of his career were reflected in the richness or the meagreness of his mathematical production from period to period." Says Major P. A. MacMahon in the biography of Sylvester published by the Royal Society: "Sylvester's first high class consisted of but one student, G. B. Halsted. This gentleman, since well known in science, had the most beneficial effect upon his master, for it was owing to his enthusiasm and persistence that Sylvester's attention was again called to the Modern Higher Algebra and the Theory of Invariants, and a fruitful crop of new discoveries was almost the immediate result. Before taking his Doctor's Degree at the Johns Hopkins University, Halsted published in 1878 his epoch-making Bibliography of Hyper-Space and Non-Euclidean Geometry, long out of print and greatly in demand, which began at once to be cited all over the world and was reproduced in Russia. Called to introduce modern higher mathematics at his ancestral college, Princeton, the papers set by Dr. Halsted on quaternions, determinants, modern higher algebra, and history of mathematics were the first ever given at Princeton. His teaching of the History of Mathematics attracted attention as early as 1881, when he had in his class Professor H. B. Fine and Professor A. L. Kimball. His work in history of mathematics has never since been discontinued. While at Princeton, Dr. Halsted produced his treatise on Mensuration which had the honor of being drawn upon by Professor Wm. Thomson for his article "Mensuration" in the ninth edition of the Encyclopaedia Britannica. He took from it, among other novelties, the steregon, the steradian, and the treatment of solid angles associated with them. Simon Newcomb wrote of Dr. Halsted: "He is the author of a treatise on Mensuration which is the most thorough and scientific with which I am acquainted."

From this brilliant field of work Dr. Halsted was called by a telegram from the University of Texas, announcing his election to the professorship of mathematics, urging his acceptance. Dr. Halsted had already produced 17 scientific papers on Mathematics and Logic. This productivity was not abated by his removal to Texas, and the titles of his papers now approach a hundred.

His Elements of Geometry which appeared in 1885, has passed through seven editions. The section headed by his phrase "Partition of a Perigon," has become classic.

Of Dr. Halsted's Elementary Synthetic Geometry, which appeared in 1893, has been said: "For more than two thousand years geometry has been founded upon, and built up by means of, congruent triangles. At last, after twenty centuries, comes a book reaching all the preceding results without making any use of congruent triangles; and so simply that, for example, all the ordinary cases of congruence of triangles are demonstrated together in eight lines."

Dr. Halsted's two books are as yet the only geometries treating spherics comparatively by considering the sphere as the analogue of the plane. "In the method employed by Dr. Halsted almost the whole of the geometry of the plane is directly applicable to the sphere." The Elementary Synthetic Geometry contains an introduction to the new Lemoine-Brocard Geometry, the only one which has appeared on the western continent.

David Eugene Smith, in his history of Modern Mathematics mentions, p. 567, as "among the most active" in securing the acceptance of the Bolyai-Lobachévski idea, Hoüel in France, Riemann, Helmholtz and Baltzer in Germany, de Tilley (1879) in Belgium, Clifford in England, and Halsted (1878) in America.

Of these all are dead but Halsted. By Halsted alone were the immortal works of Saccheri, Bolyai, Lobachévski, Vasiliev, made available to the English speaking world, and also to Japan, for his Bolyai and Lobachévski have been reissued in Tokio. Results of his travels in Japan, Hungary, Russia, added fuel to the awakening fire of interest in all things non-Euclidean, and now the majestic face of heroic Lobachévski has been worthily given to the world in the beautiful picture made from a photograph furnished by Dr. Halsted to the Open Court as frontispiece for his Life of Lobachévski in the number for July 1898.

Of John Bolyai he could find in all Hungary no picture; his lineaments are lost forever.

Professor Felix Klein.

BIOGRAPHY.

PROFESSOR FELIX KLEIN.

The eminent subject of this very imperfect sketch was born on the twenty-fifth of April, 1849, in Duesseldorf. His mother was Elise Sophie *nee* Kayser; his father, the "Landrentmeister" Caspar Klein, both of the protestant faith. For eight years, from the autumn of 1857 to the autumn of 1865 he attended the Duesseldorf Gymnasium, and went thence to the University of Bonn, for the study of mathematics and the natural sciences, especially physics. Here he had the extraordinary good fortune to come into close relations with the great Professor Pluecker, who gave him the position of assistant in the physical institute of Bonn, and used his help in writing out his profoundly original and stimulating mathematical works.

The death of Pluecker May 22nd, 1868, closed this formative period, of which the influence on Klein can not be overestimated. So mighty is the power of contact with the living spirit of research, of taking part in original work with a master, of sharing in creative authorship, that any one who has once come intimately in contact with a producer of the first rank must have had his whole mentality altered for the rest of his life.

The gradual development, high attainment, and then continuous achievement of Felix Klein are more due to Pluecker than to all other influences combined. His very mental attitude in the world of mathematics constantly recalls his great maker.

Of others whose lectures he attended, we may mention Argelander and Lipschitz, to the latter of whom particularly he has expressed his gratitude for kindly and efficient guidance and aid in his studies. Klein took his doctor's degree at Bonn on December 12th, 1868, with a dissertation "On the transformation of the general equation of the second degree between line-coordinates to a canonic form," a subject taken from the analytic line-geometry of his master Pluecker. A line-complex of the nth degree contains a triply infinite multitude of straights, which are so distributed in space, that those straights which go through a fixed point make a cone of the nth order, or, what is the same, that those straights which lie in a fixed plane envelop a curve of the nth class. Such an aggregate or form finds its analytic representation through the coordinates of the straight in space, introduced by Pluecker. According to Pluecker the straight has six homogeneous coordinates which fulfill an equation-of-condition of the second degree. By means of these the straight is determined with reference to a coordinate-tetrahedron. A homogeneous equation of the nth degree between these coordinates represents a complex of the nth degree.

The dissertation transforms the equation of the second degree between line-coordinates to a canonic form, in correspondence with a change of the coordinate-tetrahedron. It first gives the general formulas to be applied in such a transformation.

From these the problem appears algebraically as the simultaneous linear transformation of the complex to a canonic form, and of the equation-of-condition, which the line coordinates must fulfill, into itself. In carrying out these transformations, it attains to a classification of the complexes of the second degree into distinct species.

The dissertation is dedicated to Pluecker and contains eight specific references to Pluecker's "Neue Geometrie des Raumes, gegruendet auf die Betrachtung der geraden Linie als Raumelement." It is lucid and simple, but for depth and promise contrasts sharply with the great dissertation of Riemann, that "book with seven seals."

It may be interesting, as characteristic of this germinating state, to note that of his five theses the second calls attention to one of Cauchy's slips in logical rigor, slips now known to be so numerous that C. S. Pierce makes of them a paradox, maintaining that fruitfulness of Cauchy's work is essentially connected with its logical inaccuracy.

The third thesis declares the assumption of an ether unavoidable in the explanation of the phenomena of light.

The last thesis is the desirability of the introduction of newer methods in Geometry alongside the Euclidean in gymnasial teaching.

This serves, it seems, to emphasize my point that the long eight years of gymnasial so-called *training* left the seed still dormant, and only in Pluecker did it find the rain and the sun to call it to life and growth.

Within two years now the development is amazing. Already in 1870 he is working with another great genius, Sophus Lie; and in 1871 is presented to the Goettingen Academy of Science his epoch-making paper, "Ueber die sogenannte Nicht-Euklidische Geometrie." Its aim is to present the mathematical results of the non-Euclidean geometry, in so far as they pertain to the theory of parallels, in a new, intuitive way; its instrument is the mighty projective geometry, which he proves independent of all question of parallels. He perfects the projective metrics of Cayley by founding cross-ratio, after von Staudt, wholly without any use or idea of measurement. Then can be constructed a general projective expression for distance, related to an arbitrary surface of the second degree as Fundamental-surface (Cayley's Absolute). This projective metrics then gives, according to the species of Absolute used, a picture of the results of the parallel-theory in the space of Lobachevsky, of Euclid, of Riemann. But not merely a picture; they coincide to their innermost nature.

The paper begins by stating that, as well-known, the eleventh axiom of Euclid is equivalent to the theorem that the sum of the angles in a triangle equals two right angles. Legendre gave a proof that the angle-sum in a triangle cannot be greater than two right angles; but this proof, like the corresponding one in Lobachevsky, assumes the infinite length of the straight.

Drop this assumption, and the proof falls, else would it apply in surface spherics. Legendre showed further, that if in one triangle the angle-sum is two right angles, it is so in every triangle. We now know that this had been proven long before by Saccheri. But Professor Klein said that he heard the name of Saccheri for the first time in my address before the World's Science Congress. But it is claimed for Gauss that he was the first to distinctly state his conviction of the impossibility of proving the theorem of the equality of the angle-sum to two right angles. But it does not follow, as claimed by his Goettingen worshippers, that Gauss ever came to the conviction that a valid non-Euclidean geometry was possible until after it had been made simultaneously by John Bolyai and Lobachevsky, and perhaps long before by Wolfgang Bolyai. Certainly the world did not hear of it from Gauss. He published nothing on it.

In this non-Euclidean geometry there appears a certain constant characteristic for the metrics of the space. By giving this an infinite value we obtain the ordinary Euclidean geometry. But if it has a finite value, we get a quite distinct geometry, in which, for example, the following theorems hold: The angle-sum in a triangle is less than two right angles, and indeed so much the more so the greater the surface of the triangle. For a triangle whose vertices are infinitelv separated, the angle-sum is zero. Through a point without a straight one can draw two parallels to the straight, that is, lines which cut the straight on the one or the other side in a point at infinity. The straights through the point which run between the two parallels nowhere cut the given straight. But on the other hand, in Riemann's marvellous inaugural lecture, "Ueber die

Hypothesen, welche der Geometrie zu Grunde liegen," is pointed out that the unboundedness of space, which is experiential, does not carry with it the infinity of space.

It is thinkable, and would not contradict our perceptional intuition, which always relates to a finite piece of space, that space is finite and comes back into itself.

The gcometry of our space would then be like that of a tridimensioual sphere in a four dimensional manifoldness. This representation carries with it that the angle-sum in a triangle, as in ordinary spherical triangles, is greater than two right angles, and indeed the more so, the greater the triangle. The straight would then have no point at infinity, and through a given point no parallel to a given straight could be drawn. Now Cayley constructed his celebrated projective metrics to show how the ordinary Euclidean metrics may be taken as a special part of projective geometry. Klein generalizes Cayley and founds three metric geometries, the elliptic (Riemann's), the hyperbolic (Lobachevsky's), the parabolic (Euclid's).

This little paper of 1871 contains the promise of much that is most genial in the after work of a man now generally considered as the most interesting and one of the very greatest of living mathematicians. Of all those splendid and charming series of lectures with which Klein has made Goettingen so attractive to the whole world, the most delightful and epoch-making are those on non-Euclidean geometry, (Nicht-Euklidische Geometrie, I. Vorlesung, gehalten waehrend des Wintersemesters 1889-90 von F. Klein. Ausgearbeitet von Fr. Schilling. Zweiter Abdruck. Goettingen, 1893. Small Quarto, lithographed, pp. v. 365. II. Sommersemesters 1890. Zweiter Abdruck 1893, pp. iv. 238).

The World's Science Congress at Chicago was in nothing more fortunate than in the presence of Helmholtz and Felix Klein, and in the spontaneous and universal homage accorded them no idea was more often emphasized than their connection with the birth and development of that wonderful new world of pure science typified in the non-Euclidean geometry.

The narrow limits of this feeble sketch prevent the statement of how much promise, richly fulfilled in the development of this many-sided man, in totally other directions is contained in a little-known paper of 1873, "Ueber den allgemeinen Functionsbegriff und dessen Darstellung durch eine willkuerliche Curve."

Twenty years of production and achievement have not in the least dampened the ardour of this enthusiastic mind. This very summer at the great meeting of scientists in Vienna Klein seemed the busiest, the foremost of all that goodly company. — By Dr. George Bruce Halsted. From the *American Mathematical Monthly,* Vol. I, No. 12.

BIOGRAPHY.

BENJAMIN PEIRCE.

Benjamin Peirce was born at Salem, Massachusetts, April 4, 1809, and died at Cambridge, Massachusetts, October 6, 1880. He entered Harvard College, at the age of sixteen; and, at the age of twenty, he was graduated from the same College, with highest honors. He devoted himself principally to the study of Mathematics. This *favorite* study of his was pursued far beyond the limits of the curriculum of mathematical studies prescribed by the authorities of Harvard College, at that time.

As an under-graduate student, young Peirce was instructed by Nathaniel Bowditch, who soon perceived the innate mathematical genius of his pupil. Bowditch proudly predicted the future greatness of the young man. Not only did Bowditch give him valuable instruction in geometry and analysis, but also acted as his *mathematical adviser* — carefully directing him in the development of his mathematical talents and scientific powers. The lectures on higher mathematics delivered by Francis Grund he was enabled to attend, by reason of his preparation beyond the limit of the under-graduate course in mathematics. When Dr. Bowditch was publishing his translation and commentary of the *Mechanique Céleste* of Laplace, young Peirce assisted in reading the proof-sheets. This critical reading of that great work of Laplace was to him an education in itself, and may have been the prime cause that not a small part of Peirce's subsequent mathematical and scientific work was done in the great field of analytical mechanics:

In the class-room, he frequently gave original demonstrations which proved to be more direct and scientific than those given in the text-books of that day. On graduating, he went to Northampton, Massachusetts, as a teacher in Mr. Bancroft's School. As *tutor,* he returned to Harvard College, in 1831. Since Professor Farrar spent the next year in Europe, tutor Peirce was left at the *head* of the Department of Mathematics in Harvard College; and, on account of the physical inability of Professor Farrar to resume teaching, Peirce *continued* to fill his place. In fact, Peirce held this position, advancing step by step, until the time of his death. His position, in 1842, was christened "The Perkins Professorship of Mathematics and Astronomy." In the history of mathematical teaching at Harvard College, the year 1833 marks an important epoch; as it was then that Benjamin Peirce became the *professor* of Mathematics and Natural Philosophy in that institution of learning.

Professor Peirce was married in July, 1833. At the time of his death, there were living his wife, three sons, and a daughter. His eldest son, James M. Peirce, is University professor of mathematics in Harvard; Charles S. Peirce is a professor in the Johns Hopkins University; and H. H. D. Peirce is connected with the firm of Herter Brothers, New York City.

It has been said that a mere boy detected an error in Bowditch's solution of a problem. "Bring me the boy who corrects my mathematics," said Bowditch. Master Benjamin Peirce was the boy who had done the correcting; and thirty years later, this same Benjamin Peirce dedicated one of his great mathematical works "To the cherished and revered memory of my master in science, Nathaniel Bowditch, *The Father of American Geometry.*" This same title was bestowed upon Peirce, by foreign mathematicians. Sir. Wm. Thomson (Lord Kelvin), in an address before the British Association, referred to Benjamin Peirce as *"The Founder of High Mathematics in America;"* and on a similar occasion, the late Professor Cayley referred to him as *"The Father of American Mathematics."* The name of Benjamin Peirce is that of an *American*

Benjamin Peirce.

mathematician, whom no one need hesitate to rank with the names of Pythagoras, Leibnitz, Newton, Legendre, John Bernoulli, Wallis, Abel, Laplace, Lagrange, and Euler. Through the united efforts of the late Professor Wm. Chauvenet (Yale's ablest mathematician and astronomer) and Benjamin Peirce — not to speak of their worthy successors, was effected the general adoption of the *ratio-system* in American works on trigonometry.

In the reforms incident to the *New Education,* Harvard has always taken a prominent part and Benjamin Peirce was *an enthusiastic advocate* of the elective system with respect to collegiate studies. As a branch of Harvard College, there was opened, in 1842, the Lawrence Scientific School; and in this school, Professor Peirce gave instruction in higher Mathematics including analytical and celestial mechanics. Such advanced courses of mathematics, as *he* offered to students, in 1848, had never before been offered to American students by any other professor in any other American college. The second American educational institution which offered equally advanced courses of mathematics, is the Johns Hopkins University; and these courses were arranged by that *English master,* who gave a fresh and powerful impulse to mathematical study and teaching in America — *Professor J. J. Sylvester.*

The preparation of mathematical text-books was begun by Professor Peirce, immediately on beginning his career as teacher of Mathematics in Harvard College. In 1835 appeared his *Elementary Treatise on Plane Trigonometry;* in 1836, his *Elementary Treatise on Spherical Trigonometry* together with his *Elementary Treatise on Sound;* in 1837, his *Elementary Treatise on Plane and Solid Geometry* together with his *Elementary Treatise on Algebra;* during the period of 1841–46, he wrote and published in *two* volumes his *Elementary Treatise on Curves, Functions, and Forces;* and in 1855, he published his *Analytical Mechanics.* Subsequently was published his memoir on *Linear Associative Algebra;* and this memoir, according to Professor James Mills Peirce, he regarded as his great work. All of his works are models of conciseness, perspicuity, and elegance; and they all evince extraordinary originality and genius.

In 1867, Professor Peirce was made the Superintendent of the United States Coast Survey; and he held that position for seven years. He had been consulting astronomer to the *American Ephemeris and Nautical Almanac,* since 1849; and for many years, he *directed* the theoretical part of the work. In 1855, Professor Peirce was one of the men intrusted with the organization of the Dudley Observatory. For many years *before* and *after* he took charge of the United States Coast Survey, he was frequently consulted with respect to the work in that office. He received the degree of *Doctor of Laws* from the University of North Carolina, in 1847, and also from Harvard University in 1867. He was elected an Associate of the Royal Astronomical Society of London in 1849, and a member of the Royal Society of London in 1852. He was elected president of the American Association for the Advancement of Science, in 1853 (the fifth year of its existence); and he was one of the *original* members of the Royal Societies of Edinburgh, and Goettingen; Honorary Fellow of the Imperial University of St. Vladimir. at Kiev; etc.

Professor Peirce's conception of the American Social Science Association was that it should be a *university for the people,* — combining those who can contribute any thing original in social science into a temporary academical senate, to meet for some weeks in a given place and debate questions with each other, as well as to give out information for the public. In this line of thought he favored, also, the establishment of the Concord School of Philosophy, to do a similar work in the speculative studies; and he lived to see the partial realization of what he foresaw in this instance. In a Mathematical Society over which he presided for some years, each member would bring something novel in his own particular branch of study; and in the discussion which followed, it would almost invariably appear that Professor Peirce had, while the paper was being read, pushed out the author's methods to far wider results than the author had dreamed possible. The same power of extending rapidly in his own mind novel mathematical researches was exhibited at the sessions of every scientific body at which he chanced to be present. What was quite as admirable was the way in which he did it, giving the credit of the thought always to the author of the essay under discussion. His pupils thus frequently received credit for what was in reality far beyond their attainment. He robbed himself of fame in two ways: by giving the credit of his discoveries to those who had merely suggested the line of thought and by neglecting to write out and publish that which he had himself thought out.

In physical astronomy, perhaps, his greatest works were in connection with the planetary theory, his analysis of the Saturnian system, his researches regarding the lunar theory, and the *profound criticism* of the discovery of Neptune following the investigations of Adams and Leverrier. At the time of the publication of his *"System of Analytical Mechanics,"* Professor Peirce announced that the volume would be followed by three others, entitled respectively: *"Celestial Mechanics," "Potential Physics,"* and *"Analytical Morphology."* These three volumes were never published.

Professor Peirce, in a paper read before the American Association for the Advancement of Science, in 1849, showed in the vegetable world the demonstrable presence of an intellectual plan—showed that phyllotaxis (the science of the relative position of leaves) involved an algebraic idea; and this algebraic idea was subsequently shown to be the solution of a physical problem.

The higher mathematical labors of so eminent a geometer must lie beyond the course of general recognition. Among the things which give him a just claim to this title, may be mentioned: his discussion of the motions of two pendulums attached to a horizontal cord; of the motions of a top; of the fluidity and tides of Saturn's rings; of the forms of fluids enclosed in extensible sacs; of the motion of a sling; of the orbits of Uranus, Neptune, and the comet of 1843; of the criteria for rejecting doubtful observations; of a new form of binary arithmetic, of *systems* of linear and associative algebra; of various mechanical games, puzzles, etc.; of various problems in geodesy; of the lunar tables; of the occultations of the Pleiades; etc. He adapted the epicycles of Hipparchus to the analytical forms of modern science; and he, also, solved by a system of co-ordinates of his own devising, several problems concerning the involutes and evolutes of curves, which would probably have proved impregnable by any other method of mathematical approach.

None of Professor Peirce's labors lie farther above the ordinary reach of thought than his little lithographed volume on Linear and Associative Algebra. In this he discusses the nature of mathematical methods, and the characteristics which are necessary to give novelty and unity to a calculus. Then he passes to a description of seventy or eighty different kinds of simple calculus. Almost no comment is given; but the mathematical reader discovers, as he proceeds, that only *three* species of calculus, having each a unity in itself, have been hitherto used to any great extent, namely, — *ordinary algebra, differentials,* and *quaternions.* Think of it; what a wonderful volume of prophecy that is which describes seventy or eighty species of algebra, any one of which would require generation after generation of ordinary mathematicians to develop!

On both sides of the Atlantic, Professor Peirce as an author, was highly esteemed. His work on analytical mechanics was, at the time of its publication, regarded even in Germany, as the *best* of its kind. As a lecturer, Professor Peirce was highly esteemed in both scientific and popular circles. It is related that in 1843, by a series of popular lectures on astronomy, he so excited the public interest that the necessary funds were immediately supplied, for erecting an astronomical observatory at Harvard College. A remarkable series of lectures on "*Ideality in Science,*" delivered by him in 1879 before the Lowell Institute in Boston, attracted the general attention of American thinkers, on account of the thoughtful consideration of the vexed question of science and religion.

Professor Peirce was a transcendentalist in mathematics, as Agassiz was in zoology; and a certain subtle tie of affinity connected these two great men, however unlike they were in their special genius. Alike, also, they were in their enthusiasm which neither the piercing scepticism of Cambridge could wither, nor declining years chill with the frost of age. The thing he distrusted was routine and fanatical method, whether new or old; for thought, salient, vital, co-operative thought, in novel or in ancient aspects, he had nothing but respect and furtherance. Few men could suggest more while saying so little, or stimulate so much while communicating next to nothing that was tangible and comprehensible. The young man who would learn the true meaning of *apprehension* as distinct from *comprehension,* should have heard the professor lecture, after reciting to him. He was always willing to be esteemed for less than he had really accomplished; and he could join most heartily in the praise of others who even owed their impulse to him. *Modest* and *magnanimous,* but not unobservant, his ambition for personal distinction was early and easily satisfied; and he thus rid himself of what is to most men a perturbing, and too often an ignoble, element of discomfort.

Professor Peirce habitually ascribed to his listener a power of assimilation which the listener rarely possessed. He assumed his readers could follow wherever he led; and this made his lectures hard to follow, his books brief, difficult, and comprehensive. When, however, his listeners were students who had previously attained some skill as mathematicians and who had been trained in his own methods, the resulting work would be of *the highest order of excellence.* He was personally magnetic in his presence. His pupils loved and revered him; and to the young man, he always lent a helping hand in science. He inspired in them a love of truth for its own sake.

His own faith in Christianity had the simplicity of a child's; and whatever radiance could emanate from a character which combined the greatest intellectual attainment with the highest moral worth, that radiance cast its light upon those who were in his presence. *"Every portion of the material universe,"* writes Professor Peirce, *"is pervaded by the same laws of mechanical action which are incorporated into the very constitution of the human mind."* To him, then, the universe was made for the instruction of man. With this belief he approached the study of natural phenomena not in the spirit of a critic, but reverently in the mood of a sympathizing reader· and the lesson he reads is: *"There is but one God, and science is the knowledge of Him."* In his lectures and teaching he showed, as he always felt with adoring awe, that the mathematician enters (as none else can) into the intimate thought of God, sees things precisely as they are seen by the Infinite Mind, holds the scales and compasses with which the Eternal Wisdom built the earth and meted out the heavens. This consciousness had pervaded his whole scientific life. It was active in his early youth, as his coevals well remember; it gathered strength with his years; and it struck the ever recurring keynote in his latest public utterances.

Benjamin Peirce was a devout, God-fearing man; he was a Christian, in the whole aim, tenor, and habit of his life. To know Professor Peirce was simply to love him, to admire him, and to revere him. Since he was conversant with the phases of scientific infidelity, and by no means unfamiliar with the historic grounds of scepticism, it can not be regarded otherwise than with the profoundest significance, that a *mind* second to none in keen intuition, in æsthetic sensibility, in imaginative fervor, and in the capacity of close and cogent reasoning, *maintained* through life an unshaken belief and trust in the power, providence, and love of God, as beheld in his works, and as incarnate in our Lord and Savior. In one of his lectures on *Ideality in Science*, he said: "'Judge the tree by its fruit.' Is this magnificent display of ideality a human delusion? Or is it a divine record? The heavens and the earth have spoken to declare the glory of God. It is not a tale told by an idiot, signifying nothing. It is the poem of an infinite imagination, signifying immortality."

In May, 1880, Professor Peirce began to pass under the shadow of the cloud of his last illness. For some weeks there was little serious fear that it was a shadow not destined to lift: He was first confined to his chamber, on the 25th of June, 1880; and from that time, his slowly failing condition was hardly relieved even by any deceptive appearances of improvement. He died on the morning of Wednesday, October 6, 1880. Distinguished throughout his life by his freedom from the usual abhorrence of death, which he never permitted himself either to mourn when it came to others, or to dread for himself, he kept this characteristic temper to the end, through all the sad changes of his trying illness; and, two days before he ceased to breathe, it struggled into utterance in a few faintly-whispered words, which expressed and earnestly inculcated a cheerful and complete acceptance of the will of God with regard to him.

The funeral took place on Saturday, October 9, 1880, at Appleton Chapel, and was the occasion of an impressive gathering of people of great and various mark. The attendance included a very full representation of the various faculties and governing boards of the University; a large deputation of officers of the United States Coast and Geodetic Survey, headed by the superintendent and the chief assistant; delegations of eminent professors from Yale College and the Johns Hopkins University; many members of the class of 1829; and a great number of other friends of the deceased.

The pall-bearers were: President Charles W. Eliot; Ex-President Thomas Hill, Pastor of the First Parish Church, Portland, Maine; Capt. C. P. Patterson, Superintendent of the United States Coast Survey; Professor J. J. Sylvester, of the Johns Hopkins University; Hon. J. Ingersoll Bowditch; Professor Simon Newcomb, Superintendent of the American *Ephemeris and Nautical Almanac;* Dr. Oliver Wendell Holmes; Professor Joseph Lovering; and Dr. Morrill Wyman. A beautiful and simple service was conducted by the Rev. A. P. Peabody and the Rev. James Freeman Clarke.

In the career of Professor Benjamin Peirce, America has nothing to regret, but that it is now closed; while the American people have much to learn from his long, useful, and honorable life. — By. F. P. Matz. From the *American Mathematical Monthly*.

BENJAMIN PEIRCE.

For him the Architect of all
Unroofed our planet's starlit hall;
Through voids unknown to worlds unseen
His clearer vision rose serene.

With us on earth he walked by day,
His midnight path how far away!
We knew him not so well who knew
The patient eyes his soul looked through

For who his untrod realm could share
Of us that breathe this mortal air,
Or camp in that celestial tent
Whose fringes gild our firmament?

How vast the workroom where he brought
The viewless implements of thought!
The wit how subtle, how profound,
That Nature's tangled webs unwound;

That through the clouded matrix saw
The chrystal planes of shaping law,
Through these the sovereign skill that planned,—
The Father's care, the Master's hand!

To him the wandering stars revealed
The secrets in their cradle sealed;
The far-off, frozen sphere that swings
Through ether, zoned with lucid rings;

The orb that rolls in dim eclipse
Wide wheeling round its long ellipse,—
His name Urania writes with these
And stamps it on her Pleiades.

We knew him not? Ah, well we knew
The manly soul, so brave, so true,
The cheerful heart that conquered age,
The childlike silver-bearded sage.

No more his tireless thought explores
The azure sea with golden shores;
Rest, wearied frame! the stars shall keep
A loving watch where thou shalt sleep.

Farewell! the spirit needs must rise,
So long a tenant of the skies,—
Rise to that home all worlds above
Whose sun is God, whose light is love.

—*Oliver Wendell Holmes.*

BIOGRAPHY.

JAMES JOSEPH SYLVESTER, LL. D., F. R. S.

On Monday, March 15, 1897, in London, where, September 3, 1814, he was born, died the most extraordinary personage for half a century in the mathematical world.

James Joseph Sylvester was second wrangler at Cambridge in 1837. When we recall that Sylvester, Wm. Thomson, Maxwell, Clifford, J. J. Thomson were all second wranglers, we involuntarily wonder if any senior wrangler except Cayley can be ranked with them.

Yet it was characteristic of Sylvester that not to have been first was always bitter to him.

The man who beat him, Wm. N. Griffin, also a Johnian, afterwards a modest clergyman, was tremendously impressed by Sylvester, and honored him in a treatise on optics where he used Sylvester's first published paper, "Analytical development of Fresnel's optical theory of crystals," *Philosophical Magazine,* 1837.

Sylvester could not be equally generous, and explicitly rated above Griffin the fourth wrangler George Green, justly celebrated, who died in 1841.

Sylvester's second paper, "On the motion and rest of fluids," *Philosophical Magazine,* 1838 and 1839, also seemed to point to physics.

In 1838 he succeeded the Rev. Wm. Ritchie as professor of natural philosophy in University College, London.

His unwillingness to submit to the religious tests then enforced at Cambridge and to sign the 39 articles not only debarred him from his degree and from competing for the Smith's prizes, but, what was far worse, deprived him of the Fellowship morally his due. He keenly felt the injustice.

In his celebrated address at the Johns Hopkins University his denunciation of the narrowness, bigotry and intense selfishness exhibited in these compulsory creed tests, made a wonderful burst of oratory. These opinions were fully shared by De Morgan, his colleague at University College. Copies I possess of the five examination papers set by Sylvester at the June examination, session of 1839–40, show him striving as a physicist, but it was all a false start. Even his first paper shows he was always the Sylvester we knew. To the "Index of Contents" he appends the characteristic note: "Since writing this index I have made many additions more interesting than any of the propositions here cited, which will appear toward the conclusion." Ever he is borne along helpless but ecstatic in the ungovernable flood of his thought.

A physical experiment never suggests itself to the great mental experimenter. Cayley once asked for his box of drawing instruments. Sylvester answered, "I never had one." Something of this irksomeness of the outside world, the world of matter, may have made him accept, in 1841, the professorship offered him in the University of Virginia.

On his way to America he visited Rowan Hamilton at Dublin in that observatory where the maker of quaternions was as out of place as Sylvester himself would have been. The Virginians so utterly failed to understand Sylvester, his character, his aspirations, his powers, that the Rev. Dr. Dabney, of Virginia, has seriously assured me that Sylvester was actually deficient in intellect, a sort of semi-idiotic calculating boy. For the sake of the contrast, and to show the sort of civilization in which this genius had risked himself, two letters from Sylvester's tutors at Cambridge may here be of interest.

The great Colenso, Bishop of Natal, previously Fellow and Tutor of St. John's College, writes: "Having been informed that my friend and

JAMES JOSEPH SYLVESTER, LL. D., F. R. S.

former pupil, Mr. J. J. Sylvester, is a candidate for the office of professor of mathematics, I beg to state my high opinion of his character both as a mathematician and a gentleman.

"On the former point, indeed, his degree of Second Wrangler, at the University of Cambridge would be, in itself, a sufficient testimonial. But I beg to add that his powers are of a far higher order than even that degree would certify."

Philip Kelland, himself a Senior Wrangler, and then professor of mathematics in the University of Edinburgh, writes: "I have been requested to express my opinion of the qualifications of Mr. J. J. Sylvester, as a mathematician.

"Mr. Sylvester was one of my private pupils in the University of Cambridge, where he took the degree of Second Wrangler. My opinion of Mr. Sylvester then was that in originality of thought and acuteness of perception he had never been surpassed, and I predicted for him an eminent position among the mathematicians of Europe. My anticipations have been verified. Mr. Sylvester's published papers manifest a depth and originality which entitles them to the high position they occupy in the field of scientific discovery. They prove him to be a man able to grapple with the most difficult mathematical questions and are satisfactory evidence of the extent of his attainments and the vigor of his mental powers."

The five papers produced in this year, 1841, before Sylvester's departure for Virginia, show that now his keynote is really struck. They adumbrate some of his greatest discoveries.

They are: "On the relation of Sturm's auxiliary functions to the roots of an algebraic equation," British Assoc. Rep. (pt. 2), 1841; "Examples of the dialytic method of elimination as applied to ternary systems of equations," Camb. M. Jour. II., 1841; "On the amount and distribution of multiplicity in an algebraic equation," Phil. Mag. XVII., 1841; "On a new and more general theory of multiple roots," Phil. Mag. XVIII., 1841; "On a linear method of eliminating between double, treble and other systems of algebraic equations," Phil. Mag. XVIII., 1841; "On the dialytic method of elimination," Phil. Mag. XXI., Irish Acad. Proc. II.

This was left behind in Ireland, on the way to Virginia. Then suddenly occurs a complete stoppage in this wonderful productivity. Not one paper, not one word, is dated from the University of Virginia. Not until 1844 does the wounded bird begin again feebly to chirp, and indeed it is a whole decade before the song pours forth again with mellow vigor that wins a waiting world.

Disheartening was the whole experience; but the final cause of his sudden abandonment of the University of Virginia I gave in an address entitled, "Original Research and Creative Authorship the Essence of University Teaching," printed in *Science*, N. S., Vol. I., pp. 203–7, February 22, 1895.

On the return to England with heavy heart and dampened ardor, he takes up for his support the work of an actuary and then begins the study of law. In 1847 we find him at 26 Lincoln's Inn Fields, "eating his terms." On November 22, 1850, he is called to the bar and practices conveyancing.

But already in his paper dated August 12, 1850, we meet the significant names Boole, Cayley, and harvest is at hand.

The very words which must now be used to say what had already happened and what was now to happen were not then in existence. They were afterward made by Sylvester and constitute in themselves a tremendous contribution. As he himself says: "Names are, of course, all important to the progress of thought, and the invention of a really good name, of which the want, not previously perceived, is recognized, when supplied, as having ought to be felt, is entitled to rank on a level in importance, with the discovery of a new scientific theory."

Elsewhere he says of himself: "Perhaps I may without immodesty lay claim to the appellation of the Mathematical Adam, as I believe that

I have given more names (passed into general circulation) to the creatures of the mathematical reason than all the other mathematicians of the age combined."

In one year, 1851, Sylvester created a whole new continent, a new world in the universe of mathematics. Demonstration of its creation is given by the Glossary of New Terms which he gives in the *Philosophical Transactions,* Vol. 143, pp. 543–548.

Says Dr. W. Franz Meyer in his exceedingly valuable Bericht über die Fortschritte der projectiven Invariantentheorie, the best history of the subject (1892):

"Als äusseres Zeichen für den Umfang der vorgeschrittenen Entwickelung mag die ausgedehnte, grösstenteils von *Sylvester* selbst herrührende Terminologie dienen, die sich am Ende seiner grossen Abhandlung über Sturm'sche Functionen (1853) zusammengestellt findet."

Using then this new language, let us briefly say what had happened in the decade when Sylvester's genius was suffering from its Virginia wound. The birthday of the giant *Theory of Invariants* is April 28, 1841, the date attached by George Boole to a paper in the Cambridge *Mathematical Journal* where he not only proved the invariantive property of discriminants generally, but also gave a simple principle to form simultaneous invariants of a system of two functions. The paper appeared in November, 1841, and shortly after, in February, 1842, Boole showed that the polars of a form lead to a broad class of covariants. Here he extended the results of the first article to more than two Forms. Boole's papers led Cayley, nearly three years later (1845), to propose to himself the problem to determine *a priori* what functions of the coefficients of an equation possess this property of invariance, and he discovered its possession by other functions besides discriminants, for example the quadrinvariants of binary quantics, and in particular the invariant S of a quartic.

Boole next discovered the other invariant T of a quartic and the expression of the discriminant in terms of S and T. Cayley next (1846) published a symbolic method of finding invariants. Early in 1851 Boole reproduced, with additions, his paper on Linear Transformations; then at last began Sylvester. He always mourned what he called "the years he lost fighting the world"; but, after all, it was he who made the Theory of Invariants.

Says Meyer: "sehen wir in dem Cyklus *Sylvester*'scher Publicationen (1851–1854) bereits die Grundzüge einer allegemeinen Theorie erstehen, welche die Elemente von den verschiedenartigsten Zweigen der späteren Disciplin umfasst." "Sylvester beginnt damit, die Ergebnisse seiner Vorgänger unter einem einzigen Gesichtspunkte zu vereinigen."

With deepest foresight Sylvester introduced, together with the original variables, those dual to them, and created the theory of contravariants and intermediate forms. He introduced, with many other processes for producing invariantive forms, the principle of mutual differentiation.

Hilbert attributes the sudden growth of the theory to these processes for producing and handling invariantive creatures. "Die Theorie dieser Gebilde erhob sich, von speciellen Aufgaben ausgehend, rasch zu grosser Allgemeinheit — dank vor Allem dem Umstande, dass es gelang, eine Reihe von besonderen der Invariantentheorie eigenthümlichen Prozessen zu entdecken, deren Anwendung die Aufstellung und Behandlung invarianter Bildungen beträchtlich erleichterte."

"Was die Theorie der algebraischen Invarianten anbetrifft so sind die ersten Begründer derselben, *Cayley* und *Sylvester,* zugleich auch als die Vertreter der naiven Periode anzusehen: an der Aufstellung der einfachsten Invariantenbildungen und an den eleganten Anwendungen auf die Auflösung der Gleichungen der ersten 4 Grade hatten sie die unmittelbare Freude der ersten Entdeckung." It was Sylvester alone who created the theory of canonic forms and proceeded to apply it with astonishing power. What marvelous mass of brand new being he now brought forth!

Moreover he trumpeted abroad the eruption. He called for communications to himself in English, French, Italian, Latin or German, so only the "Latin character" were used.

From 1851 to 1854 he produces forty-six different memoirs. Then comes a dead silence of a whole year, broken in 1856 by a feeble chirp called "A Trifle on Projectiles."

What has happened? Some more "fighting the world." Sylvester declared himself a candidate for the vacant professorship of geometry in Gresham College, delivered a probationary lecture on the 4th of December, 1854, and was ignominiously "turned down." Let us save a couple of sentences from this lecture:

"He who would know what geometry is must venture boldly into its depths and learn to think and feel as a geometer. I believe that it is impossible to do this, to study geometry as it admits of being studied, and I am conscious it can be taught, without finding the reasoning invigorated, the invention quickened, the sentiment of the orderly and beautiful awakened and enhanced, and reverence for truth, the foundation of all integrity of character, converted into a fixed principle of the mental and moral constitution, according to the old and expressive adage '*abeunt studia in mores.*'"

But this silent year concealed still another stunning blow of precisely the same sort, as bears witness the following letter from Lord Brougham to The Lord Panmure:

"BROUGHAM,
PRIVATE. 28 Aug. 1855.

MY DEAR P.

My learned excellent friend and brother mathematician Mr. Sylvester is again a candidate for the professorship at Woolwich on the death of Mr. O'Brian who carried it against him last year.

I entreat once more your favorable consideration of this eminent man who has already to thank you for your great kindness.

Yours sincerely,
H. BROUGHAM.

On this third trial, backed by such an array of credentials as no man ever presented before, he barely scraped through, was appointed professor of mathematics at the Royal Military Academy, and served at Woolwich exactly 14 years, 10 months, and 15 days.

A single sentence of his will best express his greatest achievement there and his manner of exit thence:

"If Her most Gracious Majesty should ever be moved to recognize the palmary exploit of the writer of this note in the field of English science as having been the one successfully to resolve a question and conquer an algebraical difficulty which had exercised in vain for two centuries past, since the time of Newton, the highest mathematical intellects in Europe (Euler, Lagrange, Maclaurin, Waring among the number), by conferring upon him some honorary distinction in commemoration of the deed, he will crave the privilege of being allowed to enter the royal presence, not covered, like De Courcy, but barefooted, with rope around his waist, and a *goose*-quill behind his ear, in token of repentant humility, and as an emblem of convicted simplicity in having once supposed that on such kind of success he could found any additional title to receive fair and just consideration at the hands of Her Majesty's Government when quitting his appointment as public professor at Woolwich under the coercive operation of a non-Parliamentary retrospective and utterly unprecedented War Office enactment." Athenæum Club, January 31, 1871. Of course this means a row of barren years, 1870, 1871, 1872, 1873.

The fortunate accident of a visit paid Sylvester in the autumn of 1873 by Pafnuti Lvovich Chebyshev, of the University of St. Petersburg, reawakened our genius to produce in a single burst of enthusiasm a new branch of science.

On Friday evening, January 23, 1874, Sylvester delivered at the Royal Institution a lecture entitled "On Recent Discoveries in Mechanical Conversion of Motion," whose ideas, carried on by two of his hearers, H. Hart and A. B. Kempe, have made themselves a permanent place even in the elements of geometry and kinematics. A synopsis of this lecture was published, but so curtailed and twisted into the third person that the life and flavor are quite gone from it. I possess the unique manuscript of this epoch-making lecture as actually delivered. A few sentences will show how characteristic and inimitable was the original form:

"The air of Russia seems no less favorable to mathematical acumen than to a genius for fable and song. Lobacheffsky, the first to mitigate the severity of the Euclidean code and to beat down the bars of a supposed adamantine necessity, was born (a Russian of Russians), in the government of Nijni Novgorod; Tchebicheff [Chebyshev], the prince and conqueror of prime numbers, able to cope with their refractory character and to confine the stream of their erratic flow, their progression, within algebraic limits, in the adjacent circumscription of Moscow; and our own Cayley was cradled amidst the snows of St. Petersburg." [Sylvester himself contracted Chebyshev's limits for the distribution of primes.] "I think I may fairly affirm that a simple direct solution of the problem of the duplication of the cube by mechanical means was never accomplished down to this day. I will not say but that, by a merciful interpretation of his oracle, Apollo may have put up with the solution which the ancient geometers obtained by means of drawing two parabolic curves; but of this I feel assured that had I been then alive, and could have shown my solution, which I am about to exhibit to you, Apollo would have leaped for joy and danced (like David before the ark), with my triple cell in hand, in place of his lyre, before his own duplicated altar."

That in the very next year Sylvester was taking a more active part than has hitherto been known in the organization of the incipient Johns Hopkins University is seen from the following letter to him in London from the great Joseph Henry:

SMITHSONIAN INSTITUTION,
August 25, 1875.

MY DEAR SIR:

Your letter of the 13th inst. has just been received and in reply I have to say that I have written to President Gilman of the Hopkins University giving my views as to what it ought to be and have stated that if properly managed it may do more for the advance of literature and science in this country than any other institution ever established; it is entirely independent of public favor and may lead instead of following popular opinion.

I have advised that liberal salaries be paid to the occupants of the principal chairs and that to fill them the best men in the world who can be obtained should be secured.

I have mentioned your name prominently as one of the very first mathematicians of the day; what the result will be, however, I can not say.

The Trustees are all citizens of Baltimore and among them I have some personal friends; the President, Mr. Gilman, and one of them, came to Washington a few weeks ago to get from me any suggestions that I might have to offer.

It is to be regretted that in this country the Trustees, who control the management of bequests of this character, think it important to produce a palpable manifestation of the institution to be established by spending a large amount of the bequest in architectural displays. Against this custom I have protested and have asserted that if the proper men and necessary implements of instruction are provided, the teaching may be done in log cabins.

It would give me great pleasure to have you again as my guest, and I will do what I can to secure your election.

Very truly your friend,
JOSEPH HENRY.

We know the result.

Sylvester was offered the place; demanded a higher salary; won; came.

I was his first pupil, his first class, and he always insisted that it was I who brought him back to the Theory of Invariantive Forms. In a letter to me of September 24, 1882, he writes: "Nor can I ever be oblivious of the advantage which I derived from your well-grounded persistence in inducing me to lecture on the Modern Algebra, which had the effect of bringing my mind back to this subject, from which it had for some time previously been withdrawn, and in which I have been laboring, with a success which has considerably exceeded my anticipations, ever since."

He made this same statement at greater length in his celebrated address at the Johns Hopkins on February 22, 1877: "At this moment I happen to be engaged in a research of fascinating interest to myself, and which, if the day only responds to the promise of its dawn, will meet, I believe, a sympathetic response from the professors of our divine algebraical art wherever scattered through the world.

"There are things called Algebraical Forms; Professor Cayley calls them Quantics. These are not, properly speaking, Geometrical Forms, although capable, to some extent, of being embodied in them, but rather schemes of processes, or of operations for forming, for calling into existence, as it were, algebraic quantities.

"To every such Quantic is associated an infinite variety of other forms that may be regarded as engendered from and floating, like an atmosphere, around it; but infinite in number as are these derived existences, these emanations from the parent form, it is found that they admit of being obtained by composition, by mixture, so to say, of a certain limited number of fundamental forms, standard rays, as they might be termed, in the Algebraic Spectrum of the Quantic to which they belong; and, as it is a leading pursuit of the physicists of the present day to ascertain the fixed lines in the spectrum of every chemical substance, so it is the aim and object of a great school of mathematicians to make out the fundamental derived forms, the Covariants and Invariants, as they are called, of these Quantics.

"This is the kind of investigation in which I have, for the last month or two, been immersed, and which I entertain great hopes of bringing to a successful issue.

"Why do I mention it here? It is to illustrate my opinion as to the invaluable aid of teaching to the teacher, in throwing him back upon his own thoughts and leading him to evolve new results from ideas that would have otherwise remained passive or dormant in his mind.

"But for the persistence of a student of this university in urging upon me his desire to study with me the modern algebra I should never have been led into this investigation; and the new facts and principles which I have discovered in regard to it (important facts, I believe) would, so far as I am concerned, have remained still hidden in the womb of time. In vain I represented to this inquisitive student that he would do better to take up some other subject lying less off the beaten track of study, such as the higher parts of the Calculus or Elliptic Functions, or the theory of Substitutions, or I wot not what besides. He stuck with perfect respectfulness, but with invincible pertinacity, to his point. He would have the New Algebra (Heaven knows where he had heard about it, for it is almost unknown on this continent), that or nothing. I was obliged to yield, and what was the consequence? In trying to throw light upon an obscure explanation in our text-book my brain took fire; I plunged with requickened zeal into a subject which I had for years abandoned, and found food for thoughts which have engaged my attention for a considerable time past, and will probably occupy all my powers of contemplation advantageously for several months to come."

Another specific instance of the same thing he mentions in his paper, "Proof of the Hitherto Undemonstrated Fundamental Theorem of Invariants," dated November 13, 1877:

"I am about to demonstrate a theorem which has been waiting proof for the last quarter of a century and upwards. It is the more necessary that this should be done, because the theorem has been supposed to lead to false conclusions, and its correctness has consequently been impugned. Thus in Professor Faà de Bruno's valuable *Théorie des formes binaires*, Turin. 1876, at the foot of page 150 occurs the following passage: "Cela suppose essentiellement que les équations de condition *soient toutes inde'-*

pendantes entr'elles, ce qui n'est pas toujours le cas, ainsi qu'il résulte des recherches du Professor Gordan sur les nombres des covariants des formes quintique et sextique."

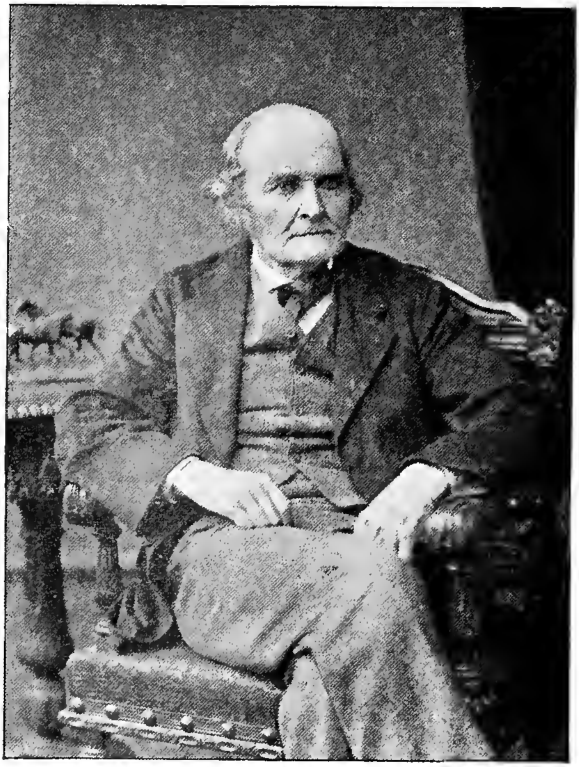
The reader is cautioned against supposing that the consequence alleged above does result from Gordan's researches, which are indubitably correct. This supposed consequence must have arisen from a misapprehension, on the part of M. de Bruno, of the nature of Professor Cayley's rectification of the error of reasoning contained in his second memoir on Quantics, which had led to results discordant with Gordan's. Thus error breeds error, unless and until the pernicious brood is stamped out for good and all under the iron heel of rigid demonstration. In the early part of this year Mr. Halsted, a fellow of Johns Hopkins University, called my attention to this passage in M. de Bruno's book; and all I could say in reply was that 'the extrinsic evidence in support of the independence of the equations which had been impugned rendered it in my mind as certain as any fact in nature could be, but that to reduce it to an exact demonstration transcended, I thought, the powers of the human understanding.'"

In 1883 Sylvester was made Savilian professor of geometry at Oxford, the first Cambridge man so honored since the appointment of Wallis in 1649.

To greet the new environment, he created a new subject for his researches — Reciprocants, which has inspired, among others, J. Hammond, of Oxford; McMahon, of Woolwich; A. R. Forsyth, of Cambridge; Leudesdorf, Elliott and Halphen.

Sylvester never solved exercise problems such as are proposed in the *Educational Times*, though he made them all his life long down to his latest years. For example, *unsolved* problems by him will be found even in Vol. LXII. and Vol. LXIII. of the *Educational Times* reprints (1895). If at the time of meeting his own problem he met also a neat solution he would communicate them together, but he never solved any. In the meagre notices that have been given of Sylvester the strangest errors abound. Thus C. S. Pierce, in the *Post*, March 16th, speaks of his accepting, "with much diffidence," a word whose meaning he never knew; and gives 1862 as the date of his retirement from Woolwich, which is eight years wrong, as this forced retirement was July 31, 1870, after his 55th birthday. Cajori, in his inadequate account (History of Mathematics, p. 326), puts the studying of law before the professorship at University College and the professorship at the University of Virginia, both of which it followed. Effect must follow cause. And strange, that of the few things he ascribes to Sylvester, he should have hit upon something not his, "the discovery of the partial differential equations satisfied by the invariants and covariants of binary quantics." But Sylvester has explicitly said in Section VI. of his "Calculus of Forms": "I alluded to the partial differential equations by which every invariant may be defined. M. Aronhold, as I collect from private information, was the first to think of the application of this method to the subject; but it was Mr. Cayley who communicated to me the equations which define the invariants of functions of two variables."

Surely he needs nothing but his very own, this marvellous man who gave so lavishly to every one devoted to mathematics, or, indeed, to the highest advance of human thought in any form. — By George Bruce Halsted. From *the American Mathematical Monthly.*



ARTHUR CAYLEY.

BIOGRAPHY.

ARTHUR CAYLEY.

Arthur Cayley was born at Richmond in Surrey, England, August the 16th, 1821. His father, Henry Cayley, was descended from the Cayleys of Brompton, in Yorkshire, but was at the time a merchant of St. Petersburg where he had married a Russian lady. In 1829 his parents took up their permanent residence at Blackheath in England; and Arthur was there educated at a private school for four years. At the age of 14 he was sent to King's College School, London; and the master of that school having observed the promise of a mathematical genius advised the father to educate his son not for his own business, but to enter the University of Cambridge.

In 1838 Arthur Cayley entered Trinity College, Cambridge, at the rather early age of 17. Throughout his undergraduate course he was first at his college examinations by an enormous interval, and he finished his undergraduate career in 1842 by carrying off the two highest honors, namely, the first place, or Senior Wrangler, in the Mathematical Tripos, and the first prize in the competition for the Smith Prizes. Immediately elected a Fellow of his College, he continued to reside at Cambridge for several years, during which time he lectured on mathematics, and also contributed papers to the *Cambridge Mathematical Journal.* His first contribution to that Journal was made, when he was an undergraduate. in 1841.

At that time it was necessary for a Fellow to take Holy Orders, or else resign the fellowship at the end of seven years. Mr. Cayley chose the latter alternative, and became by profession a conveyancer in Lincoln's Inn, London. He followed that profession for 14 years with conspicuous ability and success, and at the same time made many of his most important contributions to mathematical science.

About 1861 the Lucasian professorship of mathematics at Cambridge — the chair made illustrious by Sir Isaac Newton — fell vacant; it was filled by G. G. Stokes, already eminent for his work in mathematical physics, and Senior Wrangler the year before Cayley. However, it was felt desirable to secure Cayley also, and for this purpose the Sadlerian professorship of mathematics was created, which resulted in Cayley marrying and settling down at Cambridge, in 1863.

The duties of the Sadlerian professor were defined as follows: "to explain and teach the principles of pure mathematics, and to apply himself to the advancement of the science." In carrying out the former part of the duties Professor Cayley did not give the same course of lectures year after year, but each year took for his subject that of the memoir on which he was engaged. As a consequence his students were few, for advanced work of that kind did not pay in the great mathematical examination. How well he carried out the second part of the duties may be inferred from the fact that the Royal Society Catalogue of Scientific Papers enumerates 430 memoirs contributed by him between the years 1863 and 1883, making a total up to the latter date of 724. As he continued active to the last, it is probable that the grand total of his papers does not fall short of 1000. Some of his most celebrated contributions are: Chapters in the Analytical Geometry of (n) Dimensions, On the theory of Determinants, On the theory of linear transformations, Ten Memoirs on Quantics, Memoir on the theory of Matrices, Memoirs on Skew Surfaces, otherwise Scrolls, On the Motion of Rotation of a Solid Body, On the triple tangent planes of surfaces of the third order. Several of his achievements are elegantly referred to in a poem written by his colleague Clerk Maxwell in 1874, and addressed to the Committee of subscribers who had charge of the Cayley Portrait Fund:

O wretched race of men, to space confined!
What honor can ye pay to him whose mind
To that which lies beyond hath penetrated?
The symbols he hath formed shall sound his praise,
And lead him on through unimagined ways
To conquests new, in worlds not yet created.

First, ye Determinants, in order row
And massive column ranged, before him go,
To form a phalanx for his safe protection,
Ye powers of the *n*th root of—1!
Around his head in endless cycles run,
As unembodied spirits of direction.

And you, ye undevelopable scrolls!
Above the host wave your emblazoned rolls,
Ruled for the record of his bright inventions.
Ye cubic surfaces! by threes and nines
Draw round his camp your seven and twenty lines
The seal of Solomon in three dimensions.

March on, symbolic host! with step sublime,
Up to the flaming bounds of Space and Time!
There pause, until by Dickenson depicted,
In two dimensions, we the form may trace
Of him whose soul, too large for vulgar space,
In n dimensions flourished unrestricted.

The portrait was presented to Trinity College, and now adorns their Hall. He is represented as seated at a desk, with quill in hand, and thinking out intently some mathematical idea.

But mathematical science was advanced by Professor Cayley in yet another way. By his immense learning, his impartial judgment, and his friendly sympathy with other workers, he was eminently qualified to act as a referee on mathematical papers contributed to the various societies. Of this kind of work he did a large amount, and of his kindliness to young investigators I can speak from personal experience. Several papers which I read before the Royal Society of Edinburgh were referred to him, and he recommended their publication. Some time after I attended a meeting of the Mathematical Society of London, but the friend who would have introduced me could not be present. Professor Cayley was present, and on finding out who I was, gave me a cordial handshake, and referred in the kindest terms to the papers he had read. His was a cosmopolitan spirit, delighting only in the truth, and friendly to all seekers after the truth.

Among Cayley's papers there are several on a "Question in the Theory of Probabilities." The question was propounded by Boole, and he applied to its solution the general method of "The Laws of Thought." It was afterwards discussed by Wilbraham, Cayley and others in the *Philosophical Magazine.* My attention was drawn to the question when writing the *Principles of the Algebra of Logic,* and I ventured to contribute my idea of the question to the *Educational Times.* On mentioning the matter to Professor Kelland, he intimated pretty plainly that the discussion had been closed by Professor Cayley, and that it was temerity on my part to write anything on the subject. But the great mathematician did not think so; he wrote me a letter discussing the question and my particular way of viewing it, as well as the fundamental ideas in which I differed from Boole.

In 1882 he received a flattering invitation from the trustees of the Johns Hopkins University to deliver a course of lectures on some subject in advanced mathematics. He chose as his subject the Elliptic and Abelian functions; and the impression which his presence created has been well described by Dr. Matz in his brief notice in the January number of the MONTHLY.

Next year he was president of the British Association at the Southport meeting. In his address he spoke of the foundations of mathematics, reviewed the more important theories, traced the connection of pure with applied mathematics, and gave an outline of the vast extent of Modern Mathematics.

He regarded the complex number $a + bi$ as the fundamental quantity of mathematical analysis, and considered that with such a basis, algebra was a complete and bounded science, in which no further imaginary symbols could spring up. It is the more remarkable that he held such a view, when we consider that early in his career he made a notable contribution to space analysis. Starting from Rodrigues' formulæ for the rotation of a solid body, he arrived at the quaternion formula, and was anticipated by Hamilton only by a few months. But Cayley took a Cartesian view of analysis to the last, as is evident from the chapter which he contributed to Tait's *Treatise on Quaternions.* His aim there is to give an analytical theory of quaternions. Hamilton's aim on the other hand was to give a quaternionic theory of analysis. The difference is brought out still more strikingly in a paper printed in the last number of the *Proceedings of the Royal Society of Edinburgh.*

In 1889 the Cambridge University Press commenced the re-publication of his mathematical papers in a collected form. It was calculated that they would occupy 10 quarto volumes; 12 volumes have already appeared; and it is believed that 13 volumes will be required. No mathematician has ever had his works printed in a more handsome manner. In addition he is the author of a separate work on *Elliptic Functions.*

Space fails to enumerate the honors which he received from Universities and Scientific Academies both of the Old and of the New World. But we may mention specially, that from the Royal Society he received a Royal Medal and a Copley Medal; from the Mathematical Society of London the first DeMorgan Medal; and at the instance of the President and Members of the French Academy he was made an Officer of the Legion of Honour.

On the 26th of January, 1895, he died at Cambridge. His body was laid to rest in Mill Road Cemetery in the presence of official representatives from foreign countries and many of the most illustrious philosophers of England. His spirit still speaks to us from his works, and will continue to speak to many succeeding generations. — By Dr. Alexander Macfarlane. From the *American Mathematical Monthly,* Vol. II., No. 4. In the same number is also an interesting biography of Cayley, by Dr. George Bruce Halsted.

FALLACIES.

I. ARITHMETICAL FALLACIES.

First Fallacy.— Assume that $a=b$. Then

$$ab=a^2$$
$$\therefore\ ab-b^2=a^2-b^2$$
$$\therefore\ b(a-b)=(a+b)(a-b)$$
$$\therefore\ b=a+b$$
$$\therefore\ b=2b.$$
$$\therefore\ 1=2.$$

Second Fallacy.— Let a and b be two unequal numbers, and let c be their arithmetical mean. Then

$$a+b=2c$$
$$\therefore\ (a+b)(a-b)=2c(a-b)$$
$$\therefore\ a^2-2ac=b^2-2bc$$
$$\therefore\ a^2-2ac+c^2=b^2-2bc+c^2$$
$$\therefore\ (a-c)^2=(b-c)^2$$
$$\therefore\ a-c=b-c$$
$$\therefore\ a=b.$$

Since we assumed that a and b were unequal, where is the fallacy in our reasoning?

Third Fallacy.— We have $(-1)^2=1$. Taking logarithms,

$$2\log.(-1)=\log. 1=0$$
$$\therefore\ \log.(-1)=0$$
$$\therefore\ -1=e^0$$
$$\therefore\ -1=1,\ \text{since } e^0=1.$$

Fourth Fallacy.—We know that

$$\log.(1+x)=x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3- \ldots.$$

If $x=1$, the resulting series is convergent; hence

$$\log. 2=1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}-\tfrac{1}{6}+\tfrac{1}{7}-\tfrac{1}{8}+\tfrac{1}{9}- \ldots.$$
$$2\log. 2=2-1+\tfrac{2}{3}-\tfrac{1}{2}+\tfrac{2}{5}-\tfrac{1}{3}+\tfrac{2}{7}-\tfrac{1}{4}+\tfrac{2}{9}- \ldots.$$

Taking those terms together which have a common denominator, we obtain

$$2\log. 2=1+\tfrac{1}{3}-\tfrac{1}{2}+\tfrac{1}{5}+\tfrac{1}{7}-\tfrac{1}{4}+\tfrac{1}{9} \ldots.$$
$$=1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5} \ldots.$$
$$=\log. 2.$$
$$\therefore\ 2=1.$$

Fifth Fallacy.— We can write $\sqrt{-1}=\sqrt{-1}$ in the form

$$\sqrt{\frac{-1}{1}}=\sqrt{\frac{1}{-1}}$$
$$\therefore\ \frac{\sqrt{-1}}{\sqrt{1}}=\frac{\sqrt{1}}{\sqrt{-1}}$$
$$\therefore\ (\sqrt{-1})^2=(\sqrt{1})^2$$
$$\therefore\ -1=1.$$

Sixth Fallacy.— The mathematical theory of probability leads to various paradoxes; of these, one specimen follows: Suppose three coins to be thrown up and the fact whether each comes down head or tail to be noticed. The probability that all the coins come down head is $(\frac{1}{2})^3$, that is, $\frac{1}{8}$; similarly, the probability that all three coins come down tail is $\frac{1}{8}$; hence, the probability that all come down alike, that is, either all of them heads or all of them tails, is $\frac{1}{4}$. But, of three coins thus thrown up, at least two must come down alike. The probability that the third comes down head is $\frac{1}{2}$ and the probability that it comes down tail is $\frac{1}{2}$. Thus the probability that it comes down the same as the other two is $\frac{1}{2}$. Hence, the probability that all the coins come down alike is $\frac{1}{2}$.

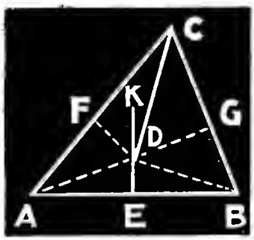
II. GEOMETRICAL FALLACIES.

Fallacious proofs of theorems are often given by the reasoner's following a figure too closely. Beginners in geometry often abandon reason and rely on the figure they have drawn to establish the truth they wish to prove. The remark that "you can see that is true from the figure," is so common, and ocular proofs are so often substituted for logical analysis, that we shall here prove the falsity of two or three propositions.

I. **Proposition.**— *All triangles are isosceles.*

II. **Given** any triangle ABC,

III. **To prove** triangle ABC is isosceles.



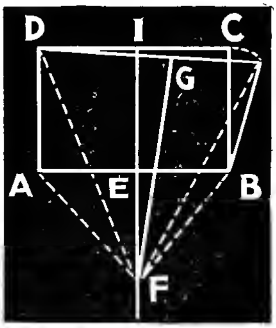
IV. **Proof.**

1. Draw KE perpendicular to AB at the middle point of AB; and
2. draw CD, the bisector of the angle C, intersecting the line, KE, in D.
3. Draw the perpendiculars, DF and DG, to the sides AC and BC, respectively.
4. Then $DG=DF$. (Why?)
5. $\therefore CF=CG$.
6. $AD=BD$. (Why?)
7. $\therefore$ the triangles ADF and DBG are equal. (Being right triangles having $AD=BD$ and $DF=DG$.)

7. ∴ $AF=BG$.
9. ∴ $AF+FC=CG+GB$, or $AC=BC$.

Q. E. D.

I. **Proposition.**— *A right angle is greater than a right angle, or right angles are unequal.*



II. ***Given*** the rectangle $ABCD$.

III. ***To prove*** the right angle A equal to the right angle B +an angle, or to prove the right angles, A and B, unequal.

IV. ***Proof.***

1. Draw IF, the perpendicular bisector of the side, DC.
2. Draw BH equal to BC, and
3. Connect D and H.
4. Draw GF, the perpendicular bisector of DH, and let it intersect IF at F.
5. Join A and F, D and F, B and F, and H and F.
6. Then $AF=BF$.
   (Oblique lines cutting off equal distances from the foot of the perpendicular.)
7. Also $DF=HF$, for the same reason.
8. ∴ Triangles DAF and HBF are equal.
   (Having the three sides of the one respectively equal to the homologous sides of the other.)
9. ∴ The angle DAF=the angle HBF.
   (Being homologous angles of equal triangles.)
10. But angle EAF=angle EBF.
    (Being base angles of an isosceles triangle.)
11. ∴ The angle DAE=the angle HBA.
    (By subtracting the equal angles FAE and FBE from the equal angles DAF and HBF.)
12. Or right angle A=right angle B + angle HBC. Q. E. D.

I. **Problem.**— *To convert a square, containing 64 square inches, into a rectangle 5 inches by 13 inches, thus containing 65 square inches.*

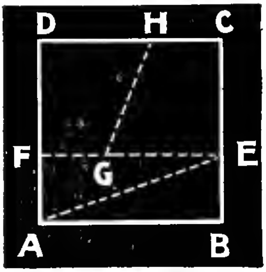


FIG. 1.

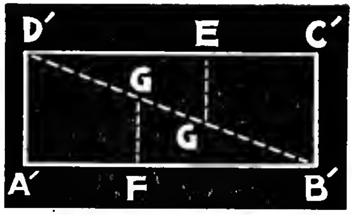


FIG. 2.

II. ***Given*** the square, *ABCD*, containing 64 sq. in.,

III. ***To convert*** this square into a rectangle, containing 65 sq. in.

IV. ***Construction.***

1. On *BC*, lay off *BE* equal to 3 inches.
2. Draw *FE* parallel to *AB*.
3. On *FE*, lay off *FG* equal to 3 inches, and on *DC* lay off *CH* equal to 3 inches.
4. Draw the lines *GH* and *AE*.
5. Then arrange the parts as shown in Fig. 2.
6. $A'B' = A'F + FB' = 8$ inches$+5$ inches$=13$ inches; and
7. $B'C'=5$ inches.
8. $\therefore$ area of $A'B'C'D'=5\times 13=65$ square inches.

Remark.—By carefully constructing the square, *ABCD*, on a piece of stiff cardboard and then cutting it into parts as indicated, and arranging these parts as shown in Fig. 2, the uninitiated will be amazed at what seems to him, a creation of something out of nothing by simply rearranging the parts of an object. Few, unacquainted with precise methods of reasoning, will detect the fallacy.

TABLE I.—*Functions of π and e.*

π = 3·1415926	π^{-1}= ·3183099	e =2·71828183
π^2= 9·8696044	π^{-2}= ·1013212	e^2 =7·38905611
π^3=31·0062761	π^{-3}= ·0322515	e^{-1}=0·3678794
$\sqrt{\pi}$= 1·7724539	$200^g \div \pi$=63^g·6619772	e^{-2}=0·1353353
$\log_{10}\pi$= 1·4971499	$180^\circ \div \pi$=57°·2957795	$\log_{10}e$ =0·43429448
$\log_e \pi$= 0·6679358	=206264″·8	$\log_e 10$ =2·30258509

TABLE II.

No.	Square root.	Cube root.
2	1·4142136	1·2599210
3	1·7320508	1·4422496
4	2·0000000	1·5874011
5	2·2360680	1·7099759
6	2·4494897	1·8171206
7	2·6457513	1·9129312
8	2·8284271	2·0000000
9	3·0000000	2·0800837
10	3·1622777	2·1544347
11	3·3166248	2·2239801
12	3·4641016	2·2894286
13	3·6055513	2·3513347
14	3·7416574	2·4101422
15	3·8729833	2·4662121
16	4·0000000	2·5198421
17	4·1231056	2·5712816
18	4·2426407	2·6207414
19	4·3588989	2·6684016
20	4·4721360	2·7144177
21	4·5825757	2·7589243
22	4·6904158	2·8020393
23	4·7958315	2·8438670
24	4·8989795	2·8844991
25	5 0000000	2·9240177
26	5·0990195	2·9624960
27	5·1961524	3·0000000
28	5·2915026	3·0365889
29	5·3851648	3·0723168
30	5·4772256	3·1072325

TABLE III.

N.	$\log_{10} N$.	$\log_e N$.
2	·3010300	·69314718
3	·4771213	1·09861229
5	·6989700	1·60943791
7	·8450980	1·94591015
11	1·0413927	2·39789527
13	1·1139434	2·56494936
17	1·2304489	2·83321334
19	1·2787536	2·94443898
23	1·3617278	3·13549422
29	1·4623980	3·36729583
31	1·4913617	3·43398720
37	1·5682017	3·61091791
41	1·6127839	3·71357207
43	1·6334685	3·76120012
47	1·6720979	3·85014760
53	1·7242759	3·97029191
59	1·7708520	4·07753744
61	1·7853298	4·11087386
67	1·8260748	4·20469262
71	1·8512583	4·26267988
73	1·8633229	4·29045944
79	1·8976271	4·36944785
83	1·9190781	4·41884061
89	1·9493900	4·48863637
97	1·9867717	4·57471098
101	2·0043214	4·61512052
103	2·0128372	4·63472899
107	2·0293838	4·67282883
109	2·0374265	4·69134788

TABLE IV.—*The Natural Logarithms (each increased by* 10) *of Numbers between* 0.00 *and* 0.99.

N.	0	1	2	3	4	5	6	7	8	9
0.0		5.395	6.088	6.493	6.781	7.004	7.187	7.341	7.474	7.592
0.1	7.697	7.793	7.880	7.960	8.034	8.103	8.167	8.228	8.285	8.339
0.2	8.391	8.439	8.846	8.530	8.573	8.614	8.653	8.691	8.727	8.762
0.3	8.796	8.829	8.861	8.891	8.921	8.950	8.978	9.006	9.032	9.058
0.4	9.084	9.108	9.132	9.156	9.179	9.201	9.223	9.245	9.266	9.287
0.5	9.307	9.327	9.346	9.365	9.384	9.402	9.420	9.438	9.455	9.472
0.6	9.489	9.506	9.522	9.538	9.554	9.569	9.584	9.600	9.614	9.629
0.7	9.643	9.658	9.671	9.685	9.699	9.712	9.726	9.739	9.752	9.764
0.8	9.777	9.789	9.802	9.814	9.826	9.837	9.849	9.861	9.872	9.883
0.9	9.895	9.906	9.917	9.927	9.938	9.949	9.959	9.970	9.980	9.990

TABLE V.—*The Natural Logarithms of Numbers between* 1.0 *and* 9.9.

N.	0	1	2	3	4	5	6	7	8	9
1	0.000	0.095	0.182	0.262	0.336	0.405	0.470	0.531	0.588	0.642
2	0.693	0.742	0.788	0.833	0.875	0.916	0.956	0.993	1.030	1.065
3	1.099	1.131	1.163	1.194	1.224	1.253	1.281	1.308	1.335	1.361
4	1.386	1.411	1.435	1.459	1.482	1.504	1.526	2.548	1.569	1.589
5	1.609	1.629	1.649	1.668	1.686	1.705	1.723	1.740	1.758	1.775
6	1.792	1.808	1.825	1.841	1.856	1.872	1.887	1.902	1.917	1.932
7	1.946	1.960	1.974	1.988	2.001	2.015	2.028	2.041	2.054	2.067
8	2.079	2.092	2.104	2.116	2.128	2.140	2.152	2.163	2.175	2.186
9	2.197	2.208	2.219	2.230	2.241	2.251	2.262	2.272	2.282	2.293

Table VI.—*The Values in Circular Measure of Angles which are given in Degrees and Minutes.*

1′	0.0003	20′	0.0058	7°	0.1222	80°	1.3963
2′	0.0006	30′	0.0087	8°	0.1396	90°	1.5708
3′	0.0009	40′	0.0116	9°	0.1571	100°	1.7453
4′	0.0012	50′	0.0145	10°	0.1745	110°	1.9199
5′	0.0015	60′ or 1°	0.0175	20°	0.3491	120°	2.0944
6′	0.0017	2°	0.0349	30°	0.5236	130°	2.2689
7′	0.0020	3°	0.0524	40°	0.6981	140°	2.4435
8′	0.0023	4°	0.0698	50°	0.8727	150°	2.6180
9′	0.0026	5°	0.0873	60°	1.0472	160°	2.7925
10′	0.0029	6°	0.1047	70°	1.2217	170°	2.9671

Table VII.—*Equivalents of Radians in Degrees, Minutes, and Seconds of Arc.*

Radians.	Equivalents.	Radians.	Equivalents.
0.0001	0° 0′ 20″.6	0.0600	3° 26′ 15″.9
0.0002	0° 0′ 41″.3	0.0700	4° 0′ 38″.5
0.0003	0° 1′ 01″.9	0.0800	4° 35′ 01″.2
0.0004	0° 1′ 22″.5	0.0900	5° 9′ 23″.8
0.0005	0° 1′ 43″.1	0.1000	5° 43′ 46″.5
0.0006	0° 2′ 03″.8	0.2000	11° 27′ 33″.0
0.0007	0° 2′ 24″.4	0.3000	17° 11′ 19″.4
0.0008	0° 2′ 45″.0	0.4000	22° 55′ 05″.9
0.0009	0° 3′ 05″.6	0.5000	28° 38′ 52″.4
0.0010	0° 3′ 26″.3	0.6000	34° 22′ 38″.9
0.0020	0° 6′ 52″.5	0.7000	40° 6′ 25″.4
0.0030	0° 10′ 18″.8	0.8000	45° 50′ 11″.8
0.0040	0° 13′ 45″.1	0.9000	51° 33′ 58″.3
0.0050	0° 17′ 11″.3	1.0000	57° 17′ 44″.8
0.0060	0° 20′ 37″.6	2.0000	114° 35′ 29″.6
0.0070	0° 24′ 03″.9	3.0000	171° 53′ 14″.4
0.0080	0° 27′ 30″.1	4.0000	229° 10′ 59″.2
0.0090	0° 30′ 56″.4	5.0000	286° 28′ 44″.0
0.0100	0° 34′ 22″.6	6.0000	343° 46′ 28″.8
0.0200	1° 8′ 45″.3	7.0000	401° 4′ 13″.6
0.0300	1° 43′ 07″.9	8.0000	458° 21′ 58″.4
0.0400	2° 17′ 30″.6	9.0000	515° 39′ 43″.3
0.0500	2° 51′ 53″.2	10.0000	572° 57′ 28″.1

To Use Table VI.—For example, express 49° 38′ 28″ in radians.

1. 40° = 0.6981 radian.
2. 9° = 0.1571 "
3. 30′ = 0·0087 "
4. 8′ = 0·0023 "
5. 28″ = $\frac{28}{60}$ of 1′ = $\frac{28}{60}$ of 0.0003 radian = 0.0001 radian.
6. Adding, 49° 38′ 28″ = 0.8643 radian.

To Use Table VII.—For example, express 1.3245 radians in degrees, minutes, and seconds.

1.	1.0000 radian	= 57°	17′	44″.3	
2.	.3000 "	= 17°	11′	19″.4	
3.	.0200 "	= 1°	8′	45″.3	
4.	.0040 "	= 0°	13′	45″.1	
5.	.0005 "	= 0°	1′	43″.1	
6.	Adding, 1.3245 radians	= 77°	53′	17″.2	

I. Find the value of θ, if $2\theta - \tan 2\theta = 2\pi$.

(See step 18, page 397.)

This is a transcendental equation, and can only be solved by trial. The method of solution is called ***Double Position,*** and is as follows:

II.

1. Assume $2\theta = \varphi'$, a value which, when substituted in the equation, gives a result a' (say), a' being less than 2π.
2. Then $\varphi' - \tan\varphi' = a'$. . . . . (1).
3. Assume $2\theta = \varphi''$, a value which, when substituted in the equation, gives a result a'' (say), a'' being greater than 2π.
4. Then $\varphi'' - \tan\varphi'' = a''$. . . . . (2).
5. $\therefore \varphi'' - \varphi' = a'' - a'$, assuming that φ' and φ'' are so nearly equal that the difference of their tangents may be omitted without serious error.
6. Also $\varphi - \varphi' = 2\pi - a'$, for same reason as in last step.
7. $\therefore \dfrac{\varphi'' - \varphi'}{\varphi - \varphi'} = \dfrac{a'' - a'}{2\pi - a'}$. . . . . (3).
8. $\therefore \varphi = \varphi' + \dfrac{2\pi - a'}{a'' - a'}(\varphi'' - \varphi')$, approximately.

Equation 3 gives the following

Rule.—*Find, by trial, two values of the unknown quantity, one giving a result too small and the other, a result too large.*

Then the difference of these results is to the difference of the two assumed numbers as the difference of the true result and either result is to the difference of the true value and the corresponding assumed value.

II.

1.

1. $\therefore$ Assume $2\theta=\frac{5}{9}\pi$. Then
2. $\frac{5}{9}\pi-\tan\frac{5}{9}\pi=1.7454+\cot 10^\circ=1.7454+5.6713=7.4167$, a result greater than 2π.
3. Assume $2\theta=\frac{7}{12}\pi$. Then
4. $\frac{7}{12}\pi-\tan\frac{7}{12}\pi=\frac{7}{12}\pi+\cot 15^\circ=1.8326+3.7321=5.5647$, a result less than 2π.
5. $\therefore 7.4167-5.5647 : \frac{5}{9}\pi-\frac{7}{12}\pi=2\pi-7.4167 : 2\theta-\frac{5}{9}\pi$.
6. $\therefore 2\theta=\frac{5}{9}\pi+\frac{2\pi-7.4167}{7.4167-5.5647}(\frac{5}{9}\pi-\frac{7}{12}\pi)=100^\circ+\frac{-1.1335}{1.752}\times(-5^\circ)=100^\circ+3^\circ.2=103^\circ+$.

2.

1. Assume $2\theta=103^\circ$. Then
2. $\frac{103}{180}\pi-\tan\frac{103}{180}\pi=1.7977+\cot 103^\circ=1.7977+4.3315=6.1292$, a result less than 2π, or 6.2832.
3. Assume $2\theta=102^\circ$. Then
4. $\frac{102}{180}\pi-\tan\frac{102}{180}\pi=1.7802+4.7046=6.4848$, a result greater than 2π.
5. $\therefore 6.4848-6.1292 : 1^\circ=6.4848-2\pi : 2\theta-102^\circ$.
6. $\therefore 2\theta=102^\circ+\frac{6.4848-2\pi}{6.4848-6.1292}\times 1^\circ=102^\circ.5$.

3.

1. Assume $2\theta=102^\circ.5$. Then
2. $\frac{102.5}{180}\pi-\tan\frac{102.5}{180}\pi=1.7889+4.5170=6.3059$, a result greater than 2π.
3. Assume $2\theta=102^\circ.6$. Then
4. $\frac{102.6}{180}\pi-\tan 102^\circ.6=1.7854+4.6057=6.2718$, a result less than 2π.
5. $\therefore 6.3059-6.2718 : 0^\circ.1=6.3059-2\pi : 2\theta-102^\circ.5$.
6. $\therefore 2\theta=102^\circ.5+\frac{6.3059-2\pi}{6.3059-6.2718}\times 0^\circ.1=102^\circ.56$.

4.

1. Assume $2\theta=102.^\circ 56$. Then
2. $\frac{102.56}{180}\pi-\tan 102^\circ.56=1.7900+4.4915=6.2815$, a result less than 2π.
3. Assume $2\theta=102^\circ.54$. Then
4. $\frac{102.54}{180}\pi-\tan 102^\circ.54=1.7897+4.4959=6.2856$, a result greater than 2π.
5. $\therefore 6.2856-6.2815 : 0^\circ.02=6.2856-2\pi : 2\theta-102^\circ.54$.
6. $\therefore 2\theta=102^\circ.54+\frac{6.2856-2\pi}{6.2856-6.2815}\times 0^\circ.02=102^\circ.55$.

5\.
1. Assume $2\theta = 102°.55$. Then
2. $\frac{102.55}{180}\pi - \tan 102°.55 = 1.7898 + 4.4922 = 6.2820$, a result less than 2π.
3. $\therefore 6.2856 - 6.2820 : 0°.01 = 6.2856 - 2\pi : 2\theta - 102°.54$.
4. $\therefore 2\theta = 102°.54 + \frac{6.2856 - 2\pi}{6.2856 - 6.2820} \times 0°.01 = 102°.54\frac{2}{3} = 102°\ 32'\ 48''$.

III. $\therefore 2\theta = 102°\ 32'\ 48''$.