EXERCISES ON EUCLID

AND IN

MODERN GEOMETRY.

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EXERCISES ON EUCLID

AND IN

MODERN GEOMETRY

FOR THE USE OF SCHOOLS, PRIVATE STUDENTS,

AND JUNIOR UNIVERSITY STUDENTS.

BY

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PREFACE.

THE present work is the result of an attempt to supply a want which has been long felt by teachers and students.

In it will be found a large collection of important propositions investigated after the manner of Euclid, which are valuable as Geometrical Exercises, and are often required in other branches of Mathematics.

The first half of the volume is confined to the Ancient Pure Geometry, and the remainder to the Modern Pure Geometry.

In the latter part will, I trust, be found a very elementary and, at the same time, a tolerably complete treatise on the Modern Geometry of the Point, Straight Line, and Circle. Indeed, throughout the work I have aimed at simplicity rather than originality or novelty, and I have been more anxious to illustrate principles than to throw together a crude and undigested mass of isolated propositions, which only tend to disgust a learner with Geometry.

The Principles of Harmonic and Anharmonic Pencils and Ranges have been fully explained, and applied to a variety of interesting and useful Propositions. Radical Axes and Centres, Geometrical Involution, Centres and Axes of Similitude, Poles and Polars, and Reciprocal Polars, have next received a proportionate share of attention.

The methods of Modern Pure Geometry have sometimes been characterised by able Mathematicians as Semi-Geometrical, because, it is asserted, they are not confined within the limits of the Ancient Pure Geometry. A very slight examination of the present work will, I believe, convince the reader that it is not liable to such a charge, but that I have strictly adhered to Euclid's methods.

As the work is intended chiefly for Schools, Private Students, and Junior University Students, I have given much more explanation than is usual in Cambridge Text Books, which are meant to be read with a private tutor.

An acquaintance with the present work will greatly facilitate the study of the excellent recent analytical works on the Higher Geometry.

J. M^{c}DOWELL.

PEMBROKE COLLEGE, CAMBRIDGE,
March 20, 1878.

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THE METHOD OF RECIPROCAL POLARS.

EXERCISES ON EUCLID

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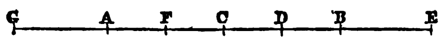
MODERN GEOMETRY.

DEF. If a point be taken on a line or a line produced, the distances of the point from the extremities of the line are called the *segments* of the line made by the point, and when the point is on the line, the line is said to be divided *internally* by the point, but when the point is on the produced part of the line, it is said to divide the line *externally*.

1. *If a line be divided internally into unequal segments, the distance of the point of section from the middle point of the line is half the difference of the segments; but if it be divided externally, the distance of the point of section from the middle point of the line is half the sum of the segments.*

Let AB be bisected in C, divided internally in D and externally in E. Make AF equal to BD, and produce BA until



AG equals BE. Because AB is bisected in C and AF equals BD, therefore FC equals CD, and FD is the difference of AD and DB. Hence CD is half the difference of the internal segments AD and DB.

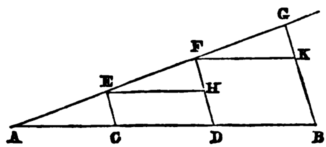
Because AG and BE are equal, and AB is bisected in C, therefore GC and CE are equal, and GE is the sum of AE and EB; therefore CE is half the sum of the external segments AE and EB.

2. *Given the sum and difference of two straight lines, find them.*

Fig. to (1). Let AB be the given sum, C the middle point of AB, and CD half the given difference of the required lines, then by (1) AD and DB are obviously the required straight lines.

3. *To divide a given straight line into any number of equal parts.*

Let AB be the given straight line. From either end (A) of it draw an indefinite straight line AG, in which take any point E, and take EF, FG, &c. each equal to AE until the number of equal parts in AG is equal to the number of parts into which AB is to be divided. Join GB, and draw EC, FD parallel to GB; then AB is divided by these parallels in C and D as required.



For draw EH, FK parallel to AB, and therefore also parallel to one another.

Since AC and EH are parallel, and AF meets them, therefore the angles EAC, FEH are equal, and because EC, FD are parallel, and AG meets them, the angles AEC and EFH are equal; also AE is equal to EF; therefore by (I. 26) AC is equal to EH and EC to FH; but ED is a parallelogram (by construction), therefore EH is equal to CD and therefore AC is equal to CD. In like manner, from the triangles EFH and FGK, it is proved that CD equals DB. Hence AB has been divided into the required number of equal parts.

Cor. Hence the straight line drawn from the middle point of any side of a triangle parallel to another side bisects the remaining side.

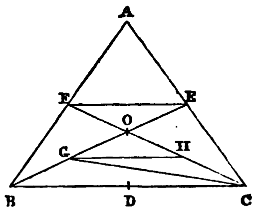
For E is the middle point of the side AF of the triangle AFD, EH is parallel to AD, and FH and HD have been each proved equal to EC, therefore FH and HD are equal.

4. *The straight line joining the middle points of two sides of a triangle is parallel to the third side and equal to half of it.*

In the Cor. to the last proposition it is proved that the straight line EH joining the middle points of the sides AF and FD of the triangle *AFD* is parallel to the third side *AD*. Also it has been proved in the same proposition that *EH* is equal to *AC* or *CD*, therefore *EH* is half of *AD*.

5. *The straight lines drawn from the vertices of a triangle to bisect the opposite sides pass through the same point and cut each other in a point of trisection, and the three triangles with this point as common vertex, and the sides as bases, are equal.*

Let *ABC* be the given triangle, *D*, *E*, *F* the middle points of its sides, and let *BE*, *CF* intersect in *O*. Bisect *BO* in *G* and *CO* in *H*; join *FE*, *GH*. Because *FE* joins the middle points of two sides of the triangle *ABC*, *FE* is parallel to *BC* and half of it, and because *GH* joins the middle points of two sides of the triangle *BOC*, *GH* is parallel to *BC* and half of it by (4), therefore *FE* and *GH* are equal and parallel, and since the straight lines *FH* and *GE* meet these two parallel lines, the angles *FEO* and *OGH*, *EFO* and *OHG* are equal, therefore (I. 26) *GO* and *OE* are equal and *HO* and *OF* are equal. But *BG* and *GO* are equal (constr.), therefore *OE* is one-third of *BE*, that is, *CF* cuts off from *BE towards AC* one-third. In the same way it can be proved that the straight line joining the points *A* and *D* cuts off from *BE* one-third part *also towards AC*, therefore the three bisectors pass through the same point *O* and cut one another in a point of trisection.

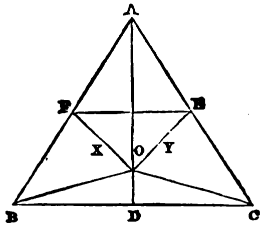


Join *GC*. Because *BG*, *GO* and *OE* are equal, the triangles *BGC*, *CGO* and *OEC* are equal, therefore the triangle *BOC* is two-thirds of *BEC*; but the triangles *BEC* and *BAE* are equal, since *AE* and *EC* are equal, therefore *BOC* is one-third of the whole triangle *ABC*; and in the same way it is shewn that if *AO* be drawn, the triangles *AOB* and *AOC* are each one-third of the triangle *ABC*.

DEF. I shall often call the straight lines drawn from the angles of a triangle to bisect the opposite sides simply *the bisectors of the sides.*

6. *The perpendiculars to the sides of a triangle through their middle points meet in the centre of the circle circumscribed about the triangle.*

Let *ABC* be the given triangle, *D*, *E*, *F* the middle points of its sides. Draw *FX*, *EY* perpendicular to *AB* and *AC* respectively. These lines must meet in a point *O*; for draw *FE*, then since the angles *AFX* and *AEY* are right, the angles *EFX* and *FEY* are together less than two right angles, and therefore *FX* and *EY* if produced must meet (Axiom 12).



Join *OD*, *AO*, *BO* and *CO*. In the triangles *AOF* and *BOF* the sides *AF* and *BF* are equal (hyp.), *FO* is common, and the angles *AFO* and *BFO* are equal being right angles, therefore the sides *AO* and *BO* are equal (I. 4). Similarly from the triangles *AOE* and *COE* the sides *AO* and *OC* are equal, therefore *BO* and *OC* are equal, and *BD* and *DC* are equal (hyp.), therefore (I. 8) the angles *BDO* and *CDO* are equal, and therefore each of them is a right angle. Hence the three perpendiculars to the sides from their middle points meet in the point *O*, which is equidistant from the three angles *A*, *B*, *C*, and is therefore the centre of the circumscribing circle.

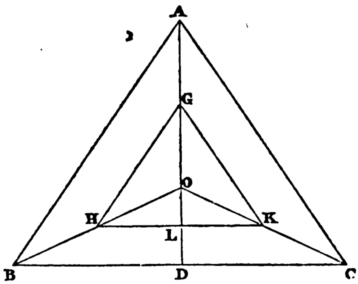
N.B. The other two cases in which the triangle *ABC* is right or obtuse-angled, may be similarly proved; but I shall not often occupy space in examining the various cases of a proposition unless when there is some peculiarity likely to present a difficulty to the mere beginner.

7. *If one triangle have its sides respectively double the sides of another triangle, the radius of its circumscribed circle is also double that of the other, and a perpendicular from the centre of its circumscribed circle on any side is double the corresponding perpendicular in the other triangle.*

Def. Sides of equiangular triangles opposite equal angles are called *corresponding* (or homologous) *sides*, and lines similarly drawn in similar figures are called *corresponding lines.*

Let *O* be the centre of the circumscribed circle of the triangle *ABC*, *G*, *H*, *K* the middle points of the radii *AO*, *BO*, *CO* respectively, and *D* the middle point of *BC*.

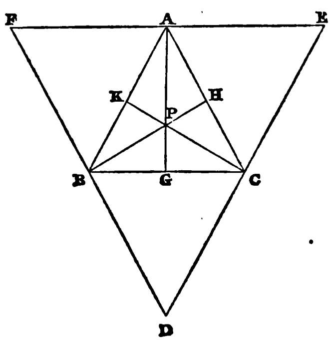
Because HK joins the middle points of two sides of the triangle BOC, HK is parallel to BC and equal to half of BC, therefore the angle OHK is equal to OBC. In like manner the side HG is half of AB and parallel to it, therefore the angle GHO is equal to ABO, and therefore the angles GHK and ABC are equal. Similarly, the angles HGK and BAC, GKH and ACB are equal, and GK is half AC. Also O is obviously the centre of the circle described about the triangle GHK, the sides of which are the halves of the sides of ABC respectively.



Because H is the middle point of BO and HK is parallel to BD, therefore (3, Cor.) OL and LD are equal. Hence the radius OB of the circle circumscribing the triangle ABC is double the radius OH of the circle circumscribing the triangle GHK, and the perpendicular OD is double the corresponding perpendicular OL.

8. *The perpendiculars from the angles of a triangle to the opposite sides pass through the same point, and the segment of any perpendicular towards an angle of the triangle is double the perpendicular from the centre of the circumscribed circle on the side opposite to that angle.*

Let ABC be the given triangle. Through the vertices draw three straight lines respectively parallel to the opposite sides so as to form the triangle DEF; then $AFBC$ and $ABCE$ are parallelograms (by construction), therefore FA and AE are each equal to the same BC, and therefore FE is bisected in A. Similarly ED is bisected in C, and DF in B.



Draw AG perpendicular to FE and meeting BC in G.

Because AG meets the two parallels FE and BC, the angles

EAG and *AGC* are together equal to two right angles, but *EAG* is a right angle, therefore *AGC* is also a right angle, or if a straight line be perpendicular to either of two parallel straight lines, it is also perpendicular to the other.

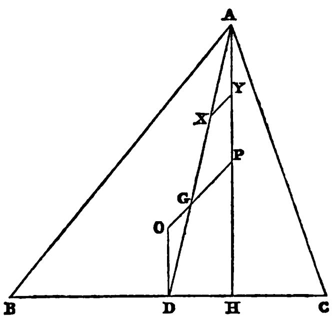
Hence the three perpendiculars to the sides of the triangle *DEF* from their middle points are also the perpendiculars from the angles to the opposite sides of the triangle *ABC*, and therefore (6) these perpendiculars meet in the same point.

Let them meet in *P*. Since (6) *P* is centre of circle circumscribed about triangle *DEF*, and *FE* in triangle *DEF* is double of corresponding side *BC* in triangle *ABC*, *PA* must (7) be double of the perpendicular from centre of circle circumscribing *ABC* to side *BC*.

DEF. The perpendiculars from the angles of a triangle to the opposite sides I shall often call simply *the perpendiculars* of the triangle. The point of intersection of these perpendiculars is called the *orthocentre* of the triangle.

9. *The centre of the circumscribed circle, the intersection of the bisectors of the sides and the intersection of the perpendiculars of a triangle, lie in the same straight line.*

Let *ABC* be the triangle, *D* the middle point of *BC*, *AH* the perpendicular on *BC*, *P* the intersection of the perpendiculars, *O* the centre of the circumscribed circle, and *AD* the bisector of the side *BC*. Join *OP* meeting *AD* in *G*.



Bisect *AG* in *X* and *AP* in *Y*, and join *XY*.

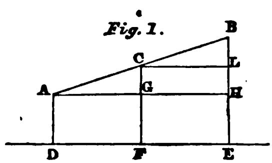
Because *XY* joins the middle points of two sides of the triangle *AGP*, *XY* is parallel to *GP* and equal to half *PG* by (4), therefore the angles *AXY* and *AGP* are equal, but *AGP* and *OGD* are also equal (I. 15), and *AY* and *OD* are equal (8).

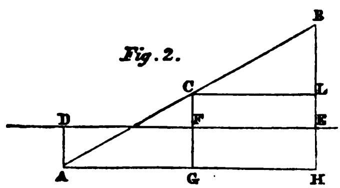
Also *AH* and *OD* are perpendicular to *BC*, and therefore they are parallel; therefore the angles *XAY* and *ODG* are equal.

Hence (I. 26) OG is equal to XY or to half GP, and GD is equal to AX or XG, therefore (5) G is the point of intersection of the bisectors of the sides. Therefore the three points are in the same straight line. Q. E. D.

10. *If from the ends and the middle point of a finite straight line three parallel straight lines be drawn meeting any indefinite straight line, the middle parallel is half the sum of the two extreme parallels when the indefinite line does not meet the finite line; but when it meets it the middle parallel is half the difference of the other parallels.*

Figs. 1 and 2. Let AB be the finite line, C its middle point, DE the indefinite line, AD, CF, and BE the three parallel straight





lines; draw AH parallel to DE meeting CF in G and BE in H, CF and BE being produced, if necessary.

Fig. 1. Since, in the triangle ABH, CG is drawn from C the middle point of AB parallel to the side BH, CG is half of BH; and because AF and GE are parallelograms, GF is equal to AD or HE, and is therefore half the sum of AD and HE. Therefore CF is half the sum of AD and BE.

Fig. 2. CG is half of BH, that is, half of BE and EH;

and FG is half of AD and EH;

therefore CF is half the difference of BE and AD.

N. B. We may write this latter part more concisely thus:

$$CG = \tfrac{1}{2}BH = \tfrac{1}{2}(BE + EH),$$

$$FG = AD \text{ or } EH = \tfrac{1}{2}(AD + EH);$$

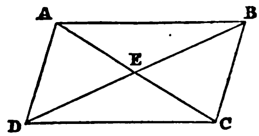
$$\therefore\ CG - FG = CF = \tfrac{1}{2}(BE - AD), \quad \text{Q. E. D.}$$

If the learner be accustomed to *vivâ voce*, which I consider by far the best method of teaching Euclid, the above symbolical method will be inadmissible; but if a proposition in Euclid is to be written out, I cannot see how it affects the reasoning whether it be written out in Algebraical language (so called) or in the Greek, French, or English language. In whatever language a proposition is written or printed, the learner should be required by the teacher to go through it *vivâ voce* in his native language.

Otherwise thus. Draw CL parallel to AH. Then $BL = LH$, and therefore by (1) in Fig. 1, LE or CF equals half the sum of BE and (EH or) AD, and in Fig. 2, LE or CF equals half the difference of BE and (EH or) AD. Q. E. D.

11. *The diagonals of a parallelogram bisect each other, and if the diagonals of a quadrilateral bisect each other it is a parallelogram. Also if the opposite sides of a quadrilateral be equal, it is a parallelogram.*

Let $ABCD$ be a parallelogram, and let its diagonals intersect in E. Because AB and DC are parallel and BD meets them, the alternate angles ABD and BDC are equal, similarly the angles BAC and DCA are equal, and the sides AB and CD are equal since they are opposite sides of a parallelogram. Therefore (I. 26) the triangles AEB and CED are equal in all respects, and therefore $AE = EC$, $BE = ED$.



Again, let $ABCD$ be a quadrilateral in which the diagonals AC and BD bisect one another, then shall $ABCD$ be a parallelogram.

Because, in the triangles AEB and CED the two sides AE, EB are respectively equal to the two CE, ED, and the contained angles are equal, therefore (I. 4) the triangles themselves are equal in all respects, and $AB = DC$, and the angle $ABE = EDC$, therefore AB and DC are parallel, and therefore $ABCD$ is a parallelogram.

Lastly, let $AB = DC$ and $AD = BC$, then shall $ABCD$ be a parallelogram. For the triangles ABC and ACD are equal in all respects by (I. 8), therefore the angles BAC and ACD, BCA and CAD are equal. Therefore AB is parallel to CD, and BC to AD. Q. E. D.

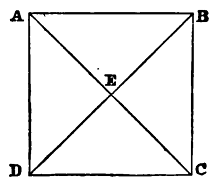
12. *The diagonals of a rectangle are equal to one another.*

In the figure to (11) suppose $ABCD$ a rectangle.

Because AB and BC are respectively equal to DC and CB, and the angles ABC and DCB are each right, therefore (I. 4) AC and DB are equal.

13. *The diagonals of a square bisect one another at right angles.*

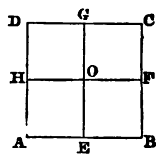
Let $ABCD$ be a square, and let its diagonals meet in E.



In the triangle ABD, $AB=AD$, and the angle BAD is right, therefore (I. 5, 32) the angles ABD and ADB are each half a right angle. Similarly, the angle BAC is proved to be half a right angle; therefore the angle AEB is a right angle, and therefore also (I. 13) BEC is a right angle, and the angles vertically opposite to these are right (I. 15). Also the diagonals bisect one another by (11). Q. E. D.

14. *The square on a straight line is four times the square on its half.*

Let $ABCD$ be a square on the straight line AB, and let the straight lines joining the middle points of its opposite sides intersect in O.

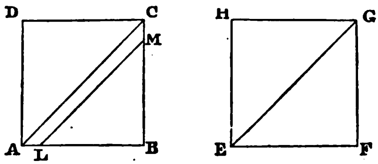


Because AE and DG are equal and parallel, therefore EG is parallel to AD or BC. In like manner HF is proved parallel to AB or CD, therefore the figures AO, BO, CO and DO are parallelograms; and since these parallelograms have the angles at A, B, C, D right, and the containing sides each equal to half of AB, they must be the squares on the half of AB. Therefore the square on AB is four times the square on its half. Q. E. D.

15. *If two squares be equal, their sides shall also be equal.*

For if possible let the equal squares BD and FH *not* have their sides equal, and suppose AB greater than EF, and make BL and BM each equal EF. Draw the diagonals AC and EG bisecting the squares (I. 34) so that the triangles ABC and EFG are equal, but the triangles LBM and EFG are also equal since the sides LB

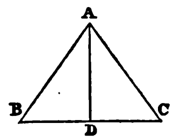
and BM are equal to EF and FG each to each, and the contained angles are equal, being right, therefore the triangle LBM is equal



the triangle ABC, which is impossible, therefore AB and EF are not unequal, that is, they are equal. Q. E. D.

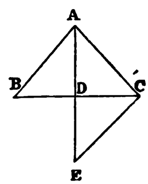
16. *The perpendicular from the vertex on the base of an isosceles triangle bisects the base and the vertical angle.*

Let ABC be an isosceles triangle, and AD the perpendicular from its vertex to its base BC. The angles ABD and ADB are respectively equal to ACD and ADC (hyp.), and AD is common to the two triangles ABD, ACD, therefore the triangles ABD and ACD are equal in all respects, and the sides BD and DC are equal, and the angles BAD and CAD.



17. *If the same straight line bisect the base and the vertical angle, the triangle is isosceles.*

Let the same straight line AD bisect the base BC and the vertical angle BAC of the triangle ABC, then shall the triangle ABC be isosceles.

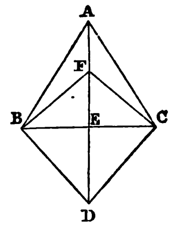


Produce AD until $DE = AD$, and join CE.

In the two triangles ADB and CDE the two sides AD, DB are equal to the two ED, DC each to each, and the contained angles at D are equal, therefore (I. 4) $AB = CE$ and the angles BAD and CED are equal; but BAD and CAD are also equal (hyp.), therefore the angles CAD and CED are equal, and therefore (I. 5) $AC = CE$, therefore $AC = AB$, or the triangle ABC is isosceles.

18. *If two isosceles triangles have a common base, the straight line (produced, if necessary) which joins their vertices bisects their common base, and is at right angles to it.*

Let ABC and DBC be isosceles triangles on the same base BC. Join AD meeting BC in E.

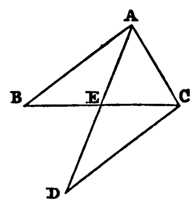


Because $AB = AC$, $BD = CD$ and AD is common to the two triangles ABD and ACD, the angles BAD and CAD are equal (I. 8).

Again, because AB and AC are equal, AE is common to the two triangles BAE and CAE; and the contained angles BAE and CAE have been proved equal, therefore (I. 4) $BE = EC$ and AE is perpendicular to BC.

COR. Hence the bisector of the vertical angle of an isosceles triangle also bisects the base perpendicularly, and the diagonals of a rhombus bisect one another at right angles.

19. *Half the base of a triangle is greater than, equal to, or less than the bisector of base, according as the vertical angle is greater than, equal to, or less than a right angle.*



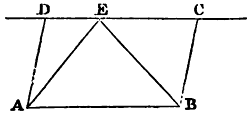
Let AE bisect the base of the triangle ABC in E, and produce AE until $DE = EA$, and join CD. Because AE, EB are respectively equal to DE, EC, and the vertically opposite angles AEB and CED are equal, therefore (I. 4) AB and CD are equal, and the angles BAE and CDE are equal, therefore (I. 27) AB and CD are parallel, and therefore (I. 29) the angles BAC and ACD are together equal to two right angles. Therefore when BAC is a right angle, ACD is also a right angle, and therefore in the triangles BAC and ACD the sides BA and AC are respectively equal to DC and CA, and the contained angles are right, therefore (I. 4) BC and AD are equal, and therefore their halves BE and AE are equal.

When the angle BAC is acute, ACD is obtuse, and therefore (I. 24) BC is less than AD, and therefore also BE less than AE; and when the angle BAC is obtuse, ACD is acute, and therefore (I. 24) BC is greater than AD, and therefore also BE greater than AE. Q. E. D.

DEF. When the conditions of a problem are not sufficient to determine a point absolutely, but restrict it to a certain line (or lines), this line is called the *locus* of the point.

20. *Given the base and area of a triangle, find the locus of its vertex.*

Upon the given base AB construct a parallelogram $ABCD$ equal to twice the given area (I. 45). DC produced indefinitely both ways is the required locus. For take any point E in DC or in its production either way, and join AE, BE.



The triangle AEB is half the parallelogram $ADCB$ (I. 41), and therefore equal to the given area. Hence, since any point taken in CD as vertex of a triangle with base AB satisfies the conditions of the problem, DC is the required locus.

DEF. A straight line is said to be *given in position* when its direction only is given, and *in magnitude and position* when both its direction and length are given. When a line is given in magnitude and position it is often simply said to be *given*. When a figure is said to be given in area, it is meant that a square or some other figure of the same area is given.

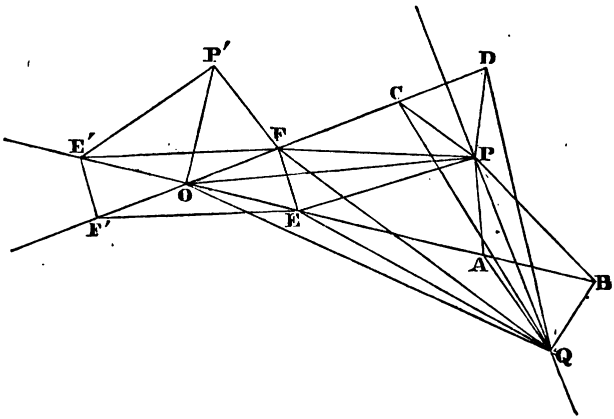
21. *Given the bases of two triangles which have a common vertex, in magnitude and position, and the sum or difference of their areas, find the locus of the common vertex.*

Let AB, CD be the given bases.

First, Fig. 1, suppose the bases, produced, if necessary, to meet in O, and let P be any point in the required locus. Make $OE = AB$ and $OF = CD$. Join the point P with the points A, B, C, D, E, F, O.

The triangles OPF and CPD, OPE and APB are equal (I. 38), therefore the figure $FPEO$ is equal to the given sum of areas, but FOE is a fixed triangle; and therefore the triangle FPE has a given area and a given base FE. Hence (20) the locus is a fixed straight line PQ parallel to EF, but when the sum of areas is given we must only take the part of this line *within* the angle DOB, for take any point Q on PQ *without* the angle DOB. Join Q with the points to which P was joined. The difference of the triangles QCD and QAB is equal to the difference of the triangles QOF and QOE, that is, to the figure $QFOE$. Therefore when the

Fig. 1.



difference of areas is given, the locus consists of the two parts of PQ produced indefinitely *outside* the angle DOB.

Again, on BO and DO produced through O, take $OE' = AB$ and $OF' = CD$ and join $E'F'$, $E'F$ and EF'. If we now take a point within the angle $E'OF'$ and on the side of $E'F'$ remote from P, we shall find, in exactly the same way as before, that a straight line parallel to $E'F'$ and at the same distance from it that PQ is from EF is another part of the required locus; that the part of this line *within* the angle $E'OF'$ belongs to the given sum of areas and the parts *outside* to the given difference of areas.

Further, suppose P' a point in the locus, and join it with the points E', O and F. In the same manner as before, it may be shewn that a straight line parallel to $E'F$ is another part of the locus, that the part of this line *within* the angle $E'OF$ belongs to the given sum of areas, and the two parts *without* this angle to the given difference of areas. Also the fourth and remaining part of the locus is the straight line parallel to EF', at the same distance from it that the locus of P' is from $E'F$, and at the side remote from P'.

Since $EFE'F'$ is a parallelogram by (11), the four distinct straight lines which constitute the locus are parallel two by two, and each pair of parallels is at the same distance from the point O.

Next, suppose the two given bases AB and CD to be parallel. First, let the bases be equal, join AC, BD, and let the given sum or difference of areas be half the parallelogram *ABDC* Suppose *P* a point in the locus. Join *P* to the points *A*, *B*, *D*, *C*, and

Fig. 2.

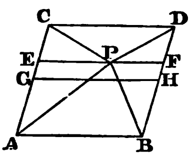
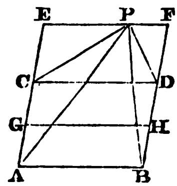


Fig. 3.



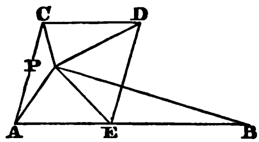
through *P* draw *EF* parallel to *AB* or *CD* and meeting *AC* and *BD*, produced, if necessary, in *E* and *F*. The triangle *APB* is half the parallelogram *AEFB* and the triangle *CPD* is half the parallelogram *CEFD*, therefore in Fig. 2, the sum of the two triangles *ABP*, *CPD* is half the parallelogram *ABDC*, and in Fig. 3 the difference of the two triangles *APB*, *CPD* is half the parallelogram *ABDC*. Hence when the two parallel bases are equal and the given sum is equal to half the parallelogram *ABDC*, the required locus is any point between the parallel bases *AB* and *CD*. If the sum of the areas *be less than* half the parallelogram *ABDC*, the problem is obviously impossible. When the given difference of areas is equal to half the parallelogram *ABDC*, the required locus is any point outside the two equal and parallel bases *AB*, *CD*.

If this difference of areas be greater than half the parallelogram *ABDC*, the problem is impossible.

The bases being still equal and parallel, let the sum of given areas be greater than half the parallelogram *ABDC*, or the given difference less than half this parallelogram. Bisect *AC* in *G* and *BD* in *H*, and join *GH*. It will be seen exactly as in (1) that the parallelogram *GEFH* is in Fig. 3 half the sum of the parallelograms *AEFB* and *CEFD*, and in Fig. 2 half their difference. Therefore in Fig. 3, the parallelogram *GEFH* is equal to the given sum of areas, and in Fig. 2 equal to the given difference of areas; hence when the sum is given, the locus consists of *two* straight lines given in position, parallel to the given bases and at equal distances from *GH*, and each straight line is *outside* the parallels *AB*, *CD*, and when the difference is given, the locus consists of two straight lines *within* *AB*, *CD*, equidistant from *GH* and parallel to the given bases.

Secondly, let the parallel bases be unequal and CD less than AB. Make $AE = CD$, and suppose P, *within* AB and CD, to be a point in the locus, when the sum of the areas is given. From P draw straight lines to A, B, C, D and E. Since the sum of the triangles AEP, CPD (as already proved) is always half the parallelogram $AEDC$, it is clear that this case of the problem is impossible when the given sum of areas is less than half the parallelogram $AEDC$. It is also obvious that the point P cannot be within the parallel bases, when the given sum of areas is greater than the triangle ADB. Let then the given sum of areas lie between those two limits. Since the sum of the two triangles APE and CPD is constant, and the sum of the triangles APB and CPD is also constant or given, therefore the area of the triangle EPB is given, and its base EB is also given; hence the locus of its vertex P is a straight line parallel to AB or CD by (20).

Fig. 4.



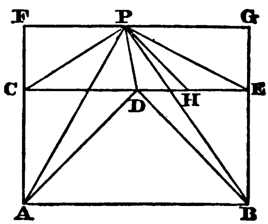
If the given sum of areas be greater than the triangle ADB, let P be a point in the locus. Join AC, and draw BE parallel to AC.

Draw through P, FG parallel to AB, meeting AC and BE produced in F and G.

Bisect DE in H, and draw the other straight lines in the figure.

The triangle ADB is half the parallelogram $ACEB$ (I. 41), which is given, and the triangle APB is half the parallelogram $AFGB$; therefore

Fig. 5.



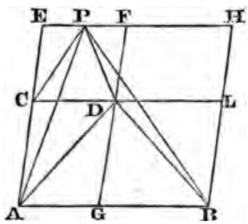
the triangle $APB - ADB$ = half the parallelogram $CFGE$

= the triangle CPE.

But the sum of the triangles APB and CPD is given, therefore the sum of the triangles CPE and CPD is also given; but as in (1) the triangle CPH is half this sum, and therefore the triangle CPH has a given area and a given base CH, therefore the locus of its vertex P is (20) a fixed parallel to AB or CD. In the same manner, it can be shewn that the remaining part of the locus in this case is a straight line parallel to AB and at the side remote from CD.

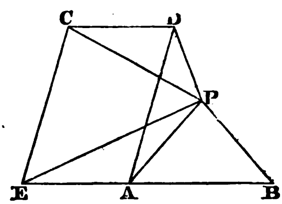
Let the given difference be greater than the triangle ADB, then it is plain that the locus cannot lie between AB and CD. Let P be a point in the locus. Take $AG=CD$. Through P draw EH parallel to AB, and through B draw BH parallel to AC and the other straight lines as in the figure. The triangle APB is half the parallelogram $AEHB$, and the triangle CPD is half the parallelogram $CEFD$ (I. 41), therefore the given difference of the triangles APB and CPD is half the parallelograms AL and LF, but the parallelogram $ACDG$ is also given, therefore the parallelogram $GFHB$ is given in area and its base GB is given, therefore FH, that is, the locus of P, is a fixed parallel to AB. A similar discussion will shew that another straight line parallel to AB, at the side remote from P, forms the remaining part of the locus in this case.

Fig. 6.



Lastly, let the given difference of areas be less than the triangle ADB. Let P be a point in the locus; on BA produced take $AE=CD$, and draw the other straight lines in the figure. The sum of the two triangles APE and CPD is half the given parallelogram $ADCE$, but the difference of the triangles APB and CPD is given. Add to this difference the two triangles APE and CPD, and the triangle EPB is given in area and its base EB is also given. Therefore the locus of P is a fixed straight line parallel to AB. When the given difference of areas is not only less than the triangle ADB but also less than the triangle CAD, another part of the locus will be a fixed straight line parallel to AB and between AB and CD. This straight line is determined in the same way as the last.

Fig. 7.



When the given difference lies between the triangles ACB and CAD, or is less than the triangle CAD, besides the straight lines already found (in Fig. 7) which are parts of the locus of P, a fixed straight line parallel to AB and at the side remote from P will form the remaining part of the locus. This may be found in a manner similar to the locus of P.

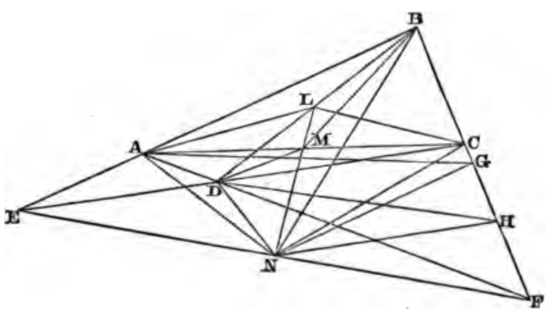
N.B. Under the first case of equal and parallel bases, it has been seen that the locus is a plane when the sum or difference of the areas of the two triangles is half the parallelogram *ABDC*. The definition of *locus* (19) may be extended so as to embrace this and similar cases, or the problem may, in such cases, be considered *indeterminate*. When the base and area of a parallelogram are given, the locus of the side parallel to the base is an indefinite straight line parallel to the base and at a given distance from it. This is manifest from (20), and is also an extension of the term *locus* as defined under (19).

In such figures as Fig. 1, where a great number of lines is likely to perplex a beginner, it will be advisable for the learner only to make, for each case, so much of the figure as that case requires.

DEF. If the two pairs of opposite sides of a quadrilateral be produced to intersect, the straight line joining the two points of intersection is called the *third diagonal* of the whole figure, which is called a *complete quadrilateral*.

22. *The middle points of the three diagonals of a complete quadrilateral lie in the same straight line.*

Let *ABCD* be the quadrilateral, *EF* its third diagonal, and *L*, *M*, *N* the middle points of its three diagonals. Join *L*, *M*, *N* with the points *A*, *B*, *C*, *D*, and draw *NG* parallel to *AB* and *NH* parallel to *DC*. Join *AG*, *DH*. The triangles *ALB* and *ALD* are equal, since they have a common vertex *A* and equal bases *BL* and *LD*. Similarly, the triangles *DLC* and *BLC* are equal, there-



fore the triangles *ALB* and *DLC*, which have the common vertex *L* and *AB* and *CD* for bases, are together equal to half the quadrilateral *ABCD*; similarly, the triangles *AMB* and *DMC* with common vertex M and the bases AB and CD are also together equal to half the quadrilateral $ABCD$, and the points L and M are *within* the bases AB and CD.

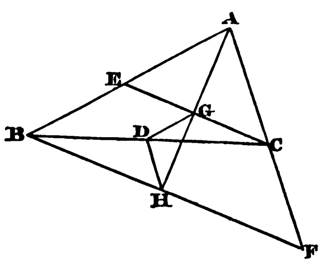
Because EN and NF are equal and NG is parallel to AB, therefore BF is bisected in G (3 Cor.). Similarly, CF is bisected in H.

The triangles ANB and AGB are on the same base AB and between the same parallels AB and NG; they are therefore equal, but BG and GF are equal, and therefore the triangle AGB is half of the triangle AFB, therefore also the triangle ANB is half of the triangle AFB. Similarly, the triangles DNC, DHC and DHF are equal, therefore the triangle DNC is half the triangle DFC.

Therefore the difference of the triangles ANB and DNC is half the difference of the triangles AFB and DFC, that is, half the quadrilateral $ABCD$, and the common vertex N of the triangles ANB, DNC is *without* the bases AB and CD; hence, by (21), the three points L, M, N lie in the *same straight line*, for when the bases AB and CD are given and the sum or difference of the areas is also given (here equal to half the quadrilateral $ABCD$), the locus of the common vertex is a fixed straight line.

23. *Given the base and the difference of the two sides of a triangle, find the locus of the foot of the perpendicular from either end of the base on the bisector of the internal vertical angle.*

Let ABC be any triangle satisfying the conditions of the problem, that is, having the given base BC, and such that the difference of the sides AB and AC is equal to the given difference.



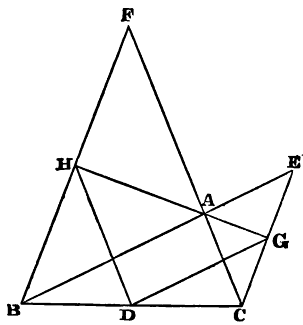
Bisect BC in D, and draw AH bisecting the vertical angle BAC. Draw CG and BH perpendicular to AH, produce CG to meet AB in E, and BH to meet AC produced in F. Join DG, DH. Because AG is common to the two triangles AGE and AGC, and the angles adjacent to AG are equal, therefore (I. 26) $EG = GC$ and $AE = AC$; therefore BE is equal to the given difference of sides; similarly, $BH = HF$, and CF is equal to the given difference of sides.

Because DG joins the middle points of the sides of the triangle BCE, DG is parallel to BE and equal to half BE (4), that is, DG is equal to half the given difference of sides.

Similarly, DH is also half the given difference of sides, and D is a fixed point, therefore the required locus is a circle with the middle point of the given base for centre and radius equal to half the given difference of sides.

24. *Given the base and the sum of the two sides of a triangle, find the locus of the foot of the perpendicular from either end of the base on the bisector of the external vertical angle.*

Let ABC be any triangle on the given base BC, and such that the sum of its sides AB and AC is equal to the given sum. Let D be the middle point of BC, and HG the bisector of the external vertical angle CAE or BAF formed by producing BA to E or CA to F.

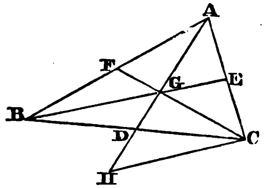


Draw BH and CG perpendicular to HG, and let them be produced to meet the sides produced in F and E. Join DG and DH. Because AG is common to the two triangles CAG, EAG, and the adjacent angles are equal, therefore $CA = AE$ and $CG = GE$; therefore BE equals the given sum of sides, and by (4) $DG = \frac{1}{2} BE$; similarly, DH equals half the given sum of sides; therefore the required locus is a circle with the middle point of the base for centre and radius equal to half the given sum of sides.

Cor. Since in this proposition and the last DG is parallel to AB and DH to AC, the feet of the perpendiculars from either end of the base on bisectors of internal and external vertical angles, and the middle point of the base lie in the same straight line; also the line joining the foot of any of the four perpendiculars, with the middle point of the base, is half the sum or half the difference of the sides according as the perpendicular is drawn to the bisector of the external or of the internal vertical angle.

25. *Given the three bisectors of the sides of a triangle; construct it.*

Suppose ABC the required triangle, and let the bisectors of its sides intersect in G so that GD is half AG (5). Produce AD until $DH = DG$, and join HC.



The two triangles BDG and CDH have the sides BD and DC, GD and DH equal and the angles at D equal, therefore BG and CH are equal.

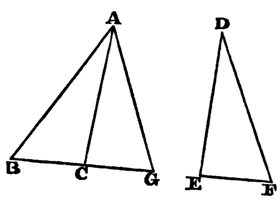
Therefore the triangle HGC has its sides respectively two-thirds the given bisectors of sides, and is therefore given; hence the following construction.

Construct the triangle HGC with its sides respectively two-thirds of the given bisectors of sides. Bisect HG in D, join CD, and produce it until $DB = DC$. Produce HG until $GA = HG$ or twice GD, and join AB, AC. ABC is the required triangle. For join BG and CG and produce these lines to meet the sides AC, AB in E and F.

Because $BD = DC$ and $DG = DH$, and the angles BDG and HDC are equal, therefore $BG = CH$ equals two-thirds one of the given bisectors, but AD bisects BC and $GD = \frac{1}{2} AG$, therefore (5) BE and CF are the bisectors of the sides AC and AB, and since BG, CG and AG are respectively two-thirds of the given bisectors, therefore the triangle ABC has the bisectors of its sides equal to the given bisectors, and therefore is the required triangle.

DEF. Two angles are called *supplemental* when their sum equals two right angles, and *complemental* when their sum equals one right angle.

26. *If two triangles have two sides of the one respectively equal to two sides of the other, and the contained angles supplemental, they are equal in area.*



Let the triangles ABC and DEF have the sides AC and CB respectively equal to DE and EF, and the angles ACB and DEF supplemental, then shall the triangles ABC and DEF be equal in area.

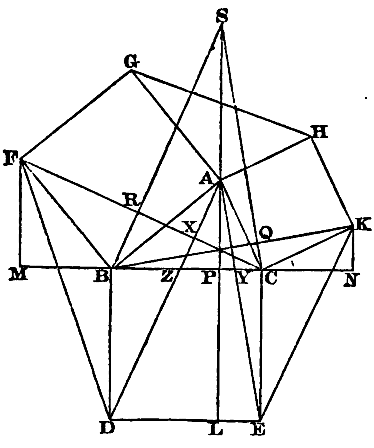
In BC produced take $CG = EF$ or BC, and join AG.

The angles ACB and DEF are equal to two right angles (hyp.), and ACB and ACG are equal to two right angles (I. 13), therefore the angles ACG and DEF are equal, and the sides about these angles are equal, therefore (I. 4) the triangles ACG and DEF are equal in all respects. But the triangles ABC and ACG are on equal bases BC and CG, and have a common vertex, therefore the triangles ABC and ACG are equal in area by (I. 38), therefore the triangles ABC and DEF are also equal.

27. *If squares be described on the sides of* any *triangle as in Euclid* I. 47, *and if* AP, FM *and* KN *be drawn perpendicular to* BC *or* BC *produced, prove that* BM *and* CN *are each equal to* AP, *and that the sum of* FM *and* KN *is equal to* BC, *that the triangles* GAH, FBD *and* KCE *are each equal to* ABC, *and that* CF *and* BK *intersect on the perpendicular* AP.

Produce PA until $AS = PL$ or CE, and join BS, CS.

In the triangles ABP and FBM the sides AB and BF are



equal, the angles APB and FMB are right, and the angles ABP and FBM are together equal to a right angle, since the angle ABF is right, but ABP and BAP are together equal to a right angle, since the angle APB is right, therefore the angles BAP and FBM are equal, and therefore the triangles ABP, FBM are equal in all respects; therefore BM and AP are equal, and FM and BP are equal. In the same manner, it can be proved, that the triangles *APC* and *KCN* are equal in all respects; and that *AP* and *CN* are equal and *CP* and *KN*.

Therefore *BM* and *CN* are each equal to *AP*, and *FM* and *KN* are together equal to *BC*.

Because the angles *ACK* and *BCE* are right angles, the angles *ACB* and *KCE* are supplemental (I. 13), and *AC*, *CB* are respectively equal to *KC*, *CE*, since the sides of a square are equal; therefore (26) the triangles *ABC* and *KCE* are equal, and in the same manner it can be proved that each of the triangles *HAG* and *FBD* is equal to the triangle *ABC*.

The two triangles *FBC* and *ABD* have the sides *FB*, *BC* respectively equal to *AB*, *BD*, and the contained angles are equal, since the angles *FBA* and *DBC* are right, therefore the triangles *FBC* and *ABD* are equal in all respects, and the angle *FCB* is equal to *BDA*.

Therefore, in the triangles *XCZ* and *BZD* the angles *XCZ* and *XZC* are respectively equal to *BDZ* and *BZD*, therefore the remaining angles *ZXC* and *DBZ* are equal; but *DBZ* is a right angle, therefore *ZXC* is also a right angle.

Again, because the alternate angles *DBP* and *SPB* are right, the straight lines *BD* and *SP* are parallel, but *BD* and *SA* are equal (constr.), therefore *BS* and *AD* are parallel (I. 33), and therefore the angles *AXC* and *SRC* are equal, but *AXC* is right, therefore *SRC* is a right angle, and in the same manner it can be proved that the angle *SQB* is a right angle, therefore *SP*, *BQ* and *CR* are the perpendiculars of the triangle *SBC*, and therefore pass through the same point by (8).

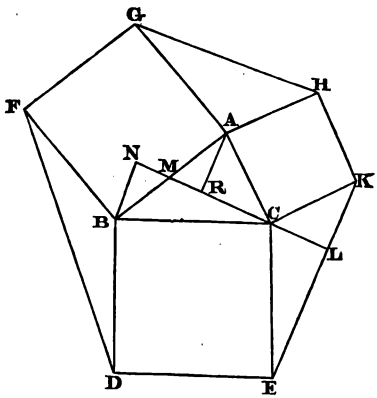
28. *If squares be described on the sides of* any *triangle* ABC, *as in* I. 47, *and if* HG, FD *and* EK *be joined, and perpendiculars drawn from the angles* A, B, C *to these lines respectively, these perpendiculars produced bisect the remote sides of the triangle* ABC, *and therefore pass through the same point, and the joining lines are respectively double the bisectors of the remote sides of the triangle* ABC.

Draw *CL* perpendicular to *EK*, and produce it to meet the remote side *AB* in *M*. Draw *AR* and *BN* perpendicular to *CM*.

In the triangles *ARC* and *CKL*, the angles *ARC* and *CLK* are right; also *ACR* and *KCL* are complemental, since *ACK* is a right angle, and ACR and RAC are also complemental, since ARC is a right angle, therefore the angles KCL and RAC are equal, and the sides AC and CK are equal, therefore (I. 26) the triangles ARC and KLC are equal in all respects; therefore AR and CL, CR and KL are equal.

In the same manner it can be proved, from the triangles BNC and CLE, that BN and CL, NC and LE are equal.

Therefore BN and AR are each equal to the same CL, and therefore they are equal to one another; therefore (I. 26) the triangles AMR and BMN are equal in all respects, and therefore AM and BM are equal, and NM and MR; therefore (1) CM is half the sum of NC and CR, that is, half the sum of EL and LK or of EK.

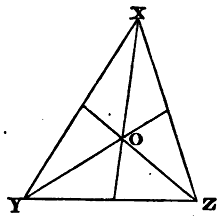


Hence the perpendicular CL produced bisects AB in M, and EK is double the bisector CM. In the same manner it can be proved that the perpendiculars from A and B to HG and FD bisect the sides BC and CA, and that GH is twice the bisector of BC, and FD twice the bisector of CA. Q. E. D.

29. *In the figure to* (28) *given in magnitude the three joining lines* HG, FD *and* EK, *construct the original triangle* ABC.

Since the lines HG, FD and EK are by (28) twice the bisectors of the sides of the required triangle, the problem is reduced to (25).

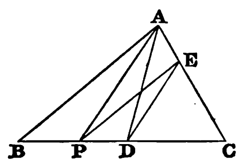
Otherwise thus—The angles ACB and KCE are together equal to two right angles, since ACK and BCE are each right angles.



Similarly, the angles GAH and BAC, FBD and ABC are together equal to two right angles, therefore the three angles HAG, FBD and ECK together with the angles of the triangle ABC are equal to six right angles, and therefore the angles HAG, FBD and ECK are together equal to four right angles; therefore these three angles can be placed with their vertices at the same point so as exactly to fill up the angular space about that point, and so that HG, FD and EK shall form a triangle, since AG and BF, BD and CE, CK and AH are equal two by two. But the triangles HAG, FBD and ECK are equal by (27). Hence by the help of (5) we have the following construction for this problem, as well as for (25).

Construct a triangle XYZ, having its sides equal to the joining lines, that is, to twice the bisectors of the sides of the required triangle. Let the bisectors of the sides of the triangle XYZ meet in O, then OX, OY, OZ are the sides of the required triangle ABC.

30. *To bisect a triangle by a straight line drawn from a given point in one of its sides.*



Let ABC be the given triangle, P the given point in its side BC, and D the middle point of BC. Join AP, AD, draw DE parallel to AP, and join PE. PE bisects the given triangle.

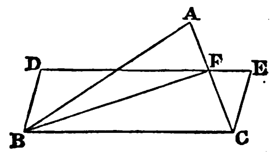
The triangles DPE and DAE are on the same base DE and between the same parallels DE and AP, therefore they are equal (I. 37).

To each of these equals add the triangle DCE, and the triangle PEC is equal to the triangle DAC, which is half the triangle ABC, since BD and DC are equal.

31. *To draw a straight line from an angle of a triangle to the opposite side, cutting off from the triangle any given area.*

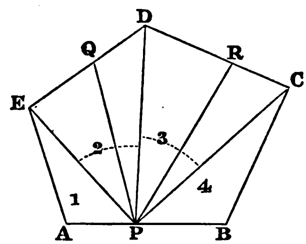
Let ABC be the given triangle, and let it be required to draw from B a straight line to AC cutting off a given area.

On *BC* construct any parallelogram *BDEC* equal to twice the given area, and let *DE* meet *AC* in *F*. Join *BF*. The triangle *BFC* is half the parallelogram *BDEC* (I. 41), and therefore *BF* cuts off *BFC* equal to the given area.



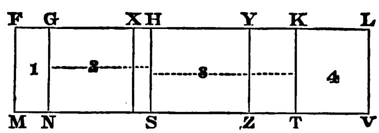
32. *To divide a polygon into any number of equal parts by straight lines drawn from a given point in one of its sides.*

Let *ABCDE* be the given polygon, and *P* the given point in one of its sides. Join *P* with the angular points of the polygon, thus dividing the polygon into the triangles 1, 2, 3, &c.



Construct parallelograms *MG*, *GS*, *SK*, *KV* respectively equal to these triangles, and so that *FL* shall be one straight line (I. 45).

Divide (3) *FL* into the required number of equal parts *FX*, *XY*, *YL*, and complete the parallelograms *MX*, *XZ*, &c. which are evidently each equal to one of the equal parts into which the given polygon is to be divided.



We see that the point of division *X* is on the side *GH* of the parallelogram *HN* or 2 corresponding to the triangle *PED* or 2.

From the triangle *PED* cut off by (31) the triangle *PQD* equal to the parallelogram *XS*, then because the figures *PAED* and *FS* are equal, if the equals *PQD* and *XS* be taken from both, the remainders *PAEQ* and *MX* are equal.

From the triangle *PDC* cut off *PRC* equal to the parallelogram *ZK*, and the figures *PQDR* and *XZ* are obviously equal, and so on.

Therefore *PQ*, *PR* divide the given polygon into the required number of equal parts.

N.B. If the polygon is required to be divided into parts which shall have given ratios to each other, divide (VI. 10) FL in X, Y,... into parts in the given ratios, and the rest is exactly the same as above.

33. *The rectangle under two lines together with the square on half their difference is equal to the square on half their sum, or the rectangle under the sum and difference of two lines is equal to the difference between their squares.*

Let AD be bisected in B, and cut unequally in C, therefore (II. 5) the rectangle $AC.CD$ together with the square on BC is equal to the square on AB.

Now consider AC and CD as two distinct lines, then AB is half their sum and BC half their difference. Hence the first enunciation is true.

Again, consider AB and BC as the lines, then AC is their sum and CD their difference, therefore the second enunciation is true. Q. E. D.

Cor. Hence the rectangle under the segments of a line is a *maximum* (that is, the greatest possible) when it is bisected, for the rectangle $AC.CD$ is less than the square on AB (which is constant) by the square on BC, and therefore this rectangle is a maximum when BC vanishes.

Hence also given the sum of two lines; find them when their rectangle is a maximum. The lines must obviously be each half the given sum.

34. *The sum of the squares on two lines is equal to twice the square on half their sum together with twice the square on half their difference, or the sum of the squares on the segments of a line is equal to the square on half the line together with the square on the distance of the point of section from the middle point of the line.*

In the figure to (33) let AC and CD be the two lines, then AB is half their sum and BC is half their difference.

Since AD is bisected in B and cut unequally in C, therefore (II. 9) the sum of the squares on AC and CD is equal to twice the square on AB together with twice the square on BC, which proves the first form of the theorem.

If we consider AD as a line divided into the two segments AC and CD, then AB is half the line and BC is the distance of the point of section from the middle point of the line. This proves the second form of the theorem.

Cor. Hence the sum of the squares on the segments of a line is a *minimum* (that is, the least possible) when the line is bisected. For the sum of the squares on AC and CD is equal to twice the squares on AB and BC, and since AB is constant, the sum of the squares on the segments must be least when BC vanishes or the point C coincides with B.

35. *Given the rectangle under two lines, find them when their sum is a* minimum.

By (34) the square on half the sum of two lines is equal to the rectangle under them together with the square on half their difference; and since the rectangle is given, the square on half their sum is a minimum when the difference vanishes or the lines are equal; therefore, when the lines are equal, their sum is also a minimum. Now, construct a square equal to the given rectangle. Each of the required lines is equal to a side of this square.

36. *Given the sum of the squares on two lines, find them when their sum is a* maximum.

By (34) twice the square on half the sum of two lines is less than the sum of their squares by twice the square on half their difference; therefore, since the sum of the squares is given, the square on half the sum is a maximum when the lines are equal, and therefore the sum of the lines is also a maximum when they are equal.

Now construct a square equal to half the given sum of squares. Each line is equal to a side of this square.

37. *The square on the sum of two lines is equal to the sum of their squares, together with twice their rectangle.*

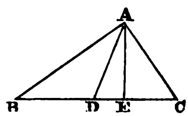
For in Fig. to (33) let AC and CD be the two lines, then (II. 4) the square on the sum AD is equal to the squares on AC, CD together with twice the rectangle AC, CD.

38. *The sum of the squares on two lines is equal to twice the rectangle under them, together with the square on their difference.*

In Fig. to (33) let AD and DC be the two lines, then AC is their difference. Since AD is divided into two parts in C, therefore (II. 7) the sum of the squares on AD and DC is equal to twice the rectangle AD, DC together with the square on AC. Q. E. D.

39. *In a right-angled triangle, the square on the perpendicular from the right angle on the hypotenuse is equal to the rectangle under the segments into which it divides the hypotenuse, and the square on either side of the triangle is equal to the rectangle under the hypotenuse and its adjacent segment.*

Let ABC be a triangle, having the angle BAC right. Draw AE perpendicular to BC, and draw AD to D the middle point of BC.



By (19) AD is equal to BD or DC, since the angle BAC is right. Because BC is bisected in D, the square on BD or AD is equal to the rectangle BE, EC together with the square on DE (II. 5), but the square on AD is equal to the squares on AE and DE (I. 47), therefore the square on AE is equal to the rectangle BE, EC.

Again, the square on AB is equal to the squares on AE and BE (I. 47), that is, to the rectangle BE, EC together with the square on BE, but the rectangle BE, EC with the square on BE is equal to the rectangle CB, BE (II. 3), therefore the square on AB is equal to the rectangle CB, BE. Q. E. D.

40. *The difference of the squares on the sides of a triangle is equal to the difference of the squares on the segments of the base, made by the perpendicular from the vertex on the base, or the base produced, and also to twice the rectangle under the base, and the distance of its middle point from the foot of the perpendicular on the base.*

Let ABC be the triangle, D the middle point of its base, and AE the perpendicular on the base. Then, in Fig. 1, DE is half the difference of the segments BE and EC, and in Fig. 2, DE is half the sum of the segments BE and EC by (1). The square on AB is equal to the squares on BE and AE, and the square on AC is equal to the squares on CE and AE (I. 47), therefore the difference of the squares on AB and AC is equal to the difference of the squares on BE and CE.

Fig. 1.

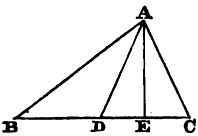
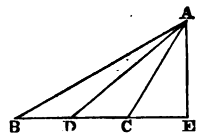


Fig. 2.



Again, the difference of the squares on BE and CE is equal to the rectangle under their sum and difference by (33), that is, to the rectangle under BC, and *twice* DE, or to twice the rectangle under BC and DE. Therefore the difference of the squares on AB and AC is equal to twice the rectangle under BC and DE. Q. E. D.

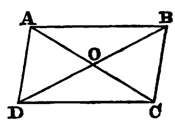
41. *The sum of the squares on the sides of a triangle is equal to twice the square on half the base, together with twice the square on the bisector of base.*

In the figures to (40), the triangle ADB is obtuse-angled at D, therefore (II. 12) the square on AB is equal to the squares on AD, DB together with twice the rectangle BD, DE. Also the triangle ADC is acute-angled at D, therefore (II. 13), the square on AC together with twice the rectangle CD, DE is equal to the squares on AD, DC. Therefore, by adding these equals, the squares on AB, AC, with twice the rectangle CD, DE, are equal to twice the squares on BD and DA, together with twice the rectangle BD, DE, but the rectangles CD, DE and BD, DE are equal, since BC is bisected in D.

Therefore the squares on AB and AC are together equal to twice the squares on BD and DA. Q. E. D.

42. *The sum of the squares on the diagonals of a parallelogram is equal to the sum of the squares on its four sides.*

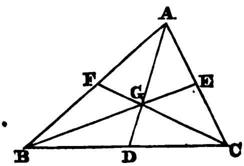
Let the diagonals of the parallelogram $ABCD$ intersect in O. By (11) O is the middle point of AC and BD, therefore by (41) the squares on AB and AD are equal to twice the squares on AO and DO, and CD and CB are equal to BA and AD respectively, therefore the sum of the squares on the four sides is equal to four times the squares on AO and DO, that is, by (14), to the squares on the diagonals AC and BD. Q. E. D.



43. *Three times the sum of the squares on the sides of a triangle is equal to four times the sum of the squares on the bisectors of the sides.*

Let the bisectors of the sides of the triangle ABC meet in G.

Since AD bisects the side BC in D, therefore by (41), twice the sum of the squares on AB and AC is equal to four times the squares on AD and DB, that is, to four times the square on AD together with the square on BC, or as I shall write it for brevity,



$$2AB^2 + 2AC^2 = 4AD^2 + BC^2.$$

Similarly, $$2AB^2 + 2BC^2 = 4BE^2 + AC^2,$$

and $$2BC^2 + 2AC^2 = 4CF^2 + AB^2;$$

therefore, adding these equals, and taking away the sum of the squares on the sides from the sums, the remainders are equal, viz. three times the sum of the squares on the sides is equal to four times the sum of the squares on the bisectors of the sides. Q. E. D.

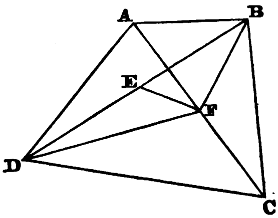
COR. The sum of the squares on the sides is equal to three times the squares on AG, BG and CG.

Since (5) AG is two-thirds of AD, therefore three times AG is equal to twice AD, and therefore four times the square on AD is equal to nine times the square on AG. Therefore three times the sum of the squares on the sides is equal to nine times the squares on AG, BG and CG, therefore the sum of the squares on the sides is equal to three times the squares on AG, BG and CG.

DEF. It may be sometimes convenient to call G (the point of intersection of the bisectors of the sides), the centre of gravity of the triangle.

44. *The sum of the squares on the sides of any quadrilateral is equal to the sum of the squares on its two diagonals, together with four times the square on the line joining the middle points of the diagonals.*

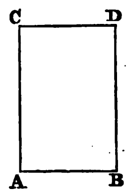
Let $ABCD$ be the quadrilateral, EF the straight line joining the middle points of its diagonals. Join BF, DF.



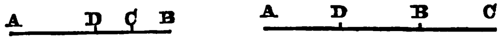
Because AC is bisected in F, the sum of the squares on AB and BC is equal to twice the squares on AF and BF (41), and the sum of the squares on AD and DC is equal to twice the squares on AF and DF, therefore the sum of the squares on the four sides is equal to four times the square on AF with twice the squares on BF and DF, or to four times the square on AF, with four times the squares on BE and EF, since E is the middle point of the base BD of the triangle BDF; also the square on AC is four times the square on AF, and the square on BD four times the square on BE, therefore the sum of the squares on the four sides is equal to the sum of the squares on the diagonals, together with four times the square on EF, joining the middle points of the diagonals. Q. E. D.

45. *Given the area of a rectangle and one side, find the adjacent side.*

To the given side AB apply the rectangle AD equal to the given area (I. 45), then AC or BD is clearly the required adjacent side.



46. *Divide a given straight line internally or externally into parts such that the difference of their squares shall be given.*



Let AB be the given straight line, D its middle point, and suppose C to be the required point of section.

The difference of the squares on AC and CB is equal to twice the rectangle under AB and CD (33); therefore the rectangle under AB and DC is given, and AB is also given.

Hence the following construction; upon AB describe a rectangle equal to half the given difference of squares (I. 45), and take DC equal to its adjacent side: C is the required point of section.

It is plain that if the given difference of squares exceed the square on AB, the point of section C will lie on AB produced. Also, in all cases, two points of section equidistant from D will answer the conditions of the problem.

47. *Divide a given straight line internally or externally into two parts, such that their rectangle shall be given.*

In the figures to (46) let AB be the given straight line, D its middle point, and suppose C the required point of section.

Since AB is bisected in D, therefore (II. 5, 6) the rectangle $AC . CB$ is equal to the difference of the squares on CD and DB, but the rectangle $AC . CB$ and the square on DB are each given, therefore the square on CD is known, and therefore the line CD is also known, therefore since the point D is given, the point C can be found by cutting off from DB or DB produced a part equal to a known line.

N.B. A different solution of this problem will be given farther on, in (59), (60).

48. *Given the base and the difference of the squares on the sides of a triangle, find the locus of its vertex.*

In the figures to (40) let BC be the given base, D its middle point, and let ABC be any triangle answering the given conditions of the problem, that is, on the given base BC and having the difference of the squares on its sides equal the given difference. Draw AE perpendicular to BC. The given difference of the squares on AB and AC is equal to twice the rectangle under BC and DE by (40); but BC is given, therefore DE is known by (45), and therefore, since D is a fixed point, the point E and the perpendicular EA are given; therefore the required locus is the perpendicular AE produced indefinitely both ways.

49. *Given the base and the sum of the squares on the sides of a triangle, find the locus of its vertex.*

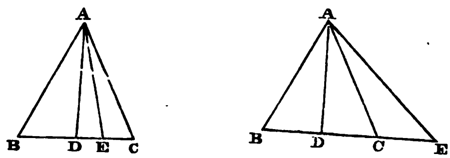
In the figures to (40) let ABC be any triangle on the given base BC, and having the sum of the squares on its sides AB and AC equal to the given sum.

Let D be the middle point of BC.

By (40) the sum of the squares on AC and BC is equal to twice the squares on BD and DA, therefore DA is known, and therefore the locus is a circle with D the middle point of the base for centre and a known radius.

50. *If a straight line be drawn from the vertex of an isosceles triangle to the base or the base produced, the difference between its square and the square on a side of the triangle is equal to the rectangle under the segments of the base.*

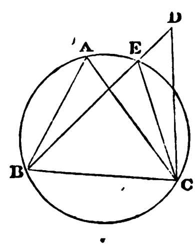
Fig. 1. Fig. 2.



Let ABC be an isosceles triangle and AE any straight line drawn to the base or the base produced. Draw AD perpendicular to BC, and therefore (16) bisecting BC in D. The perpendicular AD divides the base CE of the triangle ACE into the segments CD and DE, and therefore (40) the difference of the squares on AC and AE is equal to the difference of the squares on CD and DE or (33) to the rectangle under the sum and difference of CD and DE, that is, to the rectangle under BE and EC, since BD and DC are equal. Q. E. D.

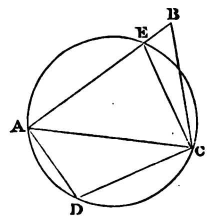
51. *If two triangles with equal vertical angles stand on the same base and at the same side of it, the circle circumscribing one of the triangles will also circumscribe the other.*

Let the triangles ABC and DBC have equal vertical angles BAC and BDC, and, if possible, let the circle circumscribing the triangle ABC not pass through D, but cut BD or BD produced in E. Join CE. Because the angles BAC and BEC are in the same segment, they are equal (III. 21), therefore the angles BEC and BDC are equal, which is impossible (I. 16), therefore the circle circumscribing the triangle ABC must pass through the vertex D of the other triangle. Q. E. D.



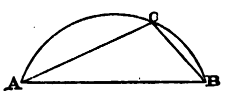
52. *If the opposite angles of a quadrilateral be supplemental, it is circumscribable by a circle.*

Let the quadrilateral $ABCD$ have its opposite angles B and D supplemental, and, if possible, let the circle circumscribing the triangle ADC not pass through B, but cut AB or AB produced in E. Join EC. Because $AECD$ is a quadrilateral in a circle, therefore (III. 22) the angles AEC and ADC are together equal to two right angles, but ABC and ADC are also equal to two right angles (hyp.), therefore the angles AEC and ABC are equal, which is impossible (I. 16). Therefore the circle described about the triangle ADC must pass through the point B. Q. E. D.



53. *Given the base and vertical angle of a triangle, find the locus of its vertex.*

On the given base AB describe a segment of a circle, ACB, capable of containing the given angle (III. 33), then the arc ACB is the required locus; for take any point C in the arc of the segment and join AC, BC. The triangle ACB is upon the given base AB, and its vertical angle ACB is equal to the given angle, therefore the segment ACB is the required locus of the vertex.

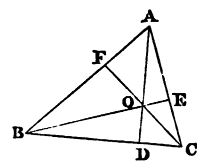


N.B. Since it is plain that an equal segment on the other side of AB will also answer the conditions of the problem, the locus really consists of two equal arcs on opposite sides of AB; but cases of this kind must be so apparent to the reader, that it will generally be unnecessary to occupy space in pointing them out.

54. *Given the base and vertical angle of a triangle, find the locus of the intersection of its perpendiculars.*

Let ABC be a triangle on the given base BC, and having its vertical angle BAC equal to the given vertical angle. Let its perpendiculars intersect in O.

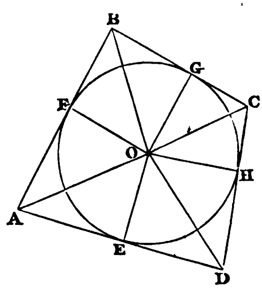
Because the four angles of a quadrilateral are together equal to four right angles (I. 32, Cor.) and the angles AEO and AFO are right angles, therefore the angles FAE and FOE



are together equal to two right angles; but the angle FAE is given, therefore the angle FOE or its equal BOC is known. Hence in the triangle BOC the base BC and the vertical angle BOC are given, and therefore (53) the locus of O is a segment upon BC containing an angle equal to the supplement of the given vertical angle.

55. *The sum of one pair of opposite sides of a quadrilateral circumscribing a circle is equal to the sum of the other pair, and the straight line which joins the middle points of the diagonals of the quadrilateral passes through the centre of the circle.*

Let the quadrilateral $ABCD$ circumscribe the circle with centre O, and touch it in the points E, F, G, H. Join O with the angular points of the quadrilateral and the points of contact.



Because the angles AFO and AEO are right (III. 18), the square on AO is equal to the squares on AF and FO, or on AE and EO (I. 47), therefore the squares on AF and AE are equal, and therefore the lines AF and AE are equal, that is, the two tangents from any point to a circle are equal.

Similarly BF and BG are equal, therefore AB is equal to AE and BG together.

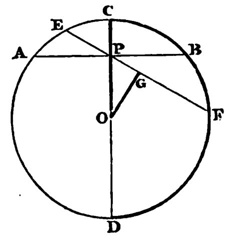
In like manner CD is equal to CG and DE together, therefore AB and CD together are equal to AD and BC together.

Because the three sides of the triangle AFO are respectively equal to the three sides of the triangle AEO, therefore (I. 8) the triangles are equal in all respects.

Similarly the triangles BFO and BGO are equal, therefore the triangle AOB is equal to the sum of the triangles AEO and BGO. Similarly the triangle DOC is equal to the triangles DEO and CGO together, therefore the triangles AOB and DOC are together equal to AOD and BOC, therefore AOB and DOC are together equal to half the quadrilateral $ABCD$, but the triangles having AB and CD for bases and the middle point of either diagonal for common vertex have been proved (22) to be together also equal to half the quadrilateral $ABCD$, therefore (21) the middle points of the diagonals and the centre O of the circumscribing circle, must lie in the same straight line which is the locus of the common vertex of two triangles on the bases AB and CD, and having the sum of their areas equal to half the quadrilateral $ABCD$. Q. E. D.

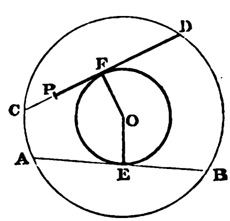
56. *Through a given point within a circle, draw the minimum chord.*

Let P be the given point within the circle whose centre is O. Draw the diameter CD, and through P draw the chord AB perpendicular to CD. AB is the minimum chord, for through P draw any other chord EF and OG perpendicular to it from the centre. In the right-angled triangle OPG, OG is less than OP, and therefore (III. 15) EF is greater than AB; therefore AB is the minimum chord through P.



57. *Through a given point within a circle to draw a chord of a given length, but which must not be less than the minimum chord through the point, nor greater than the diameter.*

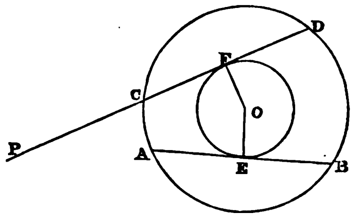
In the given circle with centre O place any chord AB of the given length, draw OE perpendicular to it, and with the centre O and radius OE describe the inner circle. Through P the given point draw the chord CD touching the inner circle at the point F. CD is the required chord.



Join OF. Because the chords AB and CD are equally distant from the centre, they are equal (III. 14). Therefore CD is of the given length, and it is drawn through the given point P.

58. *Through a given point* (P) *without a circle to draw a secant so that the intercepted chord shall be of a given length not greater than the diameter of the circle.*

Let O be the centre of the given circle. In the circle place any chord AB of the given length, and draw OE perpendicular to AB.



From the centre O at the distance OE describe the inner circle.

From P draw PD touching the inner circle at the point F (III. 17).

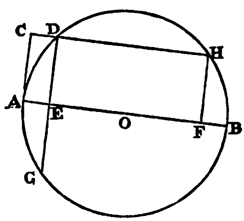
The chord CD is of the given length, for it is equal to AB, since CD and AB are at the same distance from the centre.

Cor. All chords of the outer of two concentric circles which touch the inner circle are equal and bisected at the points of contact.

59. *To divide a straight line internally into segments, such that the rectangle under the segments shall be equal to the square on a given line.*

Let AB be the line which is to be divided.

On AB as diameter describe the circle with centre O; draw AC perpendicular to AB, and equal to the other given line.



Through C draw CD parallel to AB, meeting the circle in the points D and H.

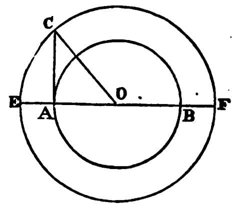
Draw DEG and HF perpendicular to AB, then AB is divided as required in either E or F. Because OE is drawn from the centre perpendicular to DG, DG is bisected in E (III. 3), therefore (III. 35) the rectangle $AE \cdot EB$ equals the rectangle $DE \cdot EG$, that is, the square on DE or AC.

Similarly the rectangle $AF \cdot FB$ equals the square on HF or AC. See (47).

Cor. Hence, if a perpendicular be drawn from any point in the circumference of a semicircle to the diameter (or base of the semicircle), the square on the perpendicular is equal to the rectangle under the segments into which it divides the diameter. See (39).

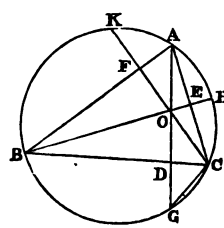
60. *To divide a given straight line externally into segments, such that their rectangle shall be equal to the square on a given straight line.*

Let AB be the given line which is to be divided externally, and let AC perpendicular to AB be the other given line. On AB as diameter describe a circle, and join its centre O with the point C. From the centre O at the distance OC, describe a circle cutting AB produced in E and F. Then AB is divided in E or F as required, for OA and OB, OE and OF are equal, therefore BE and AF are equal. Therefore the rectangle $BE \,.\, EA$ (under the segments into which AB is divided in E), is equal to the rectangle $EA \,.\, AF$, that is, by (59 Cor.) to the square on AC. See (47).



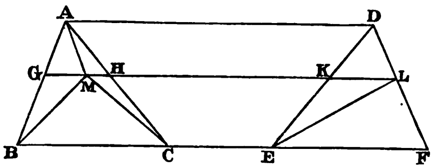
61. *If the three perpendiculars of a triangle* ABC *intersect in* O, *and (produced, if necessary,) meet the circumscribed circle in* G, H *and* K; *prove that the distances* OG, OH *and* OK *are bisected by the sides of the triangle, and that the rectangle under any perpendicular and its segment, which meets one side only, is equal to the rectangle under the segments into which the perpendicular divides that side.*

Join CG. In the triangles ODC and OFA the angles at D and F are right, and the angles at O are equal (I. 15), therefore (I. 32) the remaining angles OCD and FAO are equal, but FAO and DCG are equal, since they stand on the same arc BG, or are in the same segment $BACG$ (III. 21), therefore the angles OCD and DCG are equal, and since the angles at D are right, and the side DC common to the two triangles ODC, DCG; therefore OD and DG are equal (I. 26). Therefore the rectangle $AD \,.\, DO$ equals the rectangle $AD \,.\, DG$, which equals $BD \,.\, DC$ (III. 35). Q. E. D.



62. *If two triangles be on (the same or) equal bases and between the same parallels, the two sides of each triangle intercept equal segments on any straight line parallel to the bases.*

Let the triangles ABC, DEF be upon equal bases BC, EF



and between the same parallels BF and AD, and let GL be parallel to BF, then shall GH and KL be equal.

For, if possible, let GH be greater than KL, and make GM equal to KL.

Join AM, BM, CM and LE. Because the triangles GAM and KDL are on equal bases GM and KL and between the same parallels, they are equal (I. 38).

Similarly, the triangles GBM and KEL, BMC and ELF are equal;

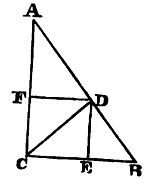
therefore the figure $ABCM$ is equal to the triangle DEF,

but the triangles ABC and DEF are equal, therefore the figure $ABCM$ is equal to the triangle ABC, which is impossible. Therefore GH and KL must be equal. In the same manner, if the parallel to the bases cut the sides produced through the vertices A and D, or through the ends of the bases BC and EF, the intercepts made on the parallel by the sides are proved to be equal. Q. E. D.

DEF. A parallelogram is said to be *inscribed* in a triangle when one of its sides is upon the base of the triangle, and the extremities of the opposite side are upon the two sides of the triangle, and when these two extremities are on the sides produced through the vertex or the ends of the base, the parallelogram is said to be *escribed* (or *exscribed*) to the triangle.

63. *To inscribe a square in a right-angled triangle having one of its angles coinciding with the right angle, and to prove that the rectangle under a side of the square and the sum of the base and altitude is equal to twice the triangle.*

Let ABC be the triangle right-angled at C. Bisect the angle ACB by the straight line CD, and draw DE and DF perpendicular to BC and CA respectively. $CEDF$ is the inscribed square. Because CD bisects the right angle ACB, DCE is half a right angle, but CED is a right angle, therefore CDE is half a right angle, and therefore CE and ED are equal (I. 5).

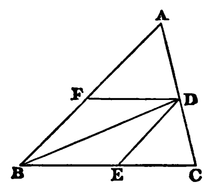


Because the angles DEC and FCE are right, therefore (I. 28) DE and CF are parallel; similarly, CE and FD are parallel, therefore EF is a parallelogram, and since the adjacent sides CE and ED have been proved equal, and the angle DEC is right, therefore EF is a square.

Again, the rectangle $DE \cdot CB$ is equal to twice the triangle CDB (I. 41), and the rectangle $DF \cdot AC$ is equal to twice the triangle ADC, therefore the rectangle under DE and the sum of AC and CB is equal to twice the triangle ACB. Q. E. D.

64. *To inscribe a rhombus in a triangle, having one of its angles coinciding with an angle of the triangle.*

Let it be required to inscribe in the triangle ABC a rhombus having an angle coinciding with the angle ABC.



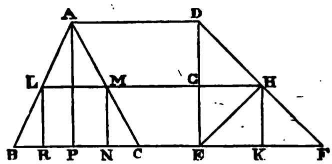
Draw BD bisecting the angle ABC, and through D draw DE and DF parallel to AB and BC respectively. Then EF is the required rhombus. Since DE and BF are parallel, therefore the alternate angles BDE and DBF are equal, but DBF and DBE are equal (constr.), therefore the angles BDE and DBE are equal, and therefore BE and ED are equal, therefore EF is a rhombus (I. 34).

65. *To inscribe a square in any triangle, and to prove that the rectangle under its side and the sum of the base and altitude of the triangle, is equal to twice the triangle.*

Let ABC be the given triangle, and AP its altitude.

In BC produced take EF equal to BC, and draw ED perpendicular to EF, meeting AD parallel to BC in D. Join DF.

Inscribe (63) the square *GK* in the right-angled triangle *EDF*. Produce *HG* to meet *AB* and *AC* in *L* and *M*, and through *L* and *M* draw *LR* and *MN* perpendicular to *BC*. Then *LMNR* is the required inscribed square. Because the triangles *ABC* and *DEF* are on equal bases *BC* and *EF*, and between the same parallels, and *LH* is parallel to *BF*, therefore (62) the intercepts *LM* and *GH* are equal; also *MN* and *GE* are equal, since *ME* is a parallelogram.



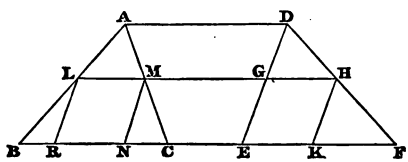
Therefore *LN* is a square equal to the square *GK*.

Again, *AP* and *DE* are equal since *AE* is a rectangle, and the rectangle under *EK* and the sum of *DE* and *EF* is equal to twice the triangle *DEF*; therefore the rectangle under a side of the square *LN* and the sum of *BC* and *AP*, the base and altitude of the triangle *ABC*, is equal to twice the triangle *ABC*.

Def. A figure is said to be given in *species* when its angles and the ratios of its sides are given.

66. *To inscribe a rhombus of given species in any triangle.*

Let *ABC* be the given triangle. In *BC* produced take *EF*

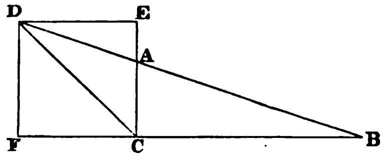


equal to *BC*, make the angle *FED* equal to one of the angles of the rhombus, and draw *AD* parallel to *BC* to meet *ED* in *D*. Join *DF*.

In the triangle *DEF* inscribe the rhombus *GK* having an angle coinciding with the angle *DEF* (64). Produce *HG* to meet *AB* and *AC* in *L* and *M*, and draw *LR*, *MN* parallel to *GE*. Then (62) *LN* is obviously equal to the rhombus *GK*, and it is inscribed in the given triangle and of the given species. Therefore *LN* is the required rhombus.

67. *To escribe a square to a right-angled triangle, the sides of the triangle being produced through an acute angle, and to prove that the rectangle under its side and the difference of the base and altitude is equal to twice the triangle.*

Let ACB be the triangle, right-angled at C. Produce CA and



BA indefinitely through A, BC through C. Draw CD bisecting the angle ACF and meeting BA in D. Draw DE and DF respectively perpendicular to CE and BF. Then EF is the required square. Because DFC is a right angle and DCF half a right angle, therefore FDC is half a right angle, and therefore DF and FC are equal; therefore EF is a square.

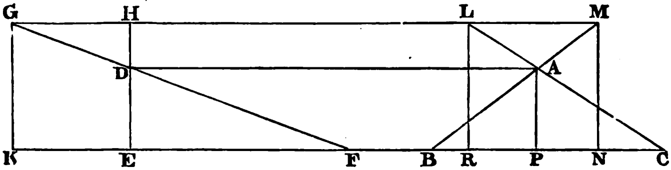
Again, the triangle BDC is half the rectangle under BC and DF (I. 41), and the triangle ADC is half the rectangle under AC and DE or DF; therefore the difference of the triangles BDC and ADC, that is, the triangle ABC, is half the rectangle under DF and the difference of BC and CA.

N.B. Since the angle BAC is greater than the interior angle ACD, and the angle ACD is half a right angle, therefore BAC is greater than half a right angle, and therefore ABC is less than half a right angle. Therefore AC is less than CB. Hence the escribed square, which has one angle coinciding with the right angle, is always upon the *greater side* about the right angle. In fact, if AC be less than CB, then the angle CAB is greater than half a right angle and therefore greater than ACD, therefore the angles ACD and DAC are together less than DAC and CAB, that is, less than two right angles, therefore (Axiom 12) CD and AD will meet as in the figure.

68. *To escribe a square to any triangle, the sides being produced through a vertex of the triangle, and to prove that the rectangle under its side and the difference between the base and altitude of the triangle is equal to twice the triangle.*

Let ABC be the given triangle, and let the side BC be greater than the altitude AP (see N.B. in 67).

In *CB* produced take *FE* equal to *BC*, and draw *ED* perpendicular to *EF*, and meeting *AD* parallel to *BC* in *D*. To the right-

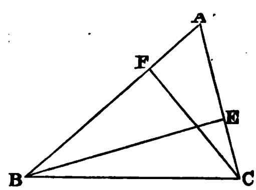


angled triangle *DEF* escribe the square *HK* (67), and produce *GH* to meet *CA* and *BA* produced in *L* and *M* respectively. Draw *LR* and *MN* perpendicular to *KC*. Thus *LMNR* is the required square, for *LM* and *GH* are equal (62).

Again, the rectangle under *HE* and the difference of *FE* and *ED* is equal to twice the triangle *DEF* (67); therefore also the rectangle under *LR* and the difference between *BC* and *AP*, the base and altitude of the triangle *ABC*, is twice the triangle *ABC*.

69. *If two sides of a triangle be unequal, the sum of the greater side and the perpendicular upon it from the opposite angle is greater than the sum of the less side, and the perpendicular upon it from the opposite angle.*

In the triangle *ABC* let the side *AB* be greater than *AC*, and let *CF* and *BE* be the perpendiculars upon these sides; then shall *AB* and *CF* together be greater than *AC* and *BE* together.



Because the angle *BAC* is common to the two triangles *BAE*, *CAF*, and the angles at *E* and *F* are right, therefore (I. 32) the two triangles are equiangular, and therefore (VI. 4),

$$AB : BE :: AC : CF;$$

therefore $AB : AB - BE :: AC : AC - CF;$

therefore, alternately, $AB : AC :: AB - BE : AC - CF;$

but *AB* is greater than *AC*, therefore $AB - BE$ is greater than $AC - CF$. Add $BE + CF$ to these unequals, then

$AB + CF$ is greater than $AC + BE$. Q. E. D.

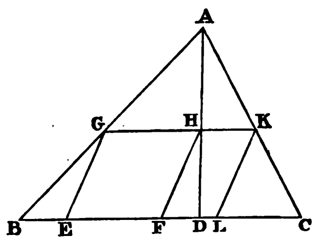
70. *If two sides of a triangle be unequal, the inscribed square which stands upon the greater side is less than the inscribed square which stands upon the less side.*

For the rectangle under a side of an inscribed square and the sum of the side on which it stands and the perpendicular is equal to twice the triangle (65), but the sum of the greater side and its perpendicular is greater than the sum of the less side and its perpendicular (69), therefore the side of the square standing upon the greater side is less than the side of the square standing upon the less side. Q. E. D.

71. *To inscribe in any triangle a parallelogram of given species.*

Let ABC be the given triangle.

Divide (VI. 10) BC in D, so that $CB : BD$ shall be in the ratio of two adjacent sides of the parallelogram.



Join AD, and in the triangle ABD inscribe (66) the rhombus $EFHG$, having the angle GEF equal to an angle of the parallelogram given in species. Produce GH (if necessary) to meet AC in K, and draw KL parallel to GE, then $GELK$ is the required parallelogram, for its angles are equal to the given angles, and $GH : GK :: BD : BC$, since GK is parallel to BC, that is, $EG : GK :: BD : BC$, but $BD : BC$ is the given ratio of the sides of the parallelogram given in species, therefore the inscribed parallelogram EK is of the given species.

N.B. Because GK is parallel to BC, therefore (I. 29) the triangles AGH and ABD, and AGK and ABC are equiangular, and therefore (VI. 4) $GH : HA :: BD : DA$, or alternately,

$$GH : BD :: HA : AD.$$

Similarly, $\quad GK : BC :: KA : AC :: HA : AD,$

but ratios which are equal to the same ratio are equal to one another (or ratios which are the same to the same ratio are the same to one another, V. 11);

therefore $GH : BD :: GK : BC$,

or alternately, $GH : GK :: BD : BC$, as assumed above.

Similarly, $HK : DC :: HA : AD$;

therefore $GH : BD :: HK : DC$,

or alternately, $GH : HK :: BD : DC$.

72. *The circle through the feet of the perpendiculars of a triangle bisects the segments of these perpendiculars towards the angles, and the sides of the triangle. Also its radius is half the radius of the circumscribed circle, and its centre is the middle point of the straight line joining the centre of the circumscribed circle and the intersection of the perpendiculars. (The Nine-Point Circle.)*

Let (Figs. 1 and 2) ABC be the given triangle, D, E, F the

Fig. 1.

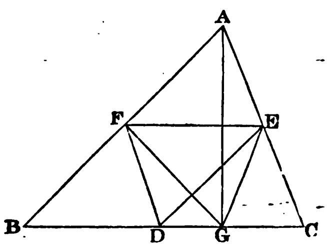
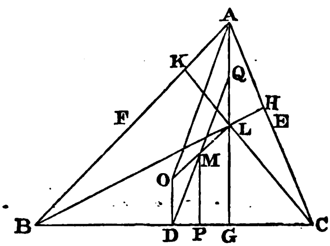


Fig. 2.



middle points of the sides; G, H, K the feet of the perpendiculars intersecting in L, and O the centre of the circumscribed circle.

Fig. 1. Because E is the middle point of AC, and the angle AGC is right, therefore AE and EG are equal, and therefore the angles AGE and EAG are equal; similarly, the angles AGF and FAG can be proved equal, therefore the angles FGE and FAE are equal.

Because $AFDE$ is a parallelogram (4), therefore the angles FAE and FDE are equal, and therefore the angles FDE and FGE are equal.

Therefore the circle through F, D, E also passes through G (51).

In the same manner the circle through F, D, E can be shewn to pass through the feet of the other two perpendiculars.

Fig. 2. Join OL, bisect DG in P, draw PM perpendicular to BC meeting OL in M, join DM and produce it to meet AL in Q.

Because DO, PM and GL are parallel, and P is the middle point of DG, therefore (see 10) M is the middle point of OL. Similarly, the perpendiculars to EH and KF through their middle points can be proved to pass through the middle point of OL, therefore these three perpendiculars intersect in M, but each of these perpendiculars passes through the centre of the circle through D, E, F, since DG, EH and KF are chords of this circle, and the perpendicular to a chord through its middle point passes through the centre of the circle (III. 1, Cor.), therefore M is the centre of the circle through the points D, E, and F.

Because OM and ML are equal, and the triangles ODM and MLQ are equiangular, for DO and QL are parallel, therefore DM and MQ are equal (I. 26), and therefore Q is a point in the circle through D, E, F, and DQ is a diameter of it. Also OD and QL are equal, but AL is twice OD (8), therefore Q is the middle point of AL, or the circle through D, E, F also passes through the middle point of AL. In the same manner it can be proved to pass through the middle points of BL and CL.

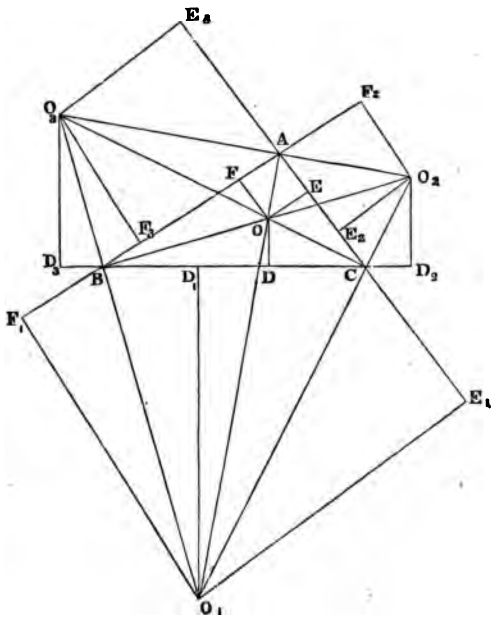
Join AO, then $AODQ$ is a parallelogram, since AQ and DO are equal and parallel, therefore AO and DQ are equal, but DQ is a *diameter* of the circle through D, E, F, and AO is a *radius* of the circumscribing circle. Q. E. D.

N. B. For a geometrical proof of the theorem that "the circle through the middle points of the sides of a triangle touches the inscribed circle and the three escribed circles" the reader is referred to a paper by the author of this work in the *Quarterly Journal of Mathematics* for June, 1862, pp. 269, et sqq.

DEF. A circle is said to be *escribed* (or *exscribed*) to any side of a triangle when it touches that side and the other two sides produced.

73. *The bisectors of the three internal angles of a triangle meet in the centre of the inscribed circle; and if two sides be produced through the extremities of the third side, the bisectors of the two external angles and of the angle opposite to the third side, meet in the centre of the circle escribed to the third side, and the straight line joining any two centres (produced, if necessary) always passes through an angle of the triangle.*

Let ABC be the given triangle. Draw BO and CO bisecting the angles ABC and ACB, and meeting in O. Join AO. Draw



CO_1, bisecting the external angle BCE_1 and meeting AO in O_1, and join BO_1. From O and O_1 draw the perpendiculars OD, OE, OF and O_1D_1, O_1E_1, O_1F_1 to the sides of the triangle ABC or those sides produced.

It is proved in (IV. 4) that O is the centre of the inscribed circle, and that D, E, F are its points of contact with the sides of the triangle ABC.

The tangents AF and AE from the same point A to the inscribed circle are equal, therefore the triangles AFO and AEO are equal in all respects (I. 8), and therefore AO bisects the angle BAC. Therefore the three bisectors of the internal angles of a triangle meet in the centre of the inscribed circle.

Again, because AO_1 bisects the angle BAC, the perpendiculars O_1E_1 and O_1F_1 are equal, and because O_1C bisects the angle D_1CE_1, the perpendiculars O_1D_1 and O_1E_1 are equal, therefore O_1D_1 and

O_1F_1 are equal, and therefore the triangles O_1D_1B and O_1F_1B are equal in all respects, therefore O_1B bisects the angle CBF_1; therefore the bisectors of the two external angles BCE_1, CBF_1 and of the internal angle BAC meet in the same point O_1, and since the three perpendiculars O_1D_1, O_1E_1, and O_1F_1 from this point to the sides are equal, therefore O_1 is the centre of the circle escribed to the side BC of the triangle ABC.

Further, produce O_1C and BO to meet in O_2, and O_1B and CO to meet in O_3.

Join AO_2 and AO_3. It can be proved, in like manner, that AO_2 and AO_3 bisect the vertically opposite angles CAF_2 and BAE_3, and therefore that AO_2 and AO_3 form one continued straight line, and that O_2 and O_3 are the centres of the circles escribed to the sides CA and AB respectively. The perpendiculars from O_2 meet the sides in D_2, E_2, F_2, and from O_3 in D_3, E_3, F_3.

It has also been proved in the above that the six lines OO_1, OO_2, OO_3, O_1O_2, O_2O_3, O_3O_1, joining the centres of the four circles, two by two, pass each through an angle of the given triangle ABC.

DEF. It will be sometimes convenient to denote the sides of the triangle ABC opposite to the angles A, B, C by a, b, c respectively; the radii of the circles escribed to these sides by r_1, r_2, r_3 and the radii of the inscribed and circumscribed circles by r and R respectively. Also put $2s = a + b + c$ so that s is the semiperimeter of the triangle.

74. *The four points in which the inscribed and three escribed circles of a triangle touch any side, and that side produced, are two by two equidistant from the middle point of that side. The distance of a point of contact of the inscribed circle from an angle is less than the semiperimeter by the side opposite to that angle. The distance of an external point of contact from the remote angle is equal to the semiperimeter. The distance between two internal points of contact, on any side, is equal to the difference of the other two sides, and the distance between the two external points of contact on any side is equal to the sum of the other two sides of the triangle. The distance between a point of contact of the inscribed circle and an external point of contact of an escribed circle is equal to the side of the triangle intersecting the line joining these points.*

N. B. When the distance between two points is mentioned, the points are, in the above, always supposed to lie in the direction of the *same side* of the triangle.

Fig. to (73). Since the tangents to a circle from the same point are equal, therefore CD and CE, BD and BF, AE and AF are equal; therefore BC and AE or AF are together equal to the semiperimeter of the triangle ABC. Therefore AE or AF is less than the semiperimeter by BC. Again, BD_3 and BF_3, AE_3 and AF_3 are equal, therefore AB is equal to the sum of BD_3 and AE_3, therefore adding AC and CB to these equals, the sum of CD_3 and CE_3 is equal to the perimeter of the triangle ABC, but CD_3 and CE_3 are equal; therefore CD_3 or CE_3 is equal to the semiperimeter of the triangle ABC. Similarly, BD_2 or BF_2 is equal to the semiperimeter.

Therefore BD_3 and CD_2 are each less than the semiperimeter by BC, and therefore $BD_3 = CD_2 = AE = AF = s - a$. Therefore also D_2 and D_3 are equidistant from the middle point of BC.

Because CD_3 and BD_2 are each equal to the semiperimeter, therefore D_3D_2 together with BC is equal to the perimeter, and therefore D_3D_2 is equal to the sum of AB and AC, or $D_2D_3 = b + c$.

It has been proved that CD and AB together are equal to the semiperimeter, that is, to CD_3, therefore by taking away CD from these equals,

the remainders DD_3 and AB are equal, or $DD_3 = c$.

Because BD_1 and BF_1 are equal, and AF_1 is equal to the semiperimeter, therefore BD_1 is less than the semiperimeter by AB, and therefore BD_1 and CD are equal. Therefore D and D_1 are equidistant from the middle point of BC. Since CD and BD_1 are equal, therefore DD_1 is the difference between CD and BD, that is, the difference between CE and BF, or between CA and BA, for AE and AF are equal, or $DD_1 = b \sim c$, (here $DD_1 = c - b$); or, which is the same thing, the distance of the middle point of the base from either *internal* point of contact is half the difference of the sides of the triangle.

The results proved above may be thus briefly stated,

$$CE = CD = BD_1 = BF_1 = AE_2 = AF_2 = s - c \quad \dots\dots\dots\dots(\alpha),$$

$$DD_1 = b \sim c \quad \dots\dots\dots\dots(\beta),$$

$$DD_2 = D_1D_3 = b \quad \dots\dots\dots\dots(\gamma),$$

$$D_2D_3 = b + c \quad \dots\dots\dots\dots(\delta).$$

75. *The circumscribed circle of a triangle bisects the six straight lines which join the centres of the inscribed and of the three escribed circles. The same circle passes through the centres of the inscribed circle, of an escribed circle, and the ends of the side between these two centres, and has its centre in the circumference of the circumscribed circle. The same circle passes through the extremities of any side, and the two centres of the escribed circles on the same side of it, and its centre is in the circumference of the circumscribed circle.*

Fig. to (73). Because the angles OAC and OAB, O_2AC and O_2AF_2 are equal, therefore (I. 13) the angle OAO_2 is a right angle. Therefore in the triangle $O_1O_2O_3$, O_1A is the perpendicular from the angle O_1 to the opposite side, similarly O_2B and O_3C are the other two perpendiculars, and OO_1, OO_2, OO_3 are the segments of these perpendiculars towards the angles, therefore (72) the circle through A, B, C, the feet of these perpendiculars, that is, the circle circumscribing the triangle ABC bisects the six lines OO_1, OO_2, OO_3, O_1O_2, O_2O_3, and O_3O_1.

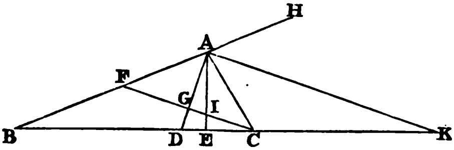
Again, since OCO_1 and OBO_1 are each right angles, therefore (52) the quadrilateral OBO_1C is circumscribable by a circle, and its centre is at the middle point of the diameter OO_1, that is, on the circumference of the circle which circumscribes the triangle ABC.

Also the angles O_3BO_2 and O_3CO_2 are right, therefore the four points O_3, B, C, O_2 lie in the circumference of a circle with its centre at the middle point of O_3O_2, which point is on the circumference of the circumscribed circle. Q. E. D.

76. *If from the greater of the two sides of a triangle a part be cut off equal to the less and conterminous with it, and if the point of section be joined with the opposite angle, each of the equal angles of the isosceles triangle thus formed is equal to half the sum of the base angles of the triangle; the angle between the base of the isosceles triangle and the base of the given triangle is half the difference of the base angles. Also the angle between the perpendicular on the base and the bisector of the internal vertical angle is half the difference of the base angles, and the angle between the base and the bisector of the external vertical angle is half the difference of the base angles.*

In the triangle ABC let the side AB be greater than AC. Cut off AF equal to AC and join CF. Draw the perpendicular AE

from A to BC, and draw AD and AK bisecting the internal and external vertical angles, and therefore including the right angle



DAK. Because AG bisects the vertical angle FAC of the isosceles triangle FAC, therefore the angle AGC is a right angle, but GAK is also a right angle, therefore AK and FC are parallel (I. 28), and therefore (I. 29) the angles ACF and CAK, AFC and HAK are equal, but CAH is equal to the sum of the base angles ABC and ACB (I. 32), therefore AFC and ACF are each half the sum of the base angles.

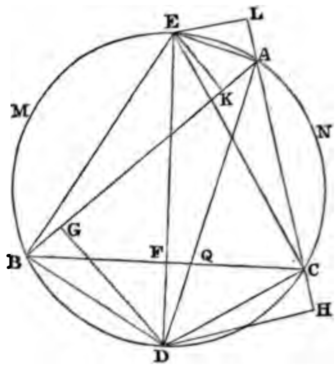
Again, AFC, the external angle of the triangle BFC, is equal to the sum of the angles ABC and BCF (I. 32), therefore ACF is also equal to the sum of the angles ABC and BCF, and therefore ACB is equal to ABC together with *twice* the angle BCF, therefore BCF is *half* the difference of the angles ACB and ABC.

In the triangles AGI and IEC the angles AGI and IEC are right, and the vertically opposite angles at I are equal, therefore the remaining angles GAI and ICE are equal, but ICE is half the difference of the base angles, therefore the angle DAE is also half the difference of the base angles. Because FC and AK are parallel, the angles FCB and AKB are equal, therefore the angle AKB is half the difference of the base angles.

77. *The bisectors of the internal and external vertical angles of a triangle, produced, meet the circumscribed circle in the middle points of the arcs of the segments into which the base divides the circle; the line joining these points is the diameter which bisects the base at right angles; and if perpendiculars be let fall from these two points on the two sides, the distances from the feet of these perpendiculars to the vertices of the triangle are either half the sum or half the difference of the two sides.*

Let ABC be the given triangle circumscribed by a circle. Let AD the bisector of the vertical angle BAC meet the circumference in D, and AE the bisector of the external vertical angle in E.

Join *DE*, *DB*, *DC*, *EB* and *EC*. Draw *DG* and *EK* perpendicular to *AB*, and *DH* and *EL* perpendicular to *AC* produced.



Because *AD* bisects the angle *BAC*, and *AE* bisects the angle *KAL*, therefore the angle *EAD* is a right angle, and therefore *ED* is a diameter.

The angles *EAL* and *EAK* are equal, but *EAL* is the exterior angle of the inscribed quadrilateral *AEBC*, therefore (III. 22) the angles *EAL* and *EBC* are equal, therefore the angles *EAB* and *EBC* are equal, and therefore (III. 26, 29) the arcs *BME* and *CNE* and the straight lines *BE* and *CE* are equal. Therefore *E* is the middle point of the arc *BEC*. Again, the angles *BAD* and *CAD* are equal, therefore the straight lines *DB* and *DC* and the arcs which they cut off are equal. Therefore *D* is the middle point of the arc *BDC*. Because *DE* joins the vertices of the two isosceles triangles *BEC* and *BDC* on the same base *BC*, therefore *DE* bisects *BC* at right angles in *F* (18).

The triangles *ADG* and *ADH* are equiangular and have the side *AD* common, therefore they are equal in all respects (I. 26). Therefore *AG* and *AH*, *DG* and *DH* are equal. *DCH* is the external angle of the quadrilateral *ABDC*, therefore the angles *DCH* and *DBG* are equal (III. 22), and the angles at *H* and *G* are right, and *DC* and *DB* are equal, therefore (I. 26) the two triangles *DCH* and *DBG* are equal in all respects, and therefore *BG* and *CH* are equal. Therefore *AG* and *AH* are together equal to *AB* and *AC*, and therefore *AG* and *AH* are each half the sum of the two sides of the triangle *ABC*. Therefore *BG* and *CH* are each half the difference of the sides *AB* and *AC*. For the difference between the greater or less of two lines and half their sum is equal to half their difference. Thus, AC is less than half the sum AH or AG by CH, and half the sum AG is less than the greater AB by BG, therefore AC is less than AB by CH and BG together; therefore CH or BG is half the difference of AB and AC.

In the same manner, as in the above, the triangles AEL and AEK, EBK and ECL are proved to be equal in all respects. Therefore BK and CL are together equal to AB and AC, and therefore BK and CL are each equal to half the sum of AB and AC, and AK and AL each equal to half the difference of AB and AC.

Cor. The angles BDG, CDH, EBA and ECA are each equal to half the difference of the base angles, and BDF, CDF, EBF and ECF are each equal to half the sum of the base angles of the triangle ABC.

For EBA and ECA are each equal to the angle EDA, since they stand upon the same arc AE. But EDA is half the difference of the base angles, since the angle between a perpendicular to the base and the bisector of the vertical angle is half the difference of the base angles (76), therefore the angles EBA and ECA are each half the difference of the base angles.

The angle EBD in a semicircle is right and therefore equal to the angles GBD and BDG taken together, therefore, taking away the common angle GBD, the angles EBA and BDG are equal. Therefore BDG and CDH are each equal to half the difference of the angles ABC and ACB. The angles CDF and DCF are together equal to the right angle EAD, but DCF and DAB are equal, since they stand on the same arc BD, therefore the angle CDF is equal to EAB, which is half the sum of the base angles. The angles CDE and CBE are equal, since they are in the same segment $CDBE$.

78. *Given the base and vertical angle of a triangle, find the locus of the centre of the inscribed circle and of the centres of the three escribed circles.*

Fig. to (73). Let ABC be a triangle on the given base BC, and with the given vertical angle BAC, the rest of the construction being the same as in (73).

Because the angles O_1AO_2, O_1BO_2 are right, therefore the quadrilateral O_1BAO_2 is circumscribable by a circle, and the angles

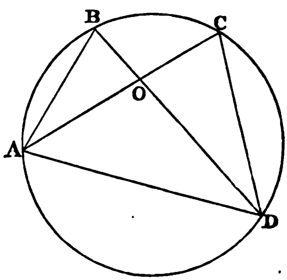
O_1O_2B and O_1AB which stand upon O_1B are equal. But O_1AB is half the given vertical angle, therefore the angle CO_2B is also half the given vertical angle, and the base BC is given, therefore (53) the locus of O_2 is a segment on BC capable of containing an angle equal to half the given vertical angle, and since the angles BO_2C and BO_3C are obviously equal, the locus of O_3 is also the same segment.

Again, because the angle O_1BO_2 is right, therefore the angle BO_1C is equal to the complement of half the vertical angle. Therefore the locus of O_1 is a segment on BC capable of containing an angle equal to the complement of half the given vertical angle.

The angle BOO_1 is equal to the sum of BAO and ABO (I. 32), and COO_1 is equal to the sum of CAO and ACO, therefore the angle BOC is equal to the angles BAC, ABO, and ACO together, that is, to half the sum of the angles of the triangle ABC together with half the angle BAC, or to a right angle together with half the given vertical angle. Therefore the locus of O is a segment upon BC capable of containing an angle equal to a right angle together with half the given vertical angle. In fact, the segments of the same circle constitute the *loci* of O and O_1, since the quadrilateral $BOCO_1$ is circumscribable by a circle.

79. *In equiangular triangles the rectangles under the non-corresponding sides about equal angles are equal* (VI. 4, 16).

Place the triangles ABO, CDO so that the equal angles at O shall be vertically opposite, and that the non-corresponding sides AO and CO, BO and DO, shall be in the same straight lines. Join AD. The angles ABO and DCO opposite the corresponding sides AO and DO are equal, therefore (51) the circle about the triangle ABD will pass through C, and therefore (III. 35) the rectangle $AO . OC$ equals the rectangle $BO . OD$. Q. E. D.



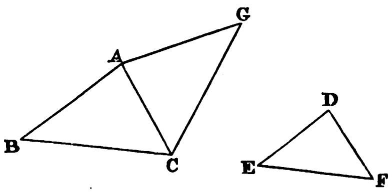
80. *If two triangles have the three pairs of rectangles under the sides about each of the three pairs of angles respectively equal, a side of each triangle being taken to form a rectangle, then shall the triangles be equiangular* (VI. 5, 16).

Let ABC and DEF be two triangles, having the rectangles $AB.DF$ and $AC.DE$, $AC.FE$ and $CB.DF$, $CB.ED$ and $BA.FE$ equal; then shall the angles A, B, C be respectively equal to the angles D, E, F.

Draw AG and CG making the angles GAC and EDF, ACG and DFE equal.

Because the triangles AGC and EDF are equiangular, therefore (79) the rectangles $AC.DE$ and $AG.DF$ are equal, but $AC.DE = AB.DF$ (hyp.), therefore $AG.DF = AB.DF$, therefore $AG = AB$.

Fig. 1. Fig. 2.



In the same manner it can be proved that CG equals CB.

Therefore (I. 8) the triangles ABC and AGC are equiangular, but the triangles ACG and DEF are equiangular (constr.), therefore ABC and DEF are also equiangular. Q. E. D.

81. *If two triangles have an angle of the one equal to an angle of the other, and the rectangles under the sides about the equal angles equal, a side of each triangle being taken to form a rectangle, the triangles shall be equiangular* (VI. 6, 16).

Fig. to (80). Let the two triangles ABC and DEF have the angles BAC and EDF equal, and let the rectangles $AB.DF$ and $AC.DE$ be equal.

Then shall the triangles ABC and DEF be equiangular.

Draw AG and CG, making the angles CAG and ACG respectively equal to EDF or BAC and DFE, so that the triangles GAC and EDF are equiangular.

Therefore AG, AC correspond to DE, DF, and therefore (79)

$$AG.DF = AC.DE, \text{ but } AB.DF = AC.DE \text{ (hyp.)},$$

therefore $AG.DF = AB.DF$, and therefore AG and AB are equal.

Therefore the two triangles BAC and GAC have the two sides AB, AC respectively equal to the two AG, AC, and the contained angles BAC and GAC equal, and therefore (I. 4) the triangles ABC and AGC are equal in all respects. But GAC and EDF are equiangular (constr.), therefore ABC and DEF are also equiangular. Q. E. D.

82. *If two triangles have one angle in each equal, the rectangles under the sides about another pair of angles equal, a side of each triangle being taken to form a rectangle; the remaining pair of angles shall be either equal or supplemental* (VI. 7, 16).

Let the two triangles ABC and DEF have the angles A and D equal, and the rectangles $AB \cdot EF$ and $BC \cdot DE$ about the angles ABC, DEF equal; then shall the angles ACB, DFE be

Fig. 1.

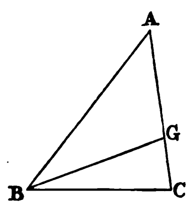
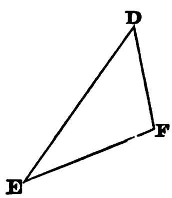


Fig. 2.

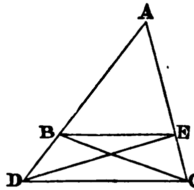


either equal or supplemental. If the angles ABC and DEF are equal, the angles ACB and DFE must be equal, but if the angles ABC and DEF be unequal, let ABC be greater than DEF. Draw BG making the angle ABG equal to DEF. Therefore the triangles ABG and DEF are equiangular, and the sides AB, BG correspond respectively to DE, EF; therefore (79) $AB \cdot EF = BG \cdot DE$. But $AB \cdot EF = BC \cdot DE$ (hyp.), therefore $BG \cdot DE = BC \cdot DE$, and therefore BG and BC are equal. Therefore the angles AGB and BCA are supplemental, but AGB and DFE are equal (constr.). Therefore the angles ACB and DFE are supplemental. Q. E. D.

N.B. If any circumstance be given which determines that the angles ACB and DFE are both acute or both obtuse, then they must be *equal*, since they cannot be *supplemental*. If one of the angles ACB, DFE be known to be right, then they are both equal and supplemental, and the triangles are equiangular.

83. *If two triangles have an angle in each equal and the rectangle under the sides about one of the equal angles equal to the rectangle under the sides about the other, the triangles are equal in area* (VI. 15, 16).

Let the two triangles ABC and ADE be placed so that the equal angles shall coincide and form the angle DAC. Let the rectangles $AB \,.\, AC$ and $AD \,.\, AE$ be equal. Then shall the triangles ABC and ADE be equal. Join BE, DC. Since $AB \,.\, AC = AD \,.\, AE$ (hyp.), therefore the triangles ABE, ADC have the rectangles under the sides about the common angle A equal, a side of each triangle being taken to form the rectangle, therefore (81) the sides AB and AE correspond to AD and AC respectively. Therefore the angles ABE and ADC, AEB and ACD are equal, and therefore BE and DC are parallel. Therefore (I. 37) the triangles BDE and BCE on the same base BE, and between the same parallels, are equal. Add the triangle ABE to these equals, then the triangles ADE and ABC are equal. Q. E. D.



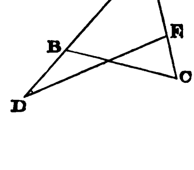
84. *If two triangles have equal vertical angles and equal areas, then shall the rectangle under the sides of the one be equal to the rectangle under the sides of the other.*

Fig. to (83). Place the equal triangles ABC, ADE with their equal vertical angles coincident and forming the angle DAC. Then shall the rectangles $AB \,.\, AC$, $AD \,.\, AE$ be equal. Join BE, DC.

Because the triangles ADE, ABC are equal, if the common triangle ABC be taken away from both, the triangles BDE and BCE are equal, and therefore (I. 39) DC and BE are parallel; therefore the triangles ABE, ADC are equiangular, and therefore (79) $AB \,.\, AC = AD \,.\, AE$.

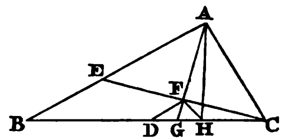
85. *Given the vertical angle and area of an isosceles triangle; construct it.*

Let DAC be the given vertical angle. Take any point D and (31) draw DE cutting off DAE equal to the given area. Find (II. 14) a line AB or AC such that the square on it shall be equal to the rectangle $AD \,.\, AE$, and join BC. The isosceles triangle ABC is (83) equal to the triangle ADE, since $AD \,.\, AE =$ the square on $AB = AB \,.\, AC$. Therefore ABC is the required triangle.



86. *Prove that the rectangle under the distances of the points in which the bisector of the vertical angle and the perpendicular on the base, meet the base, from the middle point of the base, is equal to the square on half the difference of the sides.*

Let ABC be the triangle, D the middle point of the base, AG the bisector of the vertical angle and AH the perpendicular on the base. Make $AE = AC$, and join CE, cutting AG in F. Join DF, HF.



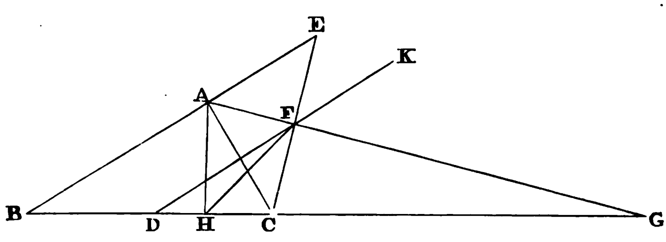
It is proved in (23) that DF is parallel to AB, and equal to half the difference of the sides AB and AC. Therefore the angles DFG and BAG or CAG are equal (I. 29). Because the angles AFC and AHC are right, therefore (51) the quadrilateral $AFHC$ is circumscribable by a circle, and therefore the external angle FHG equals the angle FAC (III. 22).

Therefore the angles DFG and FHD are equal, and therefore DF is a tangent to the circle circumscribing the triangle FGH (III. 32).

Therefore (III. 36) $HD \cdot DG$ equals the square on DF, that is, the square on half the difference of the sides.

87. *The rectangle under the distances of the points in which the bisector of the external vertical angle of a triangle and the perpendicular meet the base from the middle point of the base is equal to the square on half the sum of the sides of the triangle.*

Let ABC be the given triangle, AH the perpendicular on the



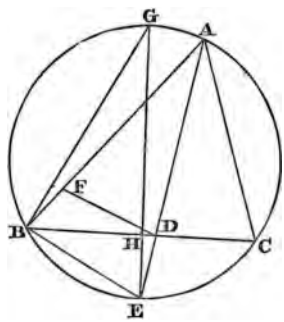
base, AG the bisector of the external vertical angle CAE, and D the middle point of the base.

Join *DF*, *HF*, and produce *DF* to *K*. It is proved in (24) that *DF* is parallel to *AB* and equal to half the sum of the sides *AB*, *AC*.

Because *DF* and *BE* are parallel, the angles *DFA* or *KFG*, and *EAG* or *CAF* are equal. Because the opposite angles *AFC* and *AHC* are right, therefore (52) the quadrilateral *AFCH* is circumscribable by a circle. Therefore the angles *FHC* and *CAF* are equal, and therefore the angles *KFG* and *FHG* are equal. Therefore *DF* is a tangent to the circle about the triangle *HFG* (III. 32), therefore (III. 36) *GD* . *DH* equals the square on *DF*, that is, equals the square on half the sum of the sides. Q. E. D.

88. *The rectangle under the sides of a triangle is equal to the square on the bisector of the vertical angle together with the rectangle under the segments of the base made by the bisector of the vertical angle* (VI. *B*.).

Let *ABC* be the given triangle, and let *AD* the bisector of the vertical angle be produced to meet the circumscribed circle in *E*. Take *AF* equal to *AC*, and join *DF*, *BE*.



In the triangles *AFD*, *ACD* the two sides *AF* and *AC* are equal, *AD* is common and the angles *FAD*, *CAD* are equal, therefore (I. 4) the angles *AFD* and *ACD* are equal, but *ACD* and *AEB* are equal, since they are in the same segment *ACEB* (III. 21), therefore the angles *AFD* and *AEB* are equal, and therefore (52) the quadrilateral *FBED* is circumscribable by a circle.

Therefore $BA \cdot AF = EA \cdot AD$ (III. 36),

but $BA \cdot AF = AB \cdot AC$,

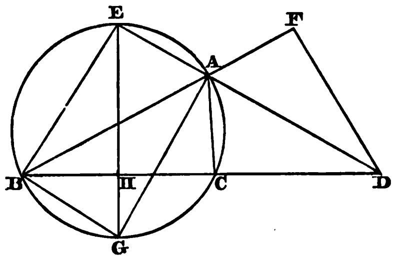
since *AF* and *AC* are equal, and *EA* . *AD* equals the square on *AD* together with the rectangle *AD* . *DE* (II. 3),

and $AD \cdot DE = BD \cdot DC$ (III. 35).

Therefore the rectangle under *AB* and *AC* is equal to the square on *AD* together with the rectangle under *BD* and *DC*. Q. E. D.

89. *The rectangle under the sides of a triangle together with the square of the bisector of the external vertical angle is equal to the rectangle under the segments into which the bisector of the external vertical angle divides the base.*

Let ABC be the given triangle, and let AD be the bisector of the external vertical angle CAF; produce AD to meet the circumscribed circle in E, and take AF equal to AC. Join BE, DF. The triangles CAD and FAD are obviously equal in all respects



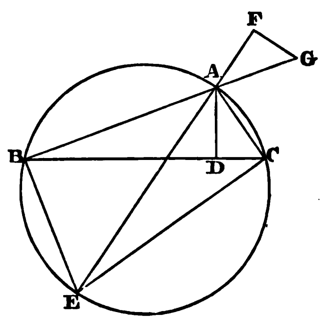
(I. 4), therefore the angles ACD and AFD are equal, but ACD and AEB are equal (III. 22), therefore the angles AEB and AFD are equal, and therefore (51) the same circle passes through the points B, E, F, D.

Therefore (III. 35) $BA \,.\, AF = EA \,.\, AD$,
that is, $BA \,.\, AC = EA \,.\, AD$.

Add to each of these equals the square on the bisector AD, then, since (II. 3) the square on AD together with the rectangle $EA \,.\, AD$ equals the rectangle $ED \,.\, DA$, and this rectangle equals $BD \,.\, DC$ (III. 36), therefore $AB \,.\, AC$ together with the square on AD equals the rectangle $BD \,.\, DC$, under the segments into which the bisector of the external vertical angle divides the base. Q.E.D.

90. *In any triangle the rectangle under the sides is equal to the rectangle under the perpendicular from the vertex on the base and the diameter of the circumscribing circle* (VI. C.).

Let ABC be the given triangle, AD the perpendicular, and AE a diameter of the circumscribing circle. Produce EA until AF equals AD, and BA until AG equals AC. Join FG, BE and EC.



The angle ACE in a semicircle is a right angle (III. 31), and therefore equal to DAC and DCA together, therefore the angles ECB and DAC are equal, but ECB and EAB are equal, since they are in the same segment $BACE$; therefore the angle DAC is equal to EAB or FAG. Therefore the triangles DAC and FAG are equal in all respects (I. 4), therefore, since ADC is a right angle, AFG is a right angle, but EBA in a semicircle is also right, therefore the four points E, B, F, G are in the same circumference (51), and therefore (III. 35),

$$BA \cdot AG = EA \cdot AF,$$

that is, $BA \cdot AC = EA \cdot AD$. Q. E. D.

91. *If the bisector of the vertical angle of a triangle be produced through the base to meet the circumscribed circle, and if the point in which it meets the circle be joined to either end of the base, the square on the joining line is equal to the rectangle under the whole produced bisector and the produced part.*

Fig. to (88). Let G be the middle point of the arc BAC so that the diameter EG bisects BC at right angles in H (77). Join BG, GA.

Because EBG in a semicircle is a right angle, and BH is perpendicular to EG, therefore (39) the square on EB is equal to the rectangle $GE \cdot EH$, but the rectangle $GE \cdot EH$ equals the rectangle $AE \cdot ED$, since the opposite angles GAD, GHD of the quadrilateral $GHDA$ are right, and it is therefore circumscribable by a circle (52). Therefore the square on EB equals the rectangle $AE \cdot ED$. Q. E. D.

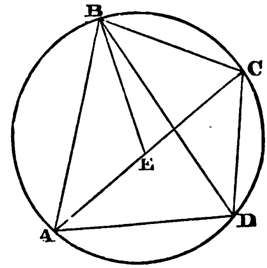
92. *If the bisector of the external vertical angle of a triangle be produced through the vertex to meet the circumscribed circle, and if the point in which it meets the circle be joined to either end of the base, the square on the joining line is equal to the rectangle under the whole produced bisector and its produced part.*

Fig. to (89). Let the bisector DA of the external vertical angle be produced to meet the circumscribed circle in E, and let G be the middle point of the arc BGC so that EG is the diameter which bisects BC at right angles in H (77). Because the angles GHD, GAD are right, therefore (51) the same circle will pass through the points G, H, A, D; therefore (III. 36) $DE \cdot EA = GE \cdot EH$, but (39) $GE \cdot EH$ equals the square on BE, since GBE and BHE are right angles. Therefore the square on EB is equal to the rectangle under DE and EA. Q. E. D.

93. *The rectangle under the diagonals of a quadrilateral inscribed in a circle is equal to the sum of the rectangles under its opposite sides* (VI. *D.*).

Let $ABCD$ be any quadrilateral inscribed in a circle.

Make the angle ABE equal to CBD. The angles BAE and BDC in the same segment $BADC$ are equal (III. 21). Therefore the triangles ABE, CBD are equiangular, and the sides AB, AE correspond to BD, DC, therefore (79),



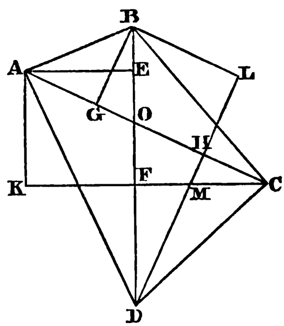
$$AB \,.\, DC = AE \,.\, BD.$$

Again, because the angles CBE and ABD, BCE and BDA are equal, the triangles CBE, BDA are equiangular, and the sides CB, CE correspond to BD, DA, therefore (79) $CB \,.\, DA = CE \,.\, BD$. Therefore

$$AB \,.\, DC + CB \,.\, DA = AE \,.\, BD + CE \,.\, BD = AC \,.\, BD. \quad \text{Q. E. D.}$$

94. *If perpendiculars be drawn from the extremities of each diagonal of any quadrilateral to the other diagonal, the sum of the perpendiculars on the first diagonal is to the sum of the perpendiculars on the second diagonal as the second diagonal is to the first.*

Let $ABCD$ be any quadrilateral, and call AC its first diagonal, BD its second. Draw AE, CF perpendicular to BD and BG, DH perpendicular to AC.



Draw AK parallel to BD meeting CF produced in K, and BL parallel to AC meeting DH produced in L. $AEFK$ and $BGHL$ are obviously rectangles.

Therefore DL is the sum of the perpendiculars on the first diagonal AC, and CK is the sum of the perpendiculars on the second diagonal BD. The triangles CMH, DMF have the angles at M equal and the angles at H and F right, therefore the remaining angles HCM and FDM are equal. Therefore the right-angled triangles BDL and ACK are equiangular, and therefore (VI. 4) $DL : DB :: CK : CA$, or alternately, $DL : CK :: DB : CA$. Q. E. D.

95. *The diagonals of a quadrilateral inscribed in a circle are as the sums of the rectangles under the pairs of sides terminated in each diagonal.*

Fig. to (94). Suppose $ABCD$ to be any quadrilateral capable of being inscribed in a circle, and let AE, CF, BG, DH be the perpendiculars from its angular points on the diagonals.

The rectangle $AB \, . \, AD$ equals the rectangle under AE and the diameter of the circumscribing circle (90). Similarly, $CB \, . \, CD$ = rectangle under CF and diameter.

Therefore $AB \, . \, AD + CB \, . \, CD$ equals rectangle under diameter and $AE + CF$.

Similarly, $BA \, . \, BC + DA \, . \, DC$ equals rectangle under diameter and $BG + DH$.

Now, the rectangle under diameter and $AE + CF$: rectangle under diameter and $BG + DH :: AE + CF : BG + DH$ (VI. 1),

and $AE + CF : BG + DH :: AC : BD$ (94).

Therefore

$AB \, . \, AD + CB \, . \, CD : BA \, . \, BC + DA \, . \, DC :: AC : BD$. Q.E.D.

96. *If* p, p′, p″ *denote the perpendiculars from the centre of the circumscribed circle to the sides of a triangle,* q, q′, q″ *the parts of these perpendiculars (produced) between the sides and the circumscribed circle, prove that*

$$q + q' + q'' = 2R - r,$$
$$\textit{and} \;\; p + p' + p'' = R + r.$$

Also prove that $r_1 + r_2 + r_3 - r = 4R$. (See 73, Def.)

Let M, O, O_1, O_2, O_3 be the centres of the circumscribed, inscribed and escribed circles of the triangle ABC. The rest of the notation and the construction are sufficiently explained by the figure.

EF is the diameter bisecting BC and also GH at right angles in D (77, 74),

and O_3G, FD and O_2H are parallel,

therefore (10) $r_2 + r_3 = 2DF$.

Also OO_1 is bisected in E and OK, ED and O_1L are parallel,

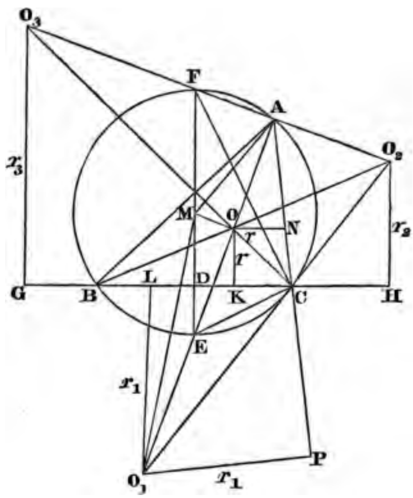
therefore (10) $r_1 - r = 2DE = 2q$.

Hence, by adding these equals,

$$r_1 + r_2 + r_3 - r = 2DF + 2DE = 2FE = 4R \ldots\ldots\ldots\ldots (\alpha).$$

It has been proved above that

$$2q = r_1 - r.$$



Similarly, $2q' = r_2 - r,$

and $2q'' = r_3 - r,$

therefore $2q + 2q' + 2q'' = r_1 + r_2 + r_3 - 3r = 4R - 2r$ by (α);

therefore $q + q' + q'' = 2R - r \ldots\ldots\ldots\ldots (\beta).$

Now, obviously $p + p' + p'' + q + q' + q'' = 3R,$

since $p + q = R$ (or $q - p = R$ if the angle BAC be obtuse),

therefore $p + p' + p'' = R + r \ldots\ldots\ldots\ldots (\gamma).$

N.B. When p, p' or p'' is drawn in the *opposite* direction to that in the figure, where M is *within* the triangle ABC, we may consider p subtractive (or *negative*). Thus the enunciation in the proposition will include every case.

97. *If* D, D_1, D_2, D_3 *denote the distances of the centre of the circumscribed circle of any triangle from the centres of the four circles touching the sides; prove that*

$$D^2 = R^2 - 2Rr,\ D_1^2 = R^2 + 2Rr_1,\ D_2^2 = R^2 + 2Rr_2,$$
$$D_3^2 = R^2 + 2Rr_3,\ \textit{and}\ D^2 + D_1^2 + D_2^2 + D_3^2 = 12R^2.$$

Fig. to (96). Since ECF is an angle in a semicircle, it is a right angle.

Also the angles EAC and EFC in the same segment are equal.

Therefore the right-angled triangles OAN and EFC, O_1AP and EFC are equiangular, and OA, ON correspond to EF, EC and O_1A, O_1P to EF, EC,

therefore (79) $OA\,.\,EC = ON\,.\,EF$ and $O_1A\,.\,EC = O_1P\,.\,EF$;

but OCO_1 is a right angle and E is the middle point of OO_1,

therefore $OA\,.\,EC = OA\,.\,OE$ and $O_1A\,.\,EC = O_1A\,.\,O_1E$.

Therefore $OA\,.\,OE = ON\,.\,EF$ and $O_1A\,.\,O_1E = O_1P\,.\,EF$.

Now, $AM = ME = R$,

therefore (50) $OA\,.\,OE = ME^2 - OM^2 = R^2 - D^2$,

and $O_1A\,.\,O_1E = O_1M^2 - ME^2 = D_1^2 - R^2$.

Therefore $R^2 - D^2 = ON\,.\,EF = 2Rr$ or $D^2 = R^2 - 2Rr$,

and $D_1^2 - R^2 = O_1P\,.\,EF = 2Rr_1$ or $D_1^2 = R^2 + 2Rr_1$.

Similarly, $D_2^2 = R^2 + 2Rr_2$ and $D_3^2 = R^2 + 2Rr_3$.

Therefore

$$D^2 + D_1^2 + D_2^2 + D_3^2 = 4R^2 + 2R(r_1 + r_2 + r_3 - r) = 12R^2 \text{ by (96).}$$

Q. E. D.

N.B. Since the triangles OAN and EFC, and O_1AP and EFC are equiangular, therefore (VI. 4) $AO : ON :: FE : (EC$ or$)\ EO$ and

$AO_1 : O_1P :: FE : (EC$ or$)\ EO_1$.

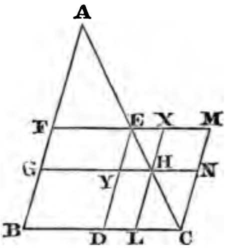
Therefore (VI. 16) $AO\,.\,EO = ON\,.\,FE$,

and $AO_1\,.\,EO_1 = O_1P\,.\,FE$, as before,

and so in all similar cases.

98. *To inscribe in a given triangle a parallelogram of given area not exceeding half the given triangle.* (See VI. 27, 28.)

Let ABC be the given triangle, D, E, F the middle points of its sides, and suppose $BGHL$ the required parallelogram.



Join FE, ED. Draw CM parallel to AB meeting FE produced in M. Produce LH to meet FM in X and GH to meet CM in N.

The parallelogram $BFED$ is clearly half the given triangle. Also the parallelograms BE and DM are equal, since they are on equal bases BD, DC and between the same parallels, but DH and HM are also equal (I. 43).

Therefore DN and LM are equal, but DN and DG are equal, therefore DG and LM are equal, and therefore LG is equal to the gnomon DNX, which is less than DM or DF by XY, therefore LG is less than half the given triangle by the parallelogram XY, therefore the area of the parallelogram XY, and therefore of the triangle XEH or EYH is known. Now the triangle EYH is obviously similar to ABC. Hence the following construction. Construct by (VI. 25) a triangle similar to ABC, and having twice its area equal to the excess of half the given triangle over the given area. From E the middle point of AC take EH equal to the side of this triangle corresponding to AC. Draw HL parallel to AB and HG to BC. Then LG is the required parallelogram.

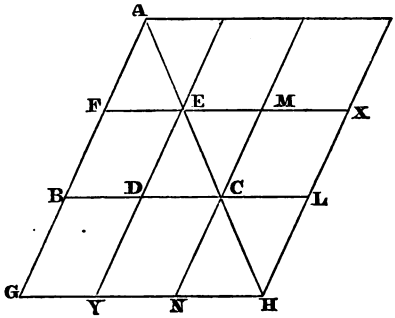
If we had taken EH in the opposite direction, the resulting parallelogram would have also been of the required area, so that two parallelograms of the required area can be inscribed.

COR. Hence the *maximum* parallelogram which can be inscribed in a given triangle has its area equal to half the triangle, and two of its vertices always at the middle points of two sides. Thus the vertices E and F must always be at the middle points of AC and AB, but we may take the side of the parallelogram, equal to FE, anywhere on BC, or the vertices D and E may remain fixed, whilst the side opposite DE and equal to DE may be taken anywhere on AB.

99. *To a given triangle to escribe a parallelogram of given area.*

Let ABC be the given triangle, D, E, F the middle points of its sides. Suppose $BGHL$ the required escribed parallelogram, and complete the other parallelograms as in the figure.

CX and CY are equal (I. 43), and DG and CY are on equal bases BD, DC and between the same parallels, therefore DG and CY are equal, and therefore DG and CX are equal. Add to these equals the parallelogram LY, then the parallelogram LG is equal to the gnomon YLM, but DM and DF are each equal to half



the given triangle; therefore the area of the parallelogram XY is known, for it exceeds the given area by half the given triangle.

Therefore construct (VI. 25) a triangle similar to ABC, and such that double its area shall be equal to the given area, together with half the given triangle. From E, the middle point of AC, take EH equal to the side of this triangle, corresponding to AC, and draw HL and HG parallel to AB and BC respectively. Then $BGHL$ is the required parallelogram. (See VI. 29.)

100. *Given the vertical angle and the sum or difference of the two sides of a triangle, the circumscribed circle (besides passing through the vertex) always passes through a fixed point on the bisector of the internal or external vertical angle.*

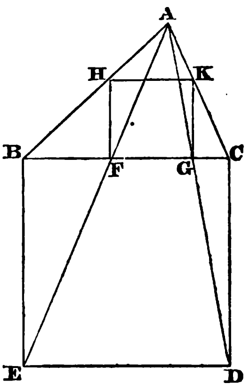
Fig. to (77). Let ABC be any triangle with the given vertical angle BAC, and the given sum or difference of sides, and let the bisectors of the internal and external vertical angles meet the circumscribed circle in D and E, the construction being precisely the same as in (77). If the sum of sides be given, then since AG half the given sum and the angles of the triangle AGD are given, therefore D is a fixed point. If the difference of the sides be given, then AK half their difference is known, and therefore the perpendicular KE meets the fixed bisector AE in the fixed point E. Q. E. D.

COR. Hence, given the vertical angle and the sum or difference of the sides, the locus of the centre of the circumscribing circle is a fixed straight line.

For AD or AE are fixed chords of all the circumscribing circles, and therefore the centres lie in the perpendicular to AD or AE, through its middle point (III. 1 Cor.).

101. *If the lower angles of a square described externally upon the base of a triangle be joined with the vertex of the triangle, the joining lines will intercept on the base* the side *of the inscribed square which stands upon the base.*

Let ABC be the given triangle, and let the joining lines intercept FG. FG is the side of the inscribed square upon BC.



Draw FH and GK perpendicular to BC, and join HK.

Because FH is parallel to BE, therefore the triangles AHF and ABE are similar. Therefore

$$EB : EA :: FH : FA,$$

or alternately, $EB : FH :: EA : FA$.

Similarly, from the similar triangles AFG, AED,

$$ED : FG :: EA : FA.$$

But ratios which are the same with the same ratio are the same with one another, therefore

$$EB : FH :: ED : FG, \text{ and } EB \text{ and } ED \text{ are equal,}$$

therefore FH and FG are equal.

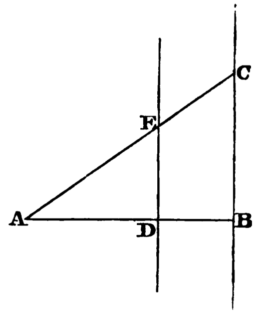
In the same manner it can be proved that FG and GK are equal. Therefore HG is a square. Q. E. D.

N.B. In the same way, AF and AG intercept on HK the side of the square inscribed in the triangle AHK, and so on.

102. *Straight lines are drawn from a given point to a given indefinite straight line and cut in a given ratio; find the locus of the points of section.*

Let A be the given point, and BC the given indefinite straight line.

Draw AB perpendicular to BC (or any line AB from A to BC), and cut AB in D in the given ratio, that is, so that $AD : DB$ is the given ratio (VI. 10). Through D draw DE parallel to BC.



DE indefinitely produced both ways is the required locus.

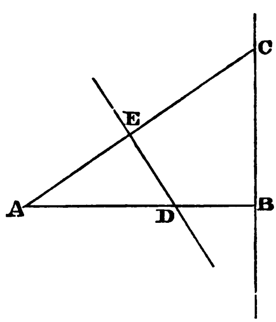
For, draw any line AC from A to BC, meeting DE in E.

Because DE is parallel to the base BC of the triangle ABC, therefore (VI. 2) $AE : EC :: AD : DB$. Hence E is a point on the locus.

103. *Straight lines are drawn from a given point to a given straight line, and cut so that the rectangle under the whole line and the distance of its point of section from the given point is constant (that is, equal to a given square); find the locus of the point of section.*

Let A be the given point, and BC the given straight line.

Draw AB perpendicular to BC, and draw AC any line from A to BC, cut AC in E, so that the rectangle $AC . AE$ is equal to the given quantity. Then E is a point on the locus. Through E draw ED perpendicular to AC, meeting AB in D. Because the angles at B and E are right, the quadrilateral $CBDE$ is circumscribable by a circle, therefore

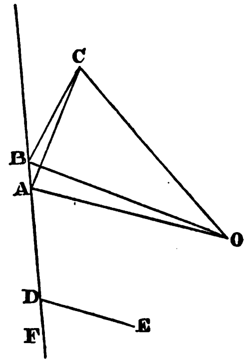


$$CA . AE = BA . AD,$$

but $CA . AE$ and BA are known, therefore also AD is a known line. Therefore the required locus is the circle on AD as diameter.

104. *One vertex of a triangle given in species turns round a fixed point, and another vertex moves along a fixed straight line; find the locus of the remaining vertex.*

Let O be the fixed point, BF the given straight line, and OBC any triangle of the given species. The locus of C is required.



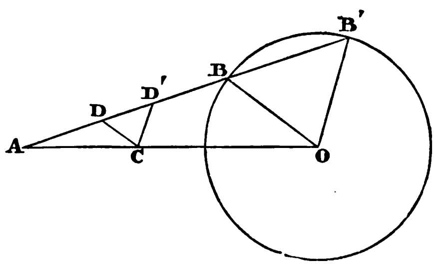
From any point D in BF, draw DE making the angle EDF equal to the angle BCO. Through O draw OA parallel to DE, and join AC.

Because the exterior angle OAD of the quadrilateral $ABCO$ is equal to the remote angle BCO, therefore (52) the quadrilateral is circumscribable by a circle, and therefore the angles OAC and OBC are equal (III. 21), but OBC is a given angle, therefore OAC is known, and since OA and the point A are fixed, therefore the locus of C is the fixed straight line AC.

105. *Straight lines are drawn from a given point to the circumference of a given circle and cut in a given ratio; find the locus of the points of section.*

Fig. 1. Let A be the given point, and O the centre of the given circle.

Fig. 1.



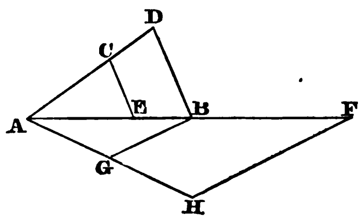
Draw any line AB to the circle and cut it in D, so that $AD : DB$ is in the given ratio. The locus of D is required.

Join AO, and make $AC : CO$ also in the given ratio. Join OB, CD.

Because $AD : DB :: AC : CO$, therefore (VI. 2) CD is parallel to OB, and therefore the triangles ADC, ABO are similar, therefore $AO : AC :: OB : CD$, but AO, AC and OB are given lines, therefore CD the fourth proportional to them is known (VI. 12).

In like manner, if AB' be cut in D' in the given ratio, it is proved that CD' is the fourth proportional to AO, AC and OB'. Therefore CD and CD' are both equal and known, and therefore the required locus is a circle with its centre at the fixed point C, and its radius of a known length.

Fig. 2.



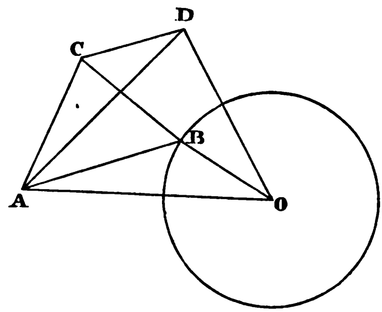
N.B. To divide a line internally or externally in a given ratio is only a particular case of (VI. 10). Thus, let AB be the given line. First, for internal section, fig. 2, draw AD, and take AC, CD equal to the terms of the given ratio. Join DB, and draw CE parallel to DB. Then AB is divided in E as required (VI. 2), since CE is parallel to the base BD of the triangle ABD. Next, for external section. Draw any line AH, and take AH equal to the greater term of the given ratio, HG equal to the less. Join GB, and draw HF parallel to GB, meeting AB produced in F. Then $AF : FB :: AH : HG$ (VI. 2), since HF is parallel to BG the base of the triangle GAB.

106. *A straight line is drawn from a given point to the circumference of a given circle and divided, so that the rectangle under the whole line and its segment between the point of section and the given point is constant; find the locus of the point of section.*

Fig. 1 to (105). Suppose AB drawn from the given point to be divided in D, so that $BA \cdot AD$ is constant. The locus of D is required. Because A is a given point, the rectangle $B'A \cdot AB$ is constant (III. 36), therefore the ratio of $B'A \cdot AB : BA \cdot AD$ is given, that is (VI. 1), $B'A : AD$. Therefore the problem is reduced to the last, and the locus is therefore a fixed circle, with its centre on the straight line passing through A and O.

107. *A triangle is given in species, one vertex turns round a fixed point, whilst another vertex moves along the circumference of a given circle; find the locus of the third vertex.*

Let A be the fixed point, O the centre of the given circle, and let ABC be any triangle of the given species. It is required to



find the locus of C. Join AO, and make the angles DAO, DOA respectively equal to the angles CAB, CBA, so that AOD is a fixed triangle. Because the angles DAO and CAB are equal (constr.), therefore the angles BAO and CAD are equal.

Because the triangles DAO and CAB are equiangular, therefore (VI. 4).

$$AO : AD :: AB : AC,$$

or alternately, $AO : AB :: AD : AC.$

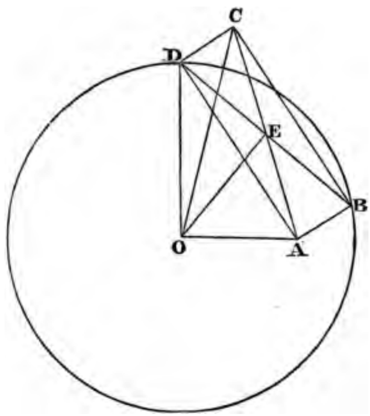
Therefore the triangles BAO, CAD have the angles BAO, CAD equal, and the sides about these angles proportional, therefore they are similar (VI. 6).

Therefore (VI. 4) $AO : OB :: AD : DC,$

that is, DC is a fourth proportional to three given lines, therefore (VI. 12) DC is a known line. Hence, the required locus is a circle with centre D and radius equal to a known line.

108. *One vertex of a rectangle turns round a fixed point and the two adjacent vertices move along a given circle; find the locus of the remaining vertex.*

Let A be the fixed point, O the centre of the given circle, and $ABCD$ any rectangle satisfying the conditions of the problem.



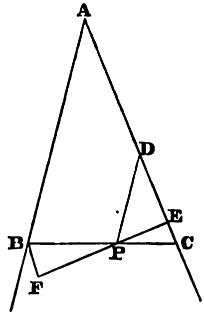
Since (11, 12) the diagonals of a rectangle bisect one another and are equal, AE, EC, BE, ED are equal, and because OE is drawn from the centre to the middle point of DB, OE is perpendicular to DB.

The sum of the squares on OA and OC is equal to twice the squares on OE and EA or ED (41), that is, to twice the square on OD (I. 47).

Therefore the square on OC is equal to the excess of twice the square on the radius OD over the square on the given line OA; therefore OC is known, and therefore the required locus is a circle concentric with the given circle.

109. *Through a given point within a given angle draw a straight line cutting the legs of the angle, so that it shall be divided at the point in a given ratio.*

Let P be the given point within the given angle BAC.



Draw PD parallel to any side AB, meeting the other side of the given angle in D. Make $AD : DC$ in the given ratio. Join CP, and produce CP to meet AB in B. Then BC is the required line.

Because PD is parallel to AB a side of the triangle ABC, therefore (VI. 2)

$$BP : PC :: AD : DC,$$

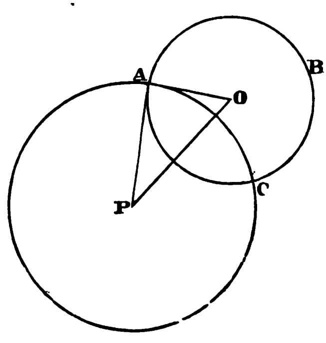
that is, in the given ratio. Otherwise thus. Draw PE perpendicular to AC, and produce EP so that $EP : PF$ shall be in the given ratio. Draw FB parallel to AC, join BP, and produce it to meet AC in C. Then BC is the required line. For the triangles EPC, BPF are clearly equiangular, therefore (VI. 4)

$$CP : PB :: EP : PF.$$

Def. The angle between two circles which intersect is measured by the angle contained by the two tangents to the circles at a point of intersection, or, which is the same thing, by the angle between two radii drawn to a point of intersection, since a radius is perpendicular to the tangent at its extremity.

Two circles are said to cut *orthogonally* (or at right angles) when a pair of radii drawn to a point of intersection contains a right angle.

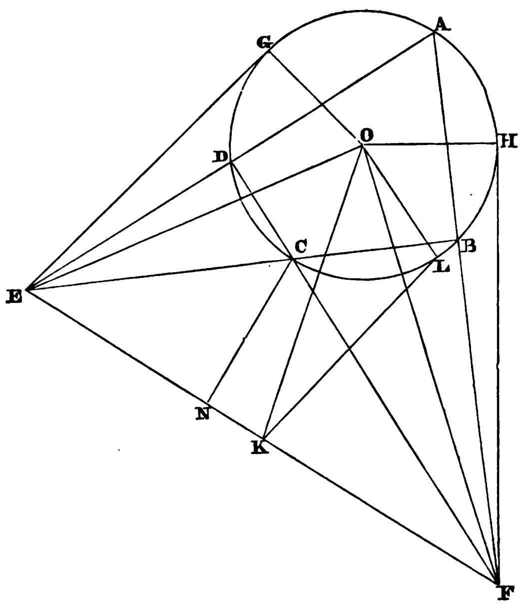
110. *If from any point a tangent be drawn to a given circle and another circle be described with the point as centre and the tangent as radius, the two circles will cut one another orthogonally.*



From P any point outside the circle with centre O, draw the tangent PA, and from the centre P at the distance PA describe the circle PAC, intersecting the other circle in A and C. Join PO, OA. Because PA is a tangent to the circle ABC, therefore (III. 18) the angle PAO is right. Q. E. D.

111. *If a quadrilateral be inscribed in a circle and the figure completed, the square on the third diagonal is equal to the sum of the squares on the two tangents from its extremities, the tangents from the middle point of the third diagonal are each equal to half the diagonal, and the circle on the third diagonal as diameter cuts the given circle orthogonally.*

Let $ABCD$ be the inscribed quadrilateral, K the middle point of EF its third diagonal, and O the centre of the circle.



Draw the tangents EG, FH and KL, and join the points of contact with O. From C draw CN making the angle CNF equal to ABC (see 104). Therefore $CBFN$ is circumscribable by a circle (III. 22). Because the angles ABC and EDC are equal

(III. 22), therefore the angles EDC and CNF are equal, and therefore $DCNE$ is circumscribable by a circle.

Because $DCNE$ is circumscribable by a circle, therefore (III. 36) $EF.FN$ equals $DF.FC$, which equals the square on the tangent FH (III. 36).

Similarly, $EF.EN$ equals $BE.EC$, which equals the square on EG.

Therefore the sum of the rectangles $EF.FN$ and $EF.EN$, that is, (II. 2), the square on EF equals the sum of the squares on FH and EG.

Because EGO and FHO are right angles, therefore (I. 47) the squares on EG, GO, OH, HF are together equal to the squares on EO, OF, but the squares on EG, HF have been proved equal to the square on EF, or to four times the square on its half EK.

Because K is the middle point of EF, therefore (41) the squares on EO, OF are together equal to twice the squares on EK and KO, but the square on KO is equal to the squares on KL and LO, since the angle KLO is right.

Therefore four times the square on EK, and the squares on GO, OH are together equal to twice the squares on EK, KL and LO.

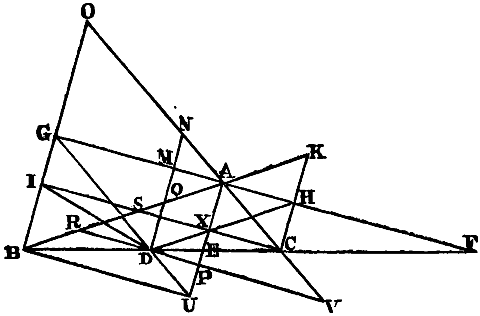
Therefore taking away equals from these equals, the remainders are equal, viz. twice the square on EK is equal to twice the square on KL.

Therefore EK and KL are equal, and therefore (110) the circle on EF as diameter cuts the original circle orthogonally. Q. E. D.

112. *The perpendiculars from the middle point of the base of a triangle on the bisectors of the internal and external vertical angles, cut off from the two sides portions equal to half the sum or half the difference of the sides.*

Let BAC be the given triangle, AU the bisector of the internal vertical angle, GF the bisector of the external vertical angle, BO and CK perpendiculars on GH, meeting the sides produced in O and K, BU and CI perpendiculars on the internal bisector AU, and let DP be drawn from D the middle point of the base perpendicular to AU, and meeting the sides in R and V. Let DM

perpendicular to GF meet the sides in Q and N. Join DI, GD, DU, DX and XH.



GDU and DXH are straight lines (24, Cor.) parallel to CA and AB respectively.

Also PR and AH are parallel since the angles APR and PAH are right angles, therefore $DRAH$ is a parallelogram. In like manner it can be proved that $DQKH$ is a parallelogram; therefore $AR = DH = QK$. But DH and DG are each half the sum of the sides (24, Cor.), and DX and DU half their difference.

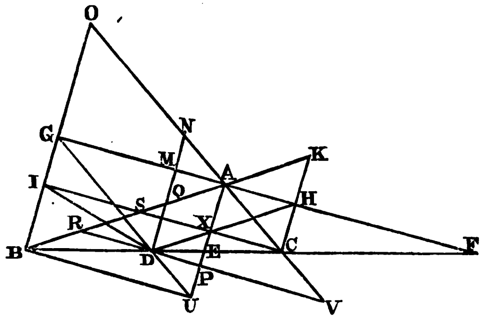
Therefore AR is half the sum of the sides.

Because $AS = AC$ and DR is parallel to CS, and D is the middle point of BC, therefore $BR = RS =$ half the difference of the sides.

Again, $DQAX$ and $DNAU$ are obviously parallelograms, therefore $AN = DU$, and $AQ = DX$. In the triangles DBU and CDV the sides BD, DC and the adjacent angles are respectively equal, therefore (I. 26) DU and CV are equal. Therefore AR, AV, BQ and CN are each half the sum of the sides, and AQ, AN, BR and CV are each half the difference of the sides.

113. *The rectangle under the perpendiculars from the extremities of the base of a triangle on the bisector of the external vertical angle, is less than the square on half the sum of the two sides, by the square on half the base; and the rectangle under the perpendiculars on the bisector of the internal vertical angle is less than the square on half the base, by the square on half the difference of the sides.*

The same construction being made as in (112), the triangles BDI and IDC are clearly isosceles, therefore (50)



$$BG \,.\, CH = BG \,.\, GI = DG^2 - DB^2,$$

and $BU \,.\, CX = IX \,.\, CX = DC^2 - DX^2$. Q. E. D.

114. *The rectangle under the perpendiculars from the ends of the base of a triangle on the bisector of the external vertical angle, is equal to the rectangle under the perpendicular from the middle point of the base on the same bisector, and the bisector of the internal vertical angle; and the rectangle under the perpendiculars on the internal bisector of the vertical angle is equal to the rectangle under the perpendicular from the middle point of the base on the same bisector, and the bisector of the external vertical angle.*

In the fig. to (113), because BA and DH, AE and HC are parallel, therefore $BF : FD :: AF : FH :: EF : FC$.

Again, because BG, DM, EA and CH are parallel, the triangles BGF, DMF, EAF and CHF are equiangular, therefore (VI. 4),

$BF : BG :: DF : DM$; or, alternately, $BF : FD :: BG : DM$,

and $EF : FC :: EA : CH$.

Therefore $BG : DM :: EA : CH$.

Therefore $BG \,.\, CH = DM \,.\, EA$, or $PA \,.\, AE$, which proves the first theorem.

Again, because the triangles BUE and DPE are equiangular, therefore

$BU : DP :: BE : ED$, but $BE : ED :: AE : EX$, because AB and DX are parallel, and $AE : EX :: AF : CX$, because the triangles AEF and XEC are equiangular. Therefore $BU : DP :: AF : CX$,

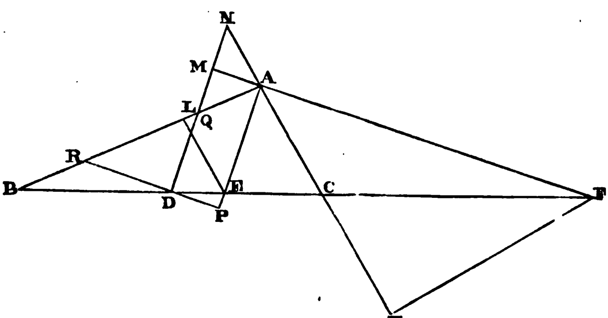
$$\text{therefore} \qquad BU \,.\, CX = AF \,.\, DP, \text{ or } AF \,.\, AM,$$

which proves the other theorem.

Cor. Hence by (113) $PA \,.\, AE$ equals the difference between the squares on half the sum of the sides and half the base, and $AF \,.\, AM$ equals the difference between the squares on half the base and half the difference of the sides.

115. *If a perpendicular be drawn from the point where the bisector of the internal vertical angle of a triangle meets the base to either side, the rectangle under the intercept between the perpendicular and the vertex, and half the sum of the sides, is less than the square on half the sum of the sides by the square on half the base; and if a perpendicular be drawn from the point where the bisector of the external vertical angle meets the base to either side, the rectangle under the intercept between the perpendicular and the vertex, and half the difference of the sides is less than the square on half the base by the square on half the difference of the sides.*

Let ABC be the given triangle, AE and AF the bisectors of the internal and external vertical angles. Draw DM perpen-



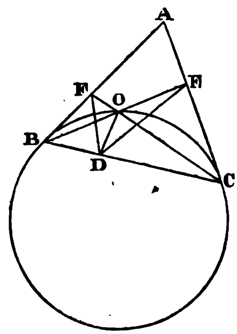
dicular to AF, DP to AE, EL to AB, and FT to AC. Then (112) AR is half the sum of the sides and AN is half their difference.

Because the opposite angles *ELR* and *EPR* of the quadrilateral *RLEP* are right angles, therefore it is circumscribable by a circle, and therefore $RA \cdot AL = PA \cdot AE$ (III. 36), but $PA \cdot AE$ is equal to the difference between the square on half the sum of the sides and the square on half the base (Cor. 114). Again, because the angles *FMN* and *FTN* are right, the four points *F*, *T*, *M*, *N*, lie on the same circumference, therefore $AT \cdot AN = FA \cdot AM$ (III. 35), but $FA \cdot AM$ is equal to the difference between the square on half the base and the square on half the difference of the sides (Cor. 114). Q. E. D.

116. *If perpendiculars be drawn from any point on the circumference of a circle to two tangents and their chord of contact, the square on the perpendicular to the chord is equal to the rectangle under the other two perpendiculars.*

Let *AB*, *AC* be two tangents to the given circle, and *BC* the straight line joining their points of contact.

Let *OD*, *OE*, *OF* be perpendiculars from any point *O* on the circumference to the two tangents and their chord of contact.



Join *ED*, *DF*, *BO* and *OC*.

Because the angles at *D*, *E* and *F* are right, the quadrilaterals *DOEC* and *DOFB* are each circumscribable by a circle.

Therefore (III. 21) the angles *ODE* and *OCE*, *OBD* and *OFD* are equal, but the angle *OCE* is equal to the angle *OBD* in the alternate segment (III. 32); therefore the angles *ODE* and *OFD* are equal, and in the same manner it can be proved that the angles *OED* and *ODF* are equal; therefore the triangles *ODE* and *ODF* are equiangular, and therefore (VI. 4)

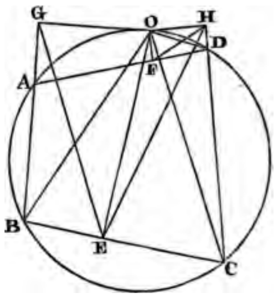
$$FO : OD :: OD : OE.$$

Therefore (VI. 17) $OD^2 = FO \cdot OE.$ Q. E. D.

117. *If perpendiculars be drawn from any point on the circumference of a circle to the sides of an inscribed quadrilateral, the rectangle under the perpendiculars on two opposite sides is equal to the rectangle under the other two perpendiculars.*

Let $ABCD$ be any quadrilateral inscribed in a circle, OE, OG, OF, OH perpendiculars on its sides from any point O on the circumference of the circle. Then shall

$$OE \cdot OF = OG \cdot OH.$$



Draw the other straight lines as in the Fig.

Because the angles at E and G are right, the quadrilateral $OEBG$ is circumscribable by a circle, therefore the angles OGE and OBE are equal, but $OBCD$ is a quadrilateral in the given circle, therefore (III. 22) the angles OBE and ODH are equal.

Also the quadrilateral $OHDF$ is circumscribable by a circle, therefore the angles ODH and OFH are equal.

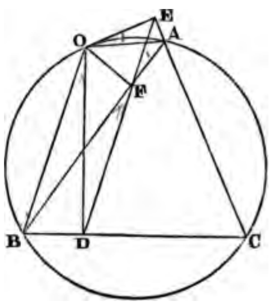
Therefore the angles OGE and OFH are equal.

Again, the angles OEG and OBG are equal, but OBG and ODF are equal, since they stand on the same arc AO, and the angles ODF and OHF are equal. Therefore the angles OEG and OHF are equal, and therefore the triangles OEG and OHF are equiangular; therefore (VI. 4) $OG : OE :: OF : OH$.

Therefore (VI. 16) $OE \cdot OF = OG \cdot OH$. Q. E. D.

118. *If perpendiculars be drawn from any point on the circumference of a circle to the sides of an inscribed triangle, their feet are in the same straight line.*

Let the perpendiculars OD, OE, OF be drawn from any point O on the circumference to the sides of the inscribed triangle ABC; then shall the points D, E, F lie in the same straight line.



Join OB, OA, DF, FE.

Because the angles at D, E and F are right, the quadrilaterals $DOEC$, $OEAF$ and $OFDB$ are circumscribable by circles; therefore the angles BOD and BFD; AOE and AFE are equal, and the angles DOE and DCE are together equal to two right angles, but BOA and DCE are also together equal to two right angles, since $BOAC$ is a quadrilateral inscribed in the given circle; therefore the angles DOE and BOA are equal; therefore the angles AOE and BOD are equal. Therefore also the angles AFE and BFD are equal, but BFA is a straight line, therefore DF and FE must form one continued straight line. Q. E. D.

COR. If the straight lines drawn from O make equal angles with the sides, *in the same order*, it can be proved in the same manner that the three points D, E, F are still in the same straight line.

119. *The circles circumscribing the four triangles formed by four intersecting straight lines all pass through the same point, and this point and the four centres lie in the same circumference.*

Let the four straight lines AE, ED, AF, FB form the four

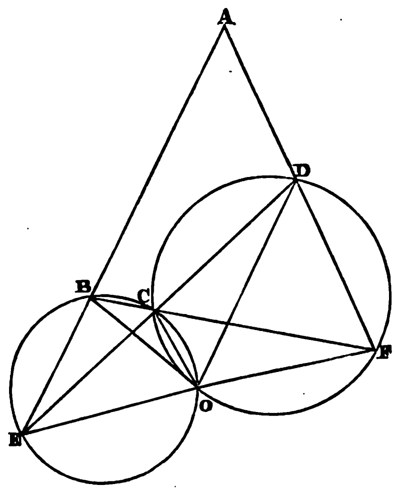


Fig. 1.

triangles AED, DFC, CEB and BAF. Let the circles circumscribing the triangles BEC and CFD intersect in O.

Join EO, BO, CO, DO, FO.

Because (Fig. 1) $BEOC$ is a quadrilateral in a circle, the angle

COE is equal to *ABF* (III. 22) and the angles *DOC* and *CFD* in the same segment are also equal; therefore the angle *DOE* is equal to the sum of the angles *ABF* and *AFB*.

To these equals add the angle *BAF*, therefore the angles *DOE* and *EAD* are together equal to the angles of the triangle *ABF*, that is, to two right angles; therefore (52) the circle circumscribing the triangle *AED* passes through the point *O*. In the same manner it can be proved that the circle circumscribing the triangle *ABF* also passes through the point *O*. Therefore the four circles pass through the same point *O*.

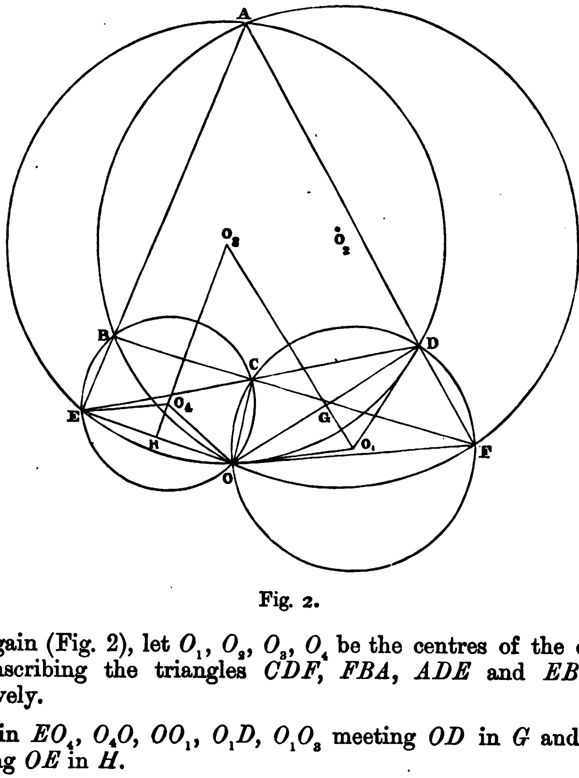


Fig. 2.

Again (Fig. 2), let O_1, O_2, O_3, O_4 be the centres of the circles circumscribing the triangles *CDF*, *FBA*, *ADE* and *EBC* respectively.

Join EO_4, O_4O, OO_1, O_1D, O_1O_3 meeting *OD* in *G* and O_3O_4 meeting *OE* in *H*.

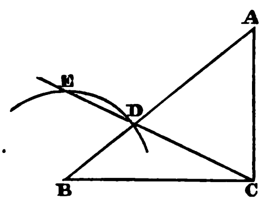
The angle EO_4O is double the angle *ECO* at the circumference (III. 20), and HO_4O is clearly half the angle EO_4O; therefore the angles HO_4O and ECO are equal, but ECO is the external angle of the inscribed quadrilateral $CDFO$; therefore the angles ECO and DFO are equal. And DFO at the circumference is equal to half the angle DO_1O at the centre, that is, to the angle GO_1O. Therefore the angles HO_4O and O_3O_1O are equal, and therefore the angles O_3O_1O and OO_4O_3 are together equal to two right angles. Therefore (52) the circle circumscribing the triangle O_1OO_4 passes through the point O_3.

In the same manner it can be proved that the circle circumscribing the triangle O_1OO_4 also passes through the point O_2. Therefore the point O in which the four circles intersect, and their four centres O_1, O_2, O_3, O_4, all lie on the same circumference. Q.E.D.

COR. Hence (118) the feet of the four perpendiculars from O on the four intersecting straight lines lie in the same straight line.

120. *Given base, difference of sides, and difference of base angles; construct the triangle.*

Let BC be the given base; draw CE making the angle BCE equal to half the given difference of base angles, and from the centre B at a distance equal to the given difference of sides, describe a circle cutting CE in D and E. Join BD and produce it to A.

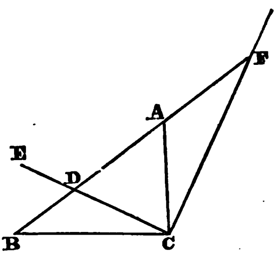


Draw CA making the angle DCA equal to CDA and meeting BD produced in A. Then ABC is the required triangle. For AD and AC are equal (I. 6). Therefore the angle DCB is half the difference of the base angles ACB and ABC, and BD is the difference of the sides. Therefore the triangle ABC has the given base, given difference of sides, and given difference of base angles. Q.E.F.

DEF. I shall sometimes call drawing a line, which shall be equal to a given straight line, from a given point to a given line, *inflecting* a line of the given length between the given point and given line.

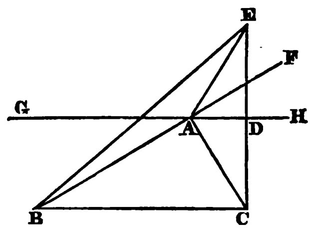
121. *Given base, sum of sides, and difference of base angles; construct the triangle.*

Let BC be the given base. Draw CE, making the angle BCE equal to half the given difference of base angles, and draw CF perpendicular to CE. From B inflect to CF the straight line BF equal to the given sum of sides, and let BF and CE meet in D. Bisect DF in A, and join AC. ABC is the required triangle for $DA = AC = AF$.



122. *Given base, difference of base angles, and area; construct the triangle.*

Since the base and area are given, the locus of the vertex is a straight line parallel to the base (20). Let GH be this locus. Draw CD perpendicular to GH, and produce it until $DE = CD$. Join BE, and on BE describe a segment of a circle capable of containing an angle the supplement of the given difference of base angles. Let this segment meet GH in A, and join AB, AC. ABC is the required triangle. For join EA and produce BA to F. The angles CAD and EAD are equal (I. 4), and DAC and ACB, FAD or GAB and ABC are equal, since BC and GH are parallel. Therefore the angle EAF is the difference of the base angles ABC and ACB. But EAF is equal to the given difference of base angles, since EAB is the supplement of the given difference of base angles.

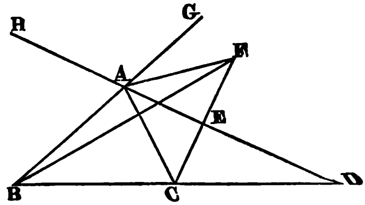


123. *Given base, difference of base angles, and locus of vertex a given straight line intersecting the base; construct the triangle.*

Let HD the given locus of vertex intersect the base BC (produced, if necessary) in D, and suppose BAC the required triangle.

Draw CE perpendicular to HD, and produce it until $EF = CE$.

Join AF, BF, and produce BA to any point G.



The angles CAE and FAE are equal (I. 4), and the angle ACB is equal to the sum of the angles ADC and CAD (I. 32), and HAB is equal to the sum of the angles ABC and ADC. Therefore, since ADC is a known angle, and the difference of the angles ACB and ABC is given, the difference of the angles HAB and CAD, or of GAD and FAE, must also be given. Therefore the angle GAF is known. Hence the following construction. On BF describe a segment of a circle, cutting HD in A, and capable of containing an angle equal to the supplement of the difference between the given difference of base angles and twice the angle ADC.

Join AB, AC. Then BAC is the required triangle.

124. *Draw a common tangent to two given circles.*

Let O and O' be the centres of the given circles. On OO' describe the semicircle OCO'. Place in the semicircle OC (Fig. 1) equal to the difference of the radii of the two given circles (IV. 1), and OC (Fig. 2) equal to the sum of two radii of the circles. Let

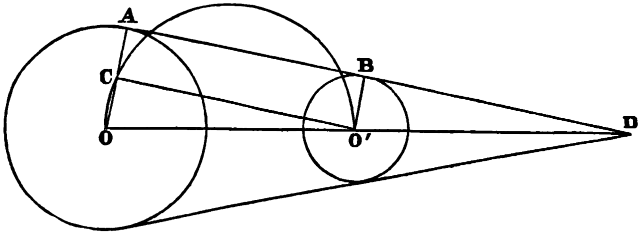


Fig. 1.

OC (produced, if necessary) meet the circle in A. Draw $O'B$ parallel to OA, and join AB, $O'C$. Then AB is a tangent to both circles. For AC and $O'B$ are clearly equal and they are parallel

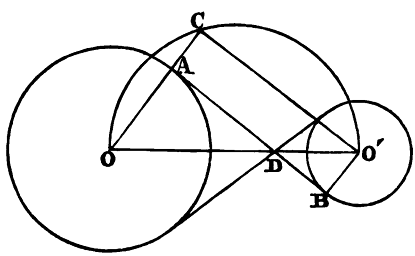


Fig. 2.

(constr.), and the angle OCO' in a semicircle is a right angle (III. 31), therefore (I. 33, 34) $ABO'C$ is a rectangle.

Therefore AB is a tangent to both circles (III. 16).

COR. Let AB and OO' (both produced, if necessary) meet in D.

Because the triangles OAD and $O'BD$ are equiangular, therefore $OD : DO' :: OA : O'B$.

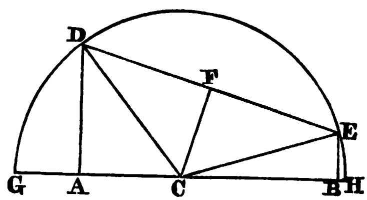
Therefore in Fig. 1, OO' is cut externally in the ratio of the radii of the two circles, and in Fig. 2, OO' is cut internally in D, in the same ratio.

It is evident that two tangents can be drawn intersecting in D in Fig. 1, and also two intersecting in D in Fig. 2. Hence, four common tangents can always be drawn to two circles, which do not intersect, and which are external to one another.

DEF. The tangents in Fig. 1 are called *direct* common tangents, and in Fig. 2 *transverse* common tangents.

125. *Produce a given finite straight line both ways, so that the rectangles under the parts into which the whole produced line is divided at the extremities of the finite line shall be equal to given squares.*

Let AB be the given finite straight line. Draw AD and BE perpendicular to AB, and equal to the sides of the given squares. Join DE and bisect it in F. Draw FC perpendicular to DE and meeting AB in C. Join CD, CE, and from the centre C at the distance CD or CE describe the semicircle $GDEH$, meeting AB produced in G and H. Then AB is produced as required.

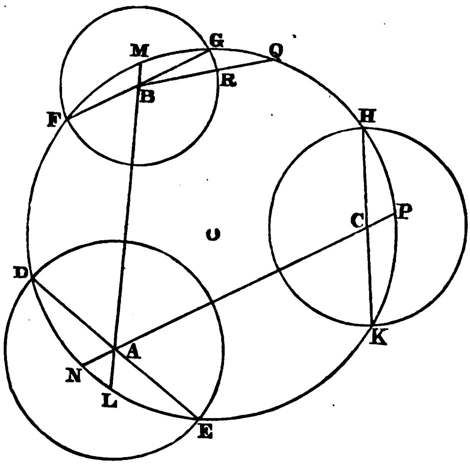


Because DA and BE are perpendicular to GH the base of the semicircle, therefore $GA . AH = AD^2$, and $GB . BH = BE^2$. Q. E. F.

126. *Describe a circle, which shall bisect three given circumferences.*

Let A, B, and C be the centres of the three given circles. Join AB, AC and (125) produce AB, AC both ways, so that $MB . BL$ equals the square on the radius of circle (B), $MA . AL$

equals the square on the radius of circle (A), $NA \cdot AP$ equals the square on the radius of circle (A), and $NC \cdot CP$ equals the square



on the radius of circle (C). Through M, L and P describe the circle whose centre is O. The circle (O) bisects the three given circumferences.

Because the rectangles $MA \cdot AL$, $NA \cdot AP$ are each equal to the square on the radius of the circle (A), therefore the circle through the points M, L, P will also pass through the point N.

Again, join FB. FB produced will pass through G, for, if possible, let FB produced meet the two circles (B) and (O) in the points R and Q respectively.

Because FBQ, MBL are two chords of the circle (O) intersecting in B, therefore $MB \cdot BL = FB \cdot BQ$ (III. 35), but $MB \cdot BL = FB \cdot BG$ (constr.); therefore $FB \cdot BG = FB \cdot BQ$, and therefore BG or $BR = BQ$, which is impossible. Therefore FB produced must pass through G. Therefore the circle (O) bisects the circumference of the circle (B). In the same manner it can be proved, that the circle (O) bisects the circumferences of the other two circles (A) and (C). Q. E. F.

127. *Describe a circle, which shall touch a given straight line, and pass through two given points on the same side of it.*

Let A, B be the two given points, and CD the given straight line.

First (Fig. 1), suppose the straight lines AB and CD parallel. Bisect AB in E, and draw EF perpendicular to AB or CD, meeting CD in F. The circle described through the points A, B, F is the required one. For EF passes through the centre of this circle (III. 1, Cor.), and therefore CD is perpendicular to a diameter through its extremity F. Therefore (III. 16, Cor.) CD touches the circle.

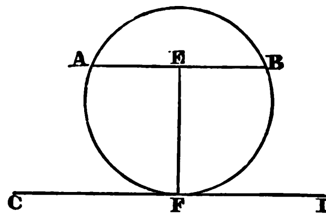


Fig. 1.

Next (Fig. 2), let AB produced meet CD in G. Take GF such that $GF^2 = BG \cdot GA$, which is a given rectangle. The circle

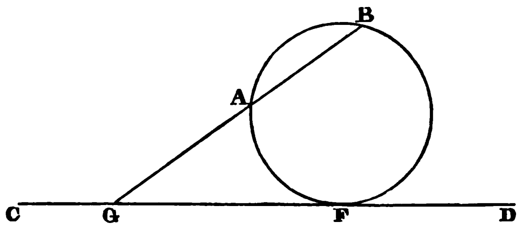


Fig. 2.

passing through the points A, B, F is clearly the required circle (III. 37).

N.B. If GF' be taken in the direction GC equal to GF it is evident that the circle through A, B and F', will also touch CD at the point F'.

Hence, *in general*, two circles can be described through two given points on the same side of a straight line so as to touch the straight line.

128. *Describe a circle, passing through a given point, and touching two given straight lines.*

First (Fig. 1), let the given straight lines LM, PQ be parallel, and let A be the given point. Through A draw FE perpendicular to LM or PQ, take BF equal to AE and $FC^2 = AF.FB$ (II. 14).

Describe the circle with centre O through the points A, B, C. This is the required circle.

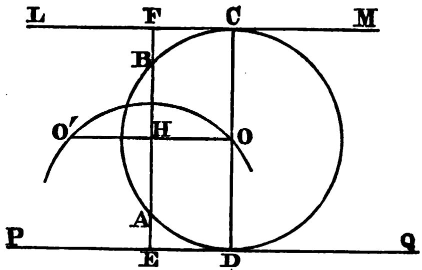


Fig. 1.

Join CO and produce CO to meet PQ in D. Bisect AB in H and join HO. It is clear that FO and OE are rectangles, and since FH and HE are equal, therefore CO and OD are equal, therefore CD is a diameter, and therefore PQ touches the circle at D.

Next (Fig. 2), let the given straight lines meet in L. Draw LH bisecting the angle MLQ, and through A draw EHF perpendicular to LH.

Make $FB = EA$, and take FC such that $FC^2 = AF.FB$ (II. 14).

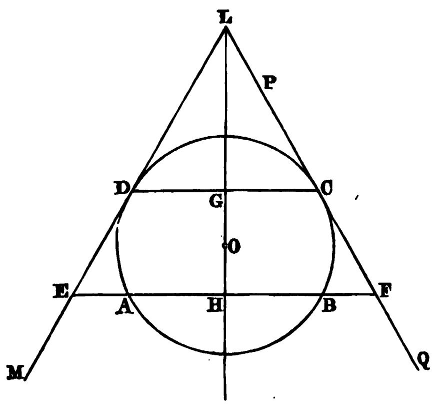


Fig. 2.

Describe a circle through A, B, C. This is the required circle.

Through C draw CGD perpendicular to LH and meeting LM in D.

Because $AF \cdot FB = CF^2$, therefore (III. 37) LQ touches the circle at the point C. It is plain (I. 26) that CG and GD, CF and DE are equal; therefore $ED^2 = BE \cdot EA$, and therefore LM touches the circle at D. Q. E. F.

N.B. It is obvious, that, by taking a line $C'F$ equal to CF in the opposite direction, the circle through the points A, B, C' also touches both the given lines. Hence, two circles can be described answering the conditions of the problem. The first case when the lines are parallel may also be solved thus. From the centre A at a distance equal to HE or HF describe a circle, cutting HO in the points O and O'. These points are the centres of the required circles.

129. *Describe a circle, touching two given straight lines and a given circle.*

Let A be the centre of the given circle, BE and CF the two given straight lines. First, let the given lines be parallel. Through A draw BC perpendicular to BE or CF, and bisect BC in D. Through D draw OO' parallel to BE, and from the centre A at a

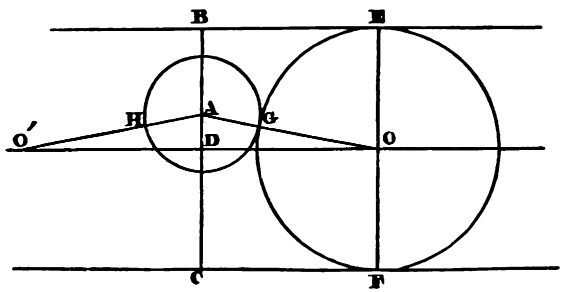


Fig. 1.

distance equal to the sum of BD, and the radius of the given circle (A), describe a circle cutting OO' in O and O'.

Join AO, meeting the given circle in G, and through O draw EF perpendicular to BE. From the centre O at the distance OG describe a circle. This circle will touch the given circle (A) at G and the two given lines at E and F.

For AO equals AG and OE together (constr.); therefore $OG = OE = OF$.

Therefore the circle (O) touches the two given lines at E and F and it also touches the circle (A) at G, since the two circles have evidently the same tangent at G. In like manner it can be proved, that the circle described from the centre O' at the distance $O'H$ also touches the two given lines and the given circle. It may be shewn, in like manner, that when the circle lies entirely between the two given lines, the circle described from the centre A at a distance equal to the *difference* between BD and the radius of the given circle cuts DO in the centres of two circles, which touch the two given lines and the given circle.

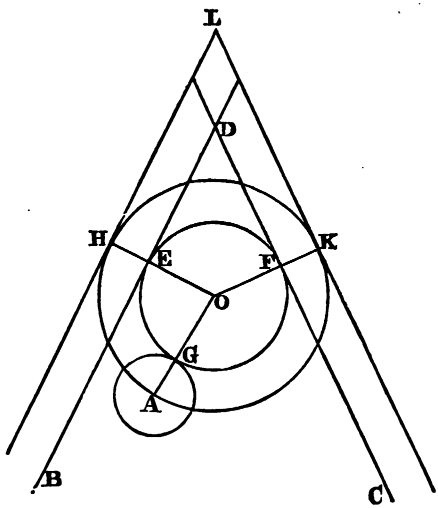


Fig. 2.

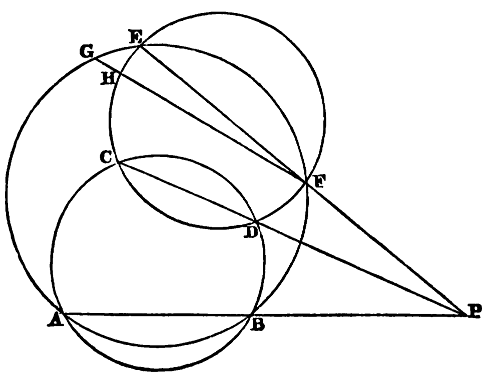
Next, let the two given lines meet in D. Draw LH and LK parallel to the two given lines BD and DC, respectively, and at a distance equal to the radius of the given circle (A). Describe (128) a circle with centre O passing through the point A, and touching LH and LK in H and K. Join OH, OK and OA, meeting the given lines in E, F and the given circle in G. The circle described from the centre O at the distance OG will touch the two given lines at E and F and the given circle at G. For $HE = FK = AG$ (constr.). Therefore $OG = OE = OF$.

N.B. When the two given lines are parallel and the given circle is wholly between them, *four* circles can be described as required. When the given circle touches the two given lines only two circles can be described. When it touches only one of the lines and lies between them, three circles can be described.

When it touches one line and lies without both, only one circle can be described as required. When it meets one or both of the given lines, two circles can be described. In Fig. 2, since (128) two circles can be described passing through the point A and touching the straight lines LH and LK, another corresponding circle can also be described passing through A and touching the given lines. If the parallels to the given lines BE and CF had been drawn on the other side of them at the same distance, two other circles could have been described touching these latter parallels and passing through A, and consequently two other corresponding concentric circles, also passing through A and touching the given lines. Thus, in all cases it will be found that four circles can be described through a given point and touching two intersecting straight lines, produced indefinitely both ways. It will be well for the learner to examine carefully all these various cases and to make separate figures for each case.

130. *If circles be described passing through two given points and cutting a given circle, the chords of intersection all pass through a fixed point on the straight line passing through the two given points or are parallel to this line.*

Let A, B be the two given points, and $CDFE$ the given circle.



Describe any circle passing through A, B, and intersecting the given circle in the points C, D. Join AB and CD, and produce AB, CD to meet in P.

Describe any other circle $ABFE$, intersecting the given circle in E and F. Join PF. PF produced will pass through E. For, if possible, let PF produced cut the given circle, and the circle $ABFE$ in H and G respectively.

From the circle $ABDC$, $AP.PB = CP.PD$ (III. 36),

but $CP.PD = HP.PF$ from the circle $CDFH$,

and $AP.PB = GP.PF$ from the circle $ABFG$.

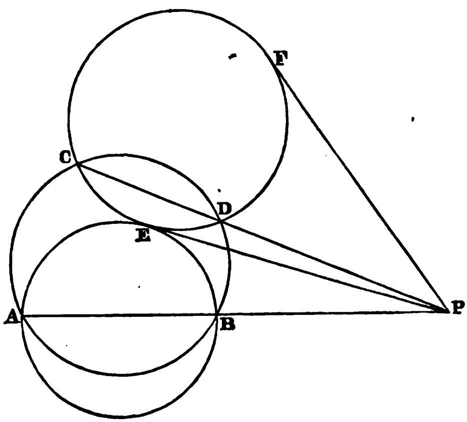
Therefore $HP.PF = GP.PF$, and therefore $HP = GP$, the less equal to the greater, which is impossible. Therefore PF produced must pass through E, that is, any chord of intersection EF passes through the fixed point P on the line joining A and B.

Since the line joining the centres of the circles $ABDC$ and $CEFD$ is perpendicular to CD their chord of intersection, if this joining line bisect AB at right angles, that is, if the line joining the centre of $CEFD$ and the middle point of AB be perpendicular to AB, it is clear that CD will be parallel to AB. Q. E. D.

131. *Describe a circle passing through two given points and touching a given circle.*

Let A, B be the given points and CDF the given circle.

Describe any circle through A and B cutting the given circle in the points C, D.



Join CD, and produce CD, AB to meet in P. From P draw PE and PF touching the given circle in the points E and F.

Describe a circle through the points A, B and E. This will touch the given circle at E.

From the circle $ABDC$, $AP.PB = CP.PD$, and from the circle CDF, $CP.PD = PE^2$. Therefore $AP.PB = PE^2$, and therefore (III. 37) PE touches the circle ABE at the point E, but it also touches the given circle at the same point. Therefore the circles ABE and CDF touch at the point E.

In the same manner, it can be proved that the circle described through the points A, B, F also touches the given circle.

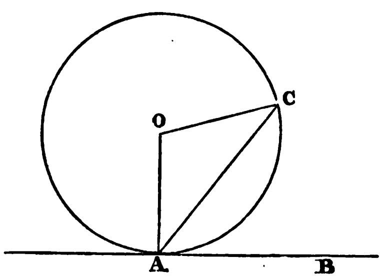
N.B. We have seen that *generally* two circles can be described through the two given points A, B touching the given circle CDF, but if AB touch the given circle, it is obvious that only one circle can be described as required. Of course, the problem is impossible, unless the two points A and B be both *outside* or both *inside* the given circle.

If the line joining the middle point of AB and the centre of the given circle be perpendicular to AB, the points E and F are the points in which the perpendicular to AB through its middle point meets the given circle CDF.

COR. Since, if we join AE, BE, the angle AEB is greater than any angle subtended by AB at any point *outside* the segment AEB, the angle AEB is the *maximum angle*, which AB subtends at any point on the circumference of the circle CDF. Also, AB subtends at F, the *minimum angle*.

132. *Describe a circle passing through a given point, and touching a given straight line at a given point.*

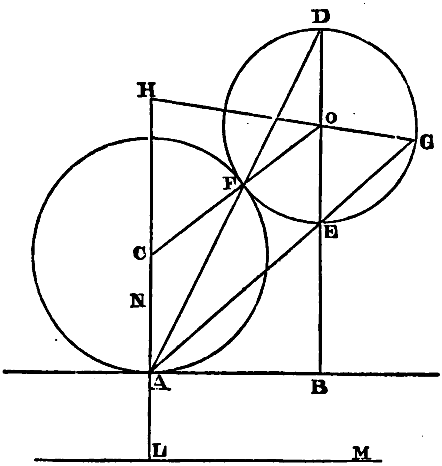
Let A be the point in the given straight line AB, and C the other point.



Join AC, and draw AO perpendicular to AB. Make the angle ACO equal to CAO. The circle described from the centre O at the distance AO or OC is evidently the required circle.

133. *Describe a circle touching a given circle, and a given straight line at a given point.*

Let O be the centre of the given circle, and A the given point in the given straight line AB. Through O draw DB perpendicular to AB and meeting the given circle in the points D and E. Join AD and AE, meeting the given circle again in F and G. Draw AC perpendicular to AB, and meeting OF and OG produced in C



and H respectively. The circles described with centres C and H and radii CF and HG respectively, will be the required circles.

Because CA and DB are perpendicular to the same line AB, they are parallel.

Therefore the angle FDO is equal to FAC. DOF is an isosceles triangle, therefore ACF is also an isosceles triangle and AC equals CF. Therefore the circle (C) clearly touches the circle (O) at F, and the given straight line at the given point A.

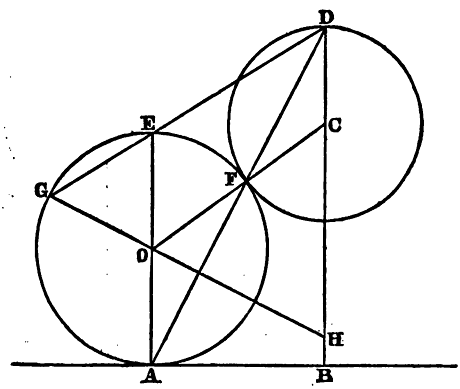
In like manner, it can be proved that the circle described from the centre H, at the distance HA, touches the given circle at G and the given straight line at A. Q. E. F.

N.B. Otherwise thus. In CA produced take $AL = OE$, and through L draw LM parallel to AB. Describe a circle touching LM at L and passing through O (132). The centre of this circle will be C, the centre of one of the required circles.

In like manner, by making $AN = OE$, and drawing through N a parallel to AB, the circle described touching this parallel at N and passing through O will have its centre at H, the centre of the other required circle.

134. *To describe a circle touching a given circle, passing through a given point and having its centre in a given straight line passing through this point.*

Let O be the centre of the given circle, D the given point, and



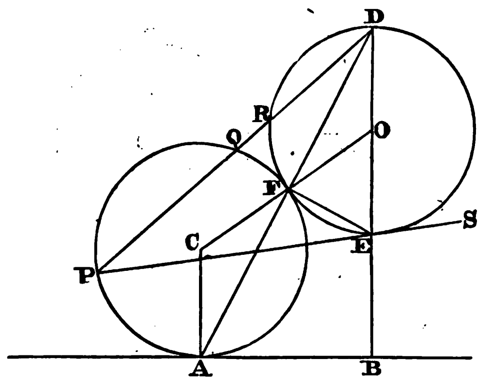
DB the given straight line. Draw the diameter AE parallel to DB, and the tangent AB. Join AD, meeting the given circle again in F, and ED meeting it in G.

Join OF, OG, and produce these lines to meet DB in C and H respectively. C and H are the centres of the required circles. For the triangles CFD and AOF are obviously equiangular, and AOF is isosceles; therefore CF and CD are equal, and the circle described from the centre C at the distance CD or CF must touch the given circle at F, for CO the distance between their centres is equal to the sum of their radii. In like manner, it can be proved that the circle described from the centre H at the distance HD or HG touches the given circle at G.

N.B. That the circles (O) and (C) touch *externally* at F may be thus shewn indirectly. For, if possible, let them have another point K in common. Draw the radii OK, CK. Therefore the triangle OKC has the two sides OK, KC together equal to the third side OC, which is impossible (I. 20). In like manner, it can be proved that, if the distance between the centres of two circles be equal to the difference of their radii, one of the circles must touch the other *internally*.

135. *To describe a circle passing through a given point, and touching a given straight line and a given circle, the circle and point lying on the same side of the straight line.*

Let P be the given point, AB the given straight line, and O



the centre of the given circle. Through O draw DEB perpendicular to AB.

Join PD, PE and make $PD \,.\, DQ = BD \,.\, DE$, which is given, and $PE \,.\, ES = DE \,.\, EB$, which is given. Through P, Q two circles can be described touching the given line AB, and through P and S two circles can also be described touching AB (127). These *four* circles will pass through the given point P, and touch the given straight line and given circle.

Let APQ be one of these circles, touching the given line at A. Join AD, and let it meet the given circle (O) in F. Join AC, CF, FO and FE. Because the angle DFE in a semicircle is a right angle, and DBA is right, therefore the quadrilateral $ABEF$ is circumscribable by a circle.

Therefore $AD \cdot DF = BD \cdot DE$ (III. 36),

but $BD \cdot DE = PD \cdot DQ$ (constr.).

Therefore $AD \cdot DF = PD \cdot DQ$, and therefore F is a point on the circle APQ.

(For, let AD meet this circle in F', then

$$PD \cdot DQ = AD \cdot DF',$$

therefore $AD \cdot DF = AD \cdot DF'$, and therefore $DF = DF'$,

that is, the points F and F' coincide, or F is also on the circle APQ.)

Again, because AC and CF, FO and OD are equal, therefore the angles CAF and CFA, OFD and ODF are equal. But CA and DB are parallel, and the straight line AD meets them, therefore the alternate angles CAF and ODF are equal; therefore the angles CFA and OFD are equal, and AFD is a straight line, therefore CFO is a straight line. Therefore the distance between the centres of the circles APQ and DFE is equal to the sum of their radii, therefore the circles touch one another externally at the point F.

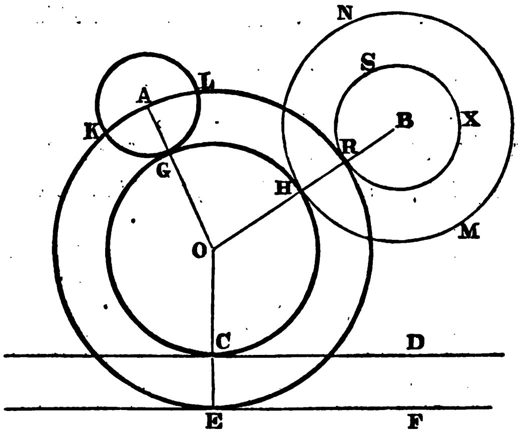
In the same manner it can be proved, that the other three circles described through P, Q and P, S touching the given line AB, also touch the given circle (O). Q. E. F.

136. *To describe a circle touching a given straight line and two given circles.*

Let CD be the given straight line, A and B the centres of the two given circles GKL and HNM respectively. Let the radius of the circle HNM be greater than that of GKL, and from the centre B at a distance equal to the difference of the radii of GKL and HNM describe the circle RSX. Draw EF parallel to CD and at a distance equal to the radius of the circle GKL.

Describe (135) a circle (O) through the point A touching EF in E and the circle RSX in R. Join OA, OE, OB, and from the centre O at the distance OH describe a circle. This will touch the given line at C and the two given circles at G and H. For CE and RH are each equal to GA (constr.). Therefore OC, OH and OG are equal, and the circle CGH manifestly touches CD at C and the given circles at G and H. Q. E. F.

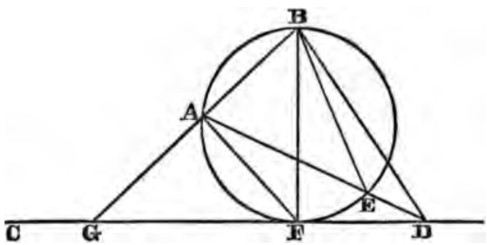
N. B. Four circles can *generally* be described touching a given straight line and two given circles on the same side of it. It is unnecessary to occupy space in discussing the different cases and modifications of the problem, as the above solution will enable the



intelligent learner to reduce every case to the problem (135) of describing a circle through a given point touching a given straight line and a given circle.

137. *Given a straight line and two points on the same side of it; find a point in the given line at which the two given points shall subtend a* maximum *angle.*

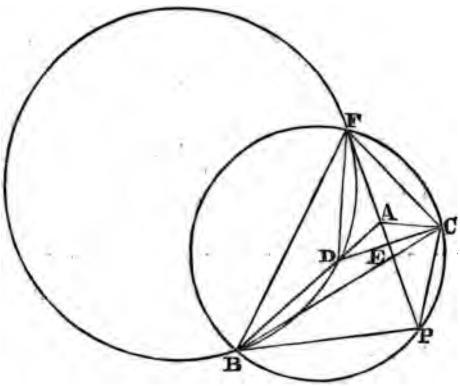
Let *CD* be the given straight line, and *A*, *B* the given points. Describe (127) a circle passing through *A*, *B* and touching *CD* at *F*. Then *F* is the required point. For, join *AF*, *FB*, *AD*, *DB*



and *BE*, *D* being any point in *CD* except *F*. The angle *AEB* is greater than *BDE* (I. 16); therefore also the angle *AFB*, which is equal to *AEB*, is greater than *ADB*.

N.B. Since (127) another circle can be described passing through *A*, *B* and touching *CD* on the side of *AB* remote from *F*, there are, *in general*, two points which satisfy the conditions of the problem. The angle *BFA* is greater than any angle *BDA* on the *same side* of *AB* as *F*, and the angle at the point of contact on the other side of *AB* is also greater than the angles subtended by *AB* at any point of *CD* on the same side of *AB* as the point of contact.

138. AE *bisects the angle* A *of a triangle* ABC, *and is produced to* P, *so that the difference of the angles* PBC *and* PCB *is a maximum, shew that their sum is half the angle* BAC.



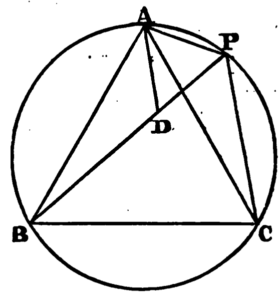
Take $AD = AC$, and join *CD*, cutting *AP* in *E*, therefore *CE* and *ED* are equal and the angles at *E* are right. Describe a circle about the triangle *BPC*, meeting *PA* produced in *F*. Join *BF*, *DF* and *CF*.

The angles *PBC* and *PFC* in the same segment are equal, and the angles *PCB* and *PFB* in the same segment are equal. Also the angles *DFE* and *CFE* are equal (I. 4). Therefore the angle *BFD* is the difference of the angles *PBC* and *PCB*, and is therefore a maximum. Hence the circle described through the points *B*, *D*, *F* touches *PF* at the point *F* (137).

Again, the angle *DFE* contained by the chord *DF* and tangent *FP* is equal to the angle *DBF* in the alternate segment of the circle *FDB*. Therefore the angle *BAP* is equal to the sum of *AFB* and *AFD*, or *AFC*, that is, half the angle *BAC* is equal to the sum of the angles *PCB* and *PBC*. Q. E. D.

139. *If any point on the circumference of a circle be joined to the three angles of an inscribed equilateral triangle, the straight line drawn to the remote angle is equal to the sum of the other two.*

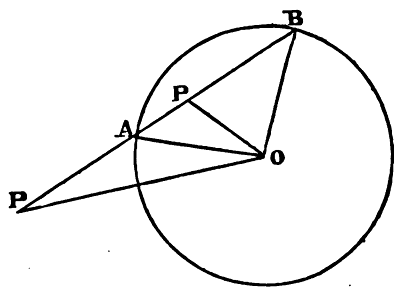
Let *ABC* be the inscribed equilateral triangle, and *P* any point on the circumference. Take *PD* equal to *PA*, and join *AD*.



Because *AP* and *PD* are equal and the angle *APD* is equal to the angle *ACB* of an equilateral triangle in the same segment, therefore *APD* is an equilateral triangle. Because the angles *BAC* and *DAP* are equal, therefore the angles *BAD* and *PAC* are equal. Therefore in the triangles *BAD*, *CAP* the two sides *BA*, *AD* and the contained angle *BAD* are respectively equal to the two sides *CA*, *AP* and the contained angle *CAP*. Therefore (I. 4) *BD* and *CP* are equal, and therefore *PB* is equal to *PA* and *PC* together. Q. E. D.

140. *Find the locus of a point such that if straight lines be drawn through it cutting a given circle, the rectangle under the intercepts between the point and the circle shall be constant.*

Let *O* be the centre of the given circle, and suppose *P* (within or without the circle) to be a point in the required locus, so that



PA . *PB* is equal to a given rectangle or square. Because *AOB* is an isosceles triangle, therefore (50) *PA* . *PB* is equal to the difference of the squares on *PO* and *AO*. But the rectangle *PA* . *PB* and the square on AO are both given, therefore also the square on PO is known, therefore PO is of a given length, and therefore the locus of P is a circle concentric with the given circle.

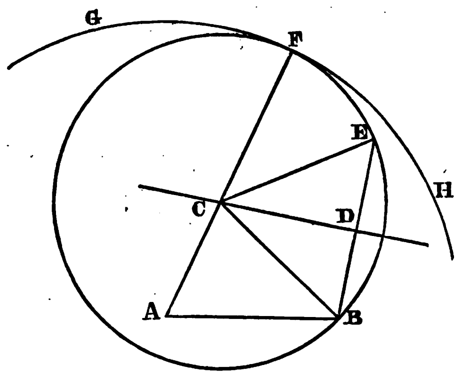
141. *Given the base of a triangle, the sum of its sides, and the locus of its vertex a fixed straight line; construct the triangle.*

Let AB be the given base, and CD the given locus of vertex.

From either extremity A of the base as centre and at a distance equal to the given sum of sides, describe the circle GFH.

Draw BD perpendicular to CD, and produce it until DE equals DB.

Through the points B and E describe (131) a circle touching the circle GFH internally at F, and join AF, meeting CD in C. Join CB.



ACB is the required triangle. Because CD bisects BE at right angles, CD passes through the centre of the circle BEF (III. 1, Cor.), and because F is the point of contact of the two circles and A the centre of one of them, therefore (III. 11) AF passes through the centre of the other circle BEF. Therefore C is the centre of the circle BEF, and therefore AC and CB together are equal to AF, the given sum of sides.

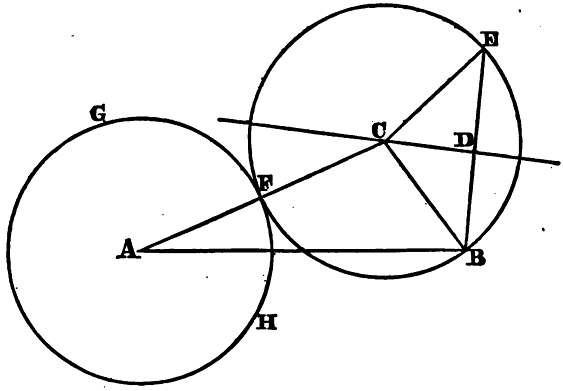
In like manner, by describing a circle from the centre B, with a radius equal to AF, and drawing a perpendicular from A to CD, another triangle can be described satisfying the conditions of the problem.

Hence, *in general*, two triangles can be described as required.

142. *Given the base of a triangle, the difference of the sides and the locus of vertex a fixed straight line; construct the triangle.*

Let AB be the given base and CD the given locus of vertex.

From the centre A at a distance equal to the given difference of sides describe the circle GFH. Draw BD perpendicular to CD



and produce it until DE equals BD. Describe a circle through B, E and touching GFH externally at F (131). Join AF, and produce it to meet CD in C. Join CB. Then ACB is the required triangle. The proof is exactly the same as in the last proposition.

If a circle with centre B and radius equal to AF be described, and if a perpendicular be drawn from A to CD, another triangle may be described satisfying the conditions of the problem. Hence, *in general*, two triangles can be described as required.

143. *In a given circle inscribe a triangle having its base parallel to a given straight line and its sides passing through two given points in this straight line.*

First (Fig. 1), suppose the two given points P, Q outside the given circle.

Make $PQ \cdot QG$ equal to the square on the tangent from Q to the given circle.

From G draw the tangents GC and GF, meeting the given circle at the points C and F respectively. Join QC and QF and produce these lines, if necessary, to meet the given circle again in A and D. Join AP, PD meeting the circle again in B and E. Join BC, FE. The triangles ABC and DEF are the required ones.

Because $PQ \cdot QG = AQ \cdot QC$ (constr.), therefore the quadrilateral $APGC$ is circumscribable by a circle, and therefore

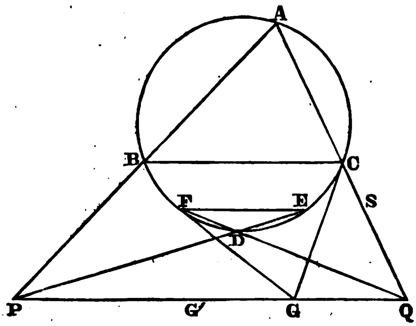


Fig. 1.

(III. 22) the angles CGQ and CAB are equal, but CG is a tangent to the circle at C, and CB is drawn from the point of contact, therefore (III. 32) the angle BCG is equal to the angle CAB in the alternate segment. Therefore the angles CGQ and BCG are equal, and therefore (I. 27) the straight lines BC and PQ are parallel. Therefore the triangle ABC has its base BC parallel to the given line PQ, and its two sides AB, AC passing through the two given points P and Q.

Again, because $PQ \cdot QG = FQ \cdot QD$ (constr.), therefore the four points F, P, G, D lie in the same circumference, and therefore the angles GFD and GPD in the same segment are equal, but FG is a tangent to the given circle at F, and FD is drawn from the point of contact, therefore (III. 32) the angles GFD and DEF are equal. Therefore the angles GPD and DEF are equal, and therefore EF and PQ are parallel. Therefore the triangle DEF is also described as required.

Next (Fig. 2), let the two given points P and Q be within the given circle.

Produce PQ indefinitely both ways to meet the circle in H and K.

On PQ produced take QG such that $PQ \cdot QG = HQ \cdot QK$. Therefore, since HQ is greater than PQ, QG is also greater than QK, therefore the point G is outside the given circle. Through G draw GC and GF touch-

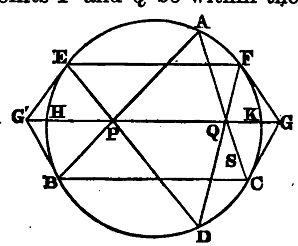


Fig. 2.

ing the circle at C and F. Join CQ, FQ and produce these lines to meet the circle again in A and D. Join AP and DP, meeting the circle again in B and E, and join BC, EF.

The two triangles ABC and DEF are the required ones.

Because $PQ \,.\, QG = AQ \,.\, QC$ (constr.), therefore the four points A, P, C, G lie on the same circumference, and therefore the angles ACG and APG in the same segment are equal, but GC touches the given circle at C, therefore the angles ACG and ABC are equal. Therefore the angles APG and ABC are equal, and therefore (I. 28) BC and PQ are parallel. In like manner, it can be proved that EF and PQ are parallel. Therefore either of the triangles ABC, DEF satisfies the conditions of the problem.

Further, the *same two* inscribed triangles are obtained by making $QP \,.\, PG' = HP \,.\, PK$, and proceeding as above. For, join $G'B$, $G'E$.

Because $G'P \,.\, PQ = AP \,.\, PB$, therefore the four points A, B, G', Q lie on the same circumference, and therefore the angles ABG' and AQG' in the same segment are equal, but AQP is equal to ACB, therefore the angle ABG' is equal to the angle ACB in the alternate segment. Therefore $G'B$ is a tangent to the given circle at B.

In the same manner, it can be proved that $G'E$ touches the given circle at E.

Hence, the same two triangles are obtained, whether tangents be drawn from G or G'. In like manner, it can be proved in Fig. 1, that if $QP \,.\, PG' = AP \,.\, PB$, the tangents from G' meet the given circle in the points B and E.

Hence, in both cases, only two distinct triangles can be inscribed as required.

It is evident, that one point P cannot be within the given circle and the other without it. The learner may examine the simple cases in which both points are on the circumference, one point on the circumference and the other on the tangent at this point, and the case in which the line PQ touches the circle at a point different from P or Q.

N.B. If the circle about the triangle APG (Figs. 1 and 2) do not pass through C let it cut CQ in S.

Therefore $\quad PQ \,.\, QG = AQ \,.\, QS$,

but $\quad PQ \,.\, QG = AQ \,.\, QC$ (constr.), therefore $QS = QC$, which is impossible;

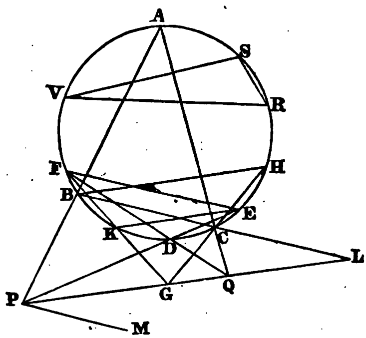
therefore if $$PQ \, . \, QG = AQ \, . \, QC,$$

the four points A, P, G, C must lie in the same circumference.

144. *In a given circle inscribe a triangle, having its base parallel to a given line and its two sides passing through two given points, not both situated on a line parallel to the given line.*

Let P, Q be the two given points, and PM the line to which the base is required to be parallel.

At any point V in the circumference of the given circle, make



the angle RVS equal to the given angle GPM, then the chord RS is constant.

Make $PQ \, . \, QG$ equal to the square on the tangent from Q to the given circle, and through G draw GH and GF such that HC and FK shall be each equal to RS (58).

Join QC and QF, and let these lines (produced, if necessary) meet the circle again in A and D. Join AP, PD, meeting the circle again in B and E. Join BC and FE. ABC and DEF are two triangles inscribed as required. For join BH and EK, and produce BC and PQ to meet in L.

Because $$PQ \, . \, QG = AQ \, . \, QC \text{ (constr.)},$$

therefore the quadrilateral $APGC$ is circumscribable by a circle, and therefore (III. 22) the angles CGQ and BAC are equal, but BAC and BHC in the same segment are equal. Therefore the angles CGQ and BHC are equal, and therefore (I. 27) BH and PQ are parallel.

Therefore the angles *HBC* (or *RVS*) and *CLP* are equal, but *HBC* and *GPM* are equal. Therefore *CLP* and *GPM* are equal, and therefore *BC* is parallel to *PM*.

Hence, *ABC* is a triangle inscribed as required.

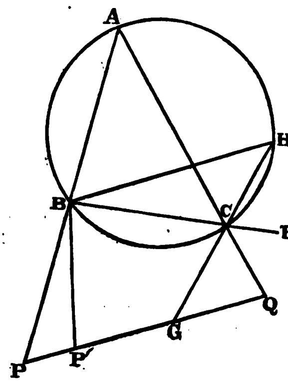
Again, because *PQ* . *QG* = *FQ* . *QD* (constr.), therefore the four points *F*, *D*, *P*, *G* lie in the same circumference, therefore the angles *FGP* and *FDP* in the same segment are equal, and therefore their supplements *FGL* and *FDE* are equal.

But *FDE* and *FKE* in the same segment are equal. Therefore the angles *FGL* and *FKE* are equal, and therefore (I. 28) *PQ* and *KE* are parallel, but the angle *FEK* is equal to *GPM*. Therefore *FE* and *PM* are parallel.

It may be shewn, in nearly the same manner as in the last problem, that if a point *G′* be taken on *PL* such that *QP* . *PG′* equals the square on the tangent from *P*, and if we proceed as above, then we shall obtain the same two triangles *ABC* and *DEF*. The learner may also treat separately, as is done in the last problem, the various possible positions of the given points *P* and *Q* with respect to the given circle.

145. *In a given circle inscribe a triangle whose sides shall pass through three given points.*

Let *P*, *Q*, *R* be the three given points.



Make *PQ* . *QG* equal to the square on the tangent from *Q* to the given circle. Inscribe (144) a triangle *BCH* in the given circle, having its two sides *BC*, *HC* passing through the fixed points *R*, *G* and its base *BH* parallel to the given line *PQ*. Join *QC*, and produce it to meet the circumference again in *A*, and join *AB*. Then *ABC* is a triangle inscribed as required. For, if possible, let *AB* produced not pass through *P* but meet *PG* in *P′*.

Because *BH* and *PQ* are parallel (constr.), therefore the angles *BHC* and *CGQ* are equal, but *BHC* and *BAC* in the same segment are equal. Therefore the angles

BAC and CGQ are equal, and therefore the quadrilateral $AP'GC$ is circumscribable by a circle, therefore $AQ \,.\, QC = P'Q \,.\, QG$,

$$\text{but} \quad PQ \,.\, QG = AQ \,.\, QC \text{ (constr.)}.$$

Therefore $P'Q \,.\, QG = PQ \,.\, QG$, and therefore $P'Q = PQ$, which is impossible. Therefore AB produced must pass through P. Therefore the triangle ABC is inscribed as required. Since, *in general*, two triangles can be inscribed in the circle, having two sides passing through G, R and the third side parallel to PQ, therefore another triangle can be inscribed as required in the given circle. Q. E. F.

146. *If two opposite sides of a parallelogram be the bases of two triangles with a common vertex, their sum is equal to half the parallelogram, when the point is within the bases, but if the point is outside the bases the difference of the triangles is equal to half the parallelogram. Also, if two adjacent sides of a parallelogram and the diagonal between them be the bases of three triangles, with a common vertex, the triangle with diagonal for base is equal to the difference or sum of the other two according as the point is within or without the angle (or its vertically opposite) contained by the two sides.*

Let $ABCD$ be the given parallelogram.

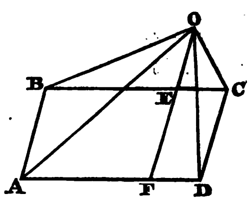


Fig. 1.

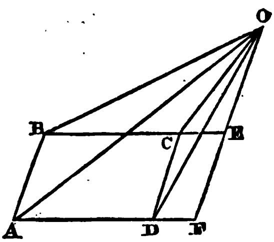


Fig. 2.

In Figs. 1 and 2 let AB and CD be the bases of triangles with common vertex O. Through O draw OEF parallel to AB, meeting the sides BC and AD (produced, if necessary) in E and F.

The triangle AOB is half the parallelogram BF (I. 41), and the triangle COD is half the parallelogram CF. Therefore in Fig. 1, the triangles AOB and COD are together half the parallelogram $ABCD$, and in Fig. 2 the difference of the triangles is half the given parallelogram.

Again, in Fig. 3, the triangles AOB and COD are together

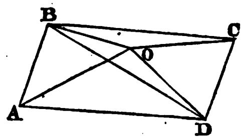


Fig. 3.

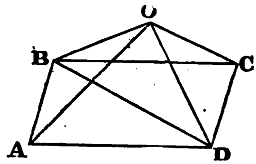


Fig. 4.

equal to the triangle BCD. Take away the common triangle COD, and there remains the triangle AOB equal to the triangles BOC and BOD. Therefore the triangle BOD is equal to the difference of the triangles AOB and BOC.

Further, in Fig. 4 the triangles AOB and COD are together equal to the triangle BCD. To these equals add the triangle BOC. Then the triangles AOB, BOC and COD are together equal to the figure $BOCD$. Take away the common triangle COD and there remains the sum of the triangles AOB and BOC equal to the triangle BOD. Q. E. D.

147. *Given an indefinite straight line and two points on the same side of it. Find a point in the line such that the straight lines drawn from it to the two given points shall be equally inclined to the given line, and prove that the sum of these straight lines is a minimum.*

Let CD be the given line, and A, B the two given points. Draw AG perpendicular to CD, and produce it until GE equals

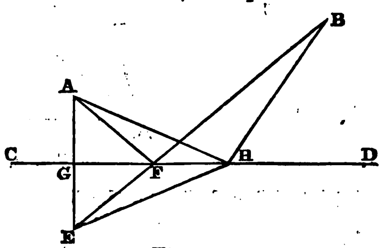


Fig. 1.

AG. Join BE, meeting CD in F. F is the required point. For, join AF, and join A and B with any other point H in CD. The angles AFG and GFE are equal (I. 4). Therefore AFG and BFD are equal (I. 15). Also, AF and FB are together equal to EB, and AH and HB are together equal to EH and HB. But EB is less than EH and HB together (I. 20). Therefore the sum of AF and FB is less than the sum of AH and HB. Q. E. D.

COR. Hence, of all triangles on the same base and with a given area the isosceles has the least perimeter. Since the base and area are given, the locus of the vertex is known (20). Let then AB be the given base, D its middle point, and CE parallel to AB the locus of vertex. Draw DC perpendicular to AB and join AC, CB, AE, EB, E being any point in CE except C.

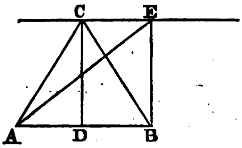
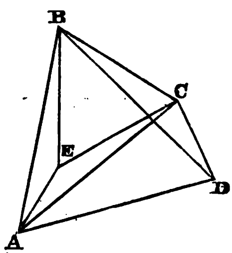


Fig. 2.

AC and CB are evidently equally inclined to CE, and therefore, by the proposition, AC and CB are together less than AE and EB. Therefore the perimeter of the isosceles triangle ACB is less than that of any other triangle AEB on the same base AB and between the same parallels. Q. E. D.

148. *If a quadrilateral be not circumscribable by a circle, the rectangle under its diagonals is less than the sum of the rectangles under its opposite sides.*

Let $ABCD$ be the given quadrilateral. Now, since it is not circumscribable, the angles BAC and BDC must be unequal. Let BAC be the greater, and draw AE, making the angle BAE equal to BDC, and BE making the angle ABE equal to CBD. Join EC.



The triangles ABE and CBD are equiangular, therefore (VI. 4)

$$AB : AE :: BD : CD.$$

Therefore (VI. 16) $AB.CD = BD.AE$.

Again, because the triangles ABE and CBD are similar,

$$AB : BE :: BD : BC, \text{ or alternately, } AB : BD :: BE : BC;$$

but the angles ABD and CBE are equal, therefore (VI. 6) the triangles ABD and CAE are similar, and therefore

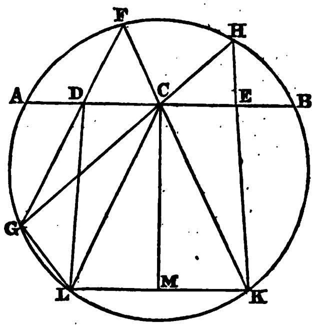
$$AD : BD :: CE : BC, \text{ therefore } AD.BC = BD.CE.$$

Therefore the sum of the rectangles under the opposite sides is equal to the rectangle under BD and the sum of AE and CE, but AC is less than the sum of AE and CE. Therefore the rectangle under the diagonals BD, AC is less than the sum of the rectangles under the opposite sides. Q. E. D.

Cor. Hence, if the rectangle under the diagonals of a quadrilateral is equal to the sum of the rectangles under the opposite sides, it is circumscribable by a circle; for if it be not circumscribable, the rectangle under its diagonals will be less than the sum of the rectangles under its opposite sides, contrary to the hypothesis.

149. *If any two chords be drawn through the middle point of a given chord of a circle, the straight lines joining their extremities, which are on opposite sides of the bisected chord, cut off equal parts from its ends.* (See 243.)

Let AB be the given chord bisected in C. Through C draw



any two chords FK and HG, and join FG, HK, meeting AB in D, E. Then shall AD and BE be equal, or CD and CE.

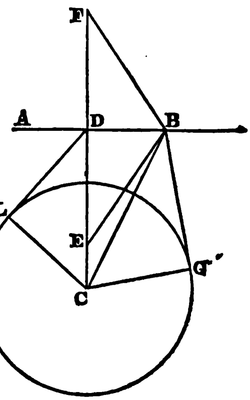
Through K draw KL parallel to AB and join CL, LD, LG. Through the centre draw a straight line perpendicular to AB and LK. This line will pass through their middle points C and M. Therefore (I. 4) the triangles CLM and CKM are equal in all respects, and therefore CL and CK are equal, and the angles CLM and CKM, LCM and KCM are equal.

Therefore the angle DCL is equal to ECK or FCD. But FCD is equal to CKL since AB and LK are parallel. Because $FGLK$ is a quadrilateral in a circle, therefore the angles CKL and DGL are together equal to two right angles. Therefore also DCL and DGL are together equal to two right angles, and therefore the quadrilateral $DGLC$ is circumscribable by a circle. Therefore the angles DGC and DLC are equal, but DGC and CKE are equal, since they stand on the same arc FH. Therefore the triangles DCL and ECK have the sides CL and CK equal, and the angles adjacent to those sides equal. Therefore (I. 26) *CD* and *EC* are equal. Q. E. D.

150. *If* CD *be the perpendicular from the centre* C *of a given circle on an indefinite straight line* AB, *and if* DE *be made equal to the tangent* DL, *then shall* BE *be always equal to the tangent* BG, *where* B *is any point on the given line* AB.

For join *CL*, *CB*, *CG*.

The square on *BE* is equal to the squares on *BD*, *DE* (I. 47). To these equals add the square on *CL*. Therefore the squares on *BE* and *CL* are together equal to the squares on *BD*, *DL* and *LC*, since *DE* is equal to *DL*. Therefore the squares on *BE* and *CL* are together equal to the squares on *BD*, *DC*, that is, to the square on *BC* or the squares on *BG* and *GC*. Therefore the square on *BE* is equal to the square on *BG*, and therefore *BE* and *BG* are equal. Q. E. D.

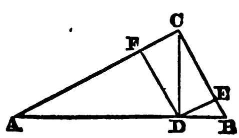


N.B. Take *DF* equal to *DE* or *DL*, and join *FB*. Since *BF* is equal to *BE*, it is evident that the tangent from any point *B* is always equal to *BF*.

Def. In geometry, the *reciprocal* of a line is the third proportional to it, and any assumed line which is called the *linear unit.*

151. *If a perpendicular be drawn from the right angle of a triangle to the hypotenuse, the square on its reciprocal is equal to the sum of the squares on the reciprocals of the sides.*

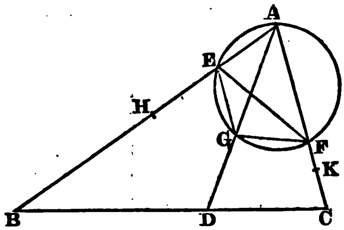
Let *ABC* be a triangle having the angle *ACB* right. Draw *CD* perpendicular to the hypotenuse, and take it as the linear unit. Draw *DF*, *DE* perpendicular to *AC*, *CB*. Therefore $AC \cdot CF = CD^2$ (39), therefore *CF* is the reciprocal of *AC*, and similarly *CE* is the reciprocal of *BC*. Also *CD* is



obviously its own reciprocal, and the square on CD is equal to the squares on CF, CE, since $CEDF$ is a rectangle. Q. E. D.

152. *Given the vertical angle of a triangle in magnitude and position, and the sum of the reciprocals of the sides; prove that the base always passes through a fixed point on the bisector of the vertical angle.*

Let BAC be the given vertical angle, AD its bisector, AH or



AK the linear unit, and ABC any triangle satisfying the given conditions. Make $BA . AE = AH^2$ and $CA . AF = AK^2$. Therefore AE is the reciprocal of AB and AF of AC. Therefore the sum of AE and AF is given (hyp.). Therefore (100) the circle circumscribing the triangle AEF always passes through a fixed point G on the bisector of the vertical angle.

Because $BA . AE$ and $CA . AF$ are each equal to the square on the linear unit, therefore $EBCF$ is circumscribable by a circle.

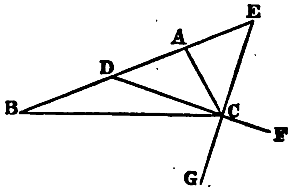
Therefore the angle AEF is equal to FCD, but AEF is also equal to AGF in the same segment. Therefore the angles AGF and FCD are equal, and therefore $DGFC$ is circumscribable. Therefore $DA . AG = CA . AF = AK^2$. But AG and AK are known lines, therefore also AD is a known line, and therefore the base BC always passes through the fixed point D. In the same manner it can be proved, when the difference of the reciprocals of the sides is given, that the base always passes through a fixed point on the bisector of the external vertical angle. Q. E. D.

153. *Given base of a triangle, vertical angle, and sum or difference of sides; construct the triangle.*

First, when the sum of the sides is given. Draw EB equal to the given sum of sides and the indefinite line EG, making the angle BEG equal to half the given vertical angle. Inflect from B to EG, the straight line BC equal to the given base, and draw

CA, making the angle *ECA* equal to *CEA*. Then *BAC* is the required triangle, for it obviously has the given base, given vertical angle and given sum of sides.

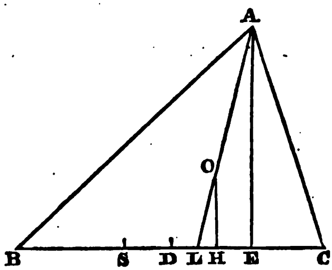
Next, when the difference of sides is given. Draw *DB* equal to the given difference of sides, and *DF* making the angle *ADF* equal to half the supplement of the given vertical angle. Inflect *BC* from *B* to *DF* equal to the given base, and draw *CA* making the angle *ACD* equal to *ADC*. *BAC* is evidently the required triangle.



N.B. This problem is clearly equivalent to the following. *Find a point in the arc of a given segment, so that the sum or difference of the straight lines which join the point to the extremities of the base of the segment shall be given.* In both cases, the learner will have no difficulty in investigating the limits of the problem, within which a solution is possible.

154. *If* D *be the middle point of the base* BC *of any triangle* BAC, E *and* L *the points where the perpendicular on the base, and the bisector of the vertical angle, meet the base, and* H, S *the points of contact of the inscribed and escribed circles; prove that the rectangles under the segments into which* L *divides* SH *and* DE *are equal, and that* DH . HE = DE . HL.

DS and *DH* are each equal to half the difference of the sides *AB* and *AC* (74).



Therefore (86) $DL \,.\, DE = DH^2$ or DS^2.

Now $SL \,.\, LH + DL^2 = DH^2$ (II. 5),

and $DL \,.\, DE = DL^2 + DL \,.\, LE$, (II. 3).

Therefore $SL . LH + DL^2 = DL^2 + DL . LE$,

and therefore $SL . LH = DL . LE$ which proves one part of the proposition.

Again $DE . DH = DE . HL + DE . DL$, (II. 1),

and $DH . DE = DH^2 + DH . HE$, (II. 3).

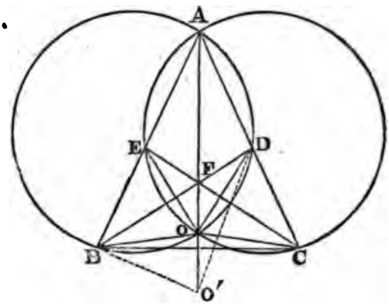
Therefore $DH^2 + DH . HE = DE . HL + DE . DL$,

but $DH^2 = DE . DL$,

therefore $DH . HE = DE . HL$. Q. E. D.

155. *If the bisectors of the base angles of a triangle be equal, the triangle is isosceles.*

Let ABC be the triangle, BD and CE the equal bisectors of the



base angles, then shall AB be equal to AC.

About the triangle AEC describe a circle meeting AF produced in O.

The circle about the triangle ABD must also pass through O. For, if possible, let it meet AF in O' and join OB, OE, OD, OC, $O'B$ and $O'D$.

Because the chords CE, BD are equal and subtend the same angle BAC, therefore (III. 24) the segments of circles CAE and BAD are equal, and therefore the remaining segments on CE and BD at the side remote from A are equal. Therefore also the chords of these equal circles, which subtend the equal angles BAF, CAF are equal, viz. the chords OE, OC, $O'B$, $O'D$. In the triangle EAC the bisector of the vertical angle meets the circumscribed circle in O, therefore (91) $AO . OF = OE^2$.

Similarly, $AO' . O'F = O'B^2$.

But OE and $O'B$ are equal, therefore $AO \cdot OF = AO' \cdot O'F$, which is impossible.

Therefore the circle about the triangle ABD must also intersect AF in O, and therefore the four straight lines OB, OE, OD, OC are equal. Therefore the angles OBE and OEB are equal, but OEB is equal to the angle OCA, since the quadrilateral $AEOC$ is in a circle. Therefore the angles OBE and OCA are equal, but OBC and OCB are also equal, since OB and OC are equal.

Therefore the angles ABC and ACB are equal. Q. E. D.

156. *Through a given point within a given angle, to draw a straight line cutting the sides of the given angle, so that the rectangle under the intercepts between the point and the sides of the given angle may be equal to a given rectangle.*

Let BAC be the given angle, and D the given point. Join AD, and produce it so that $AD \cdot DE$ shall be equal to the given rectangle. On DE describe a segment of a circle capable of containing an angle equal to the angle BAD, and let this segment cut AC in C. Join CD, and produce it to meet AB in B. BC is the required line.

Because the angles BCE and BAE are equal, therefore the figure $ABEC$ is circumscribable by a circle, and therefore $BD \cdot DC = AD \cdot DE$. Q. E. F.

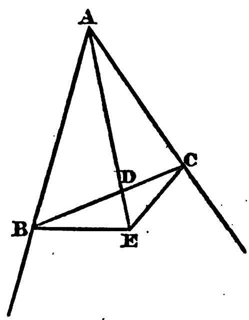


Fig. 1.

N.B. Let the given point D be *outside* the given angle BAC. Make $AD \cdot DE$ equal to the given rectangle, and on DE describe a segment of a circle containing an angle equal to DAB, and cutting AC in C. Join CD, meeting AB in B. Then DC is the required line. For $AEBC$ is circumscribable by a circle, and therefore $CD \cdot DB = AD \cdot DE$, the given rectangle.

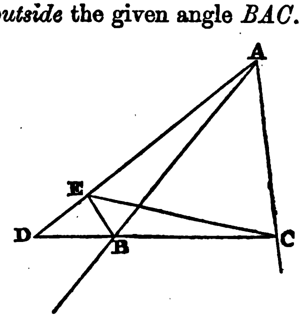
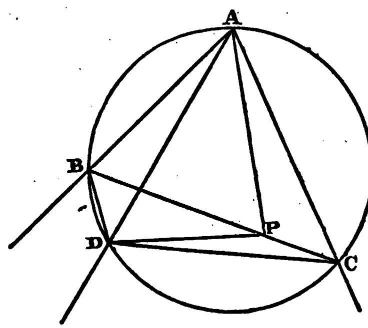


Fig. 2.

The learner should be careful to use the same letters at the corresponding points of the figures, since, in this way, he will generally find it easy to deduce one case of a proposition from another.

157. *Through a given point, to draw a straight line so as to form with the sides of a given angle a triangle of given area.*

Suppose P the given point, and BAC the required triangle.



Since the area of the triangle ABC and its vertical angle BAC are given, therefore (84) the rectangle under its sides AB and AC is known.

Join AP, and draw AD making the angle BAD equal to CAP, and make the rectangle $AP.AD$ equal to $AB.AC$. Join DB, DP, DC.

Because $AB.AC = AD.AP$ (constr.),

therefore (VI. 16) $AB : AD :: AP : AC$;

but the angles BAD, CAP are equal, therefore (VI. 6), the triangles BAD and CAP are similar, and therefore the angles BDA and BCA are equal.

Therefore the figure $ABDC$ is circumscribable by a circle, and therefore the angles BAD and BCD are equal. Hence the following construction. On DP describe a segment containing an angle equal to BAD, and cutting AC in C. Join CP, and produce it to meet AB in B. Then BAC is the required triangle.

The solution is similar, when the point P is without the given angle BAC. (See the remarks on the last problem.)

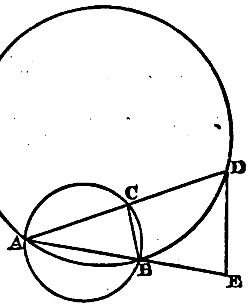
COR. Hence, if from the vertex of a triangle two straight lines be drawn equally inclined to the sides, one meeting the base, and the other the circumscribed circle, the rectangle under them is equal to the rectangle under the sides.

$$\text{For} \qquad AP \,.\, AD = AB \,.\, AC.$$

VI. *B* is a particular case of this Corollary.

158. AB *is a common chord of two circles; draw the straight line* ACD, *meeting the two circles in* C *and* D, *so that* AC . AD *shall be given.*

Produce *AB* so that *AB* . *AE* shall be equal to the given rect-



angle, and draw *ED*, making the angle *BED* equal to an angle in the given segment *ACB*, and meeting the circle *ABD* in *D*.

Join *AD*. This is the required line. For the quadrilateral *CBED* is clearly circumscribable by a circle, and therefore

$$AC \,.\, AD = AB \,.\, AE. \qquad \text{Q. E. F.}$$

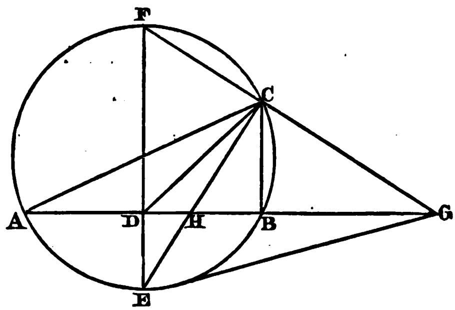
159. *Given the rectangle under the sides, the bisector of the base, and the difference of the base angles; construct the triangle.*

Suppose *ABC* the required triangle circumscribed by a circle. Draw *CE* and *CG*, bisecting the internal and external vertical angles, and meeting the circle in *E* and *F*.

Join *EF*, which is a diameter, bisecting the base *AB* at right angles in *D* (77). Join *EG*.

Because the angles *EDG* and *ECG* are right, therefore the figure *EDCG* is circumscribable by a circle. Therefore the angle

DEC is equal to the angle *DGC*, which is half the given difference of the base angles (76). Therefore the angle *DHC* is given,

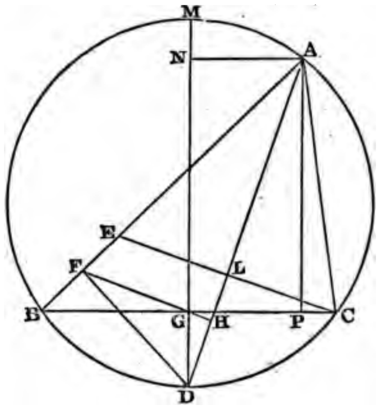


being equal to a right angle, together with half the difference of base angles.

Also, $EC \cdot CH = AC \cdot CB$ is given. Hence the following construction. Describe on *CD* the given bisector of base as common chord, two segments of circles containing angles equal to half the given difference of base angles, and the sum of half this given difference and a right angle. From *C* draw by (158) *CE* cutting those segments in *H* and *E*, so that $CH \cdot CE$ shall be equal to the given rectangle under the sides. Join *DH*, and produce it indefinitely both ways towards *A* and *B*. Join *ED*, and produce it to meet *CF*, drawn perpendicular to *CE* in *F*. Describe a circle about *ECF*, meeting *AB* in *A* and *B*. Join *AC*, *BC*. Then *ACB* is the required triangle. For it has clearly the given bisector of base, given difference of base angles, and given rectangle under the sides. Q.E.F.

160. *Given the vertical angle, the perpendicular on the base, and the sum of the two sides; construct the triangle.*

Let *BAC* be the vertical angle, given in magnitude and position, and suppose *BAC* the required triangle. Draw *AD*, bisecting the vertical angle and meeting the circumscribed circle in *D*, and *DF* perpendicular to *AB*, so that *AF* is half the sum of the sides, and *BF* half their difference (77). Therefore if *AE* equals *AC*, *BF* equals *FE*. Draw the diameter *DM*, bisecting *BC* at right angles in *G* (77), and draw *AN* perpendicular to *DM*, and *AP* perpendicular to *BC*. Because *FG* joins the middle points of the sides of the triangle *BEC*, it is parallel to *EC*, and therefore *FG* produced meets *AD* at right angles in *H*, since *AD* cuts *EC* at right angles in L. Therefore from the right-angled triangle AFD,



$$FD^2 = AD \, . \, DH = ND \, . \, DG;$$

but NG is equal to the given perpendicular AP, and FD is a given line. Therefore also DG is a known line, and H is a fixed point. Hence the triangle DGH can be constructed. Now draw the tangent BC through G to the circle with centre A, and radius equal to the given perpendicular, and this tangent will cut off the required triangle.

The complete construction then is simply this. Take AF equal to half the given sum of sides, and draw FD perpendicular to AB, meeting the bisector of the given vertical angle in D. Draw FH perpendicular to AD. Produce the given perpendicular until the rectangle under the whole produced line and the produced part is equal to the square on DF. From D inflect to FH a line DG, equal to the produced part of the given perpendicular. Through G draw the tangent BC to the circle with centre A, and radius equal to the given perpendicular.

Then ABC is the required triangle.

The synthetical proof of this construction may be deduced from the analysis given above. Q. E. F.

N.B. The solution is similar when the difference of the sides is given, but instead of a fixed point D on the bisector of the internal vertical angle, through which point the circumscribed circle always passes, it will pass through a fixed point M on the bisector of the external vertical angle. (See 152 and 77.)

The remainder of this work will be devoted chiefly to Modern Geometry.

I will now give some explanations of principles and abbreviated modes of expression, several of which will be often employed in the following pages.

$AB . CD$ is to be read, the rectangle under or contained by the straight lines AB and CD.

AB^2 is to be read, the square on AB.

$AD . DE = AB . AC$ is to be read, the rectangle under AD and DE is equal to the rectangle under AB and AC.

If r, r_1, r_2, r_3 and R represent straight lines, then

$$r_1 + r_2 + r_3 - r = 4R$$

is to be read the sum of r_1, r_2, and r_3 exceeds r by four times R, *or* the sum of r_1, r_2, and r_3 is equal to r, together with four times R.

$AB . DC + CB . DA = AE . BD + CE . BD = AC . BD$ is to be read, the sum of the rectangles under AB and DC and under CB and DA is equal to the sum of the rectangles under AE and BD, and CE and BD, which is equal to the rectangle under AC and BD.

$DD_1 = b \sim c$ is to be read, the straight line DD_1 is equal to the difference of the two straight lines b and c.

$$\tfrac{1}{2}(AC + CB) : \tfrac{1}{2}(AC - CB) :: \tfrac{1}{2}(AD + DB) : \tfrac{1}{2}(AD - DB)$$

is to be read, half the sum of AC and CB is to half their difference as half the sum of AD and DB to half their difference.

$AB : BE :: AC : CF$ is to be read

AB is to BE as AC is to CF.

$$BF : FC :: \begin{cases} BD : DA \\ AE : EC \end{cases}$$ is to be read,

BF is to FC in the ratio compounded of the ratios of BD to DA and of AE to EC.

Euclid has proved (VI. 20) that similar polygons are in the duplicate ratio of their homologous sides. Now squares are similar figures, therefore the similar and similarly placed figures upon the straight lines AB and CD are as the squares upon AB and CD.

"A *parallelopiped* (as defined by Dr Simson, XI. Def. A) is a solid figure contained by six quadrilateral figures, whereof every opposite two are parallel." It is clear that all the six *quadrilateral* figures are parallelograms. We may consider any two opposite and parallel *faces* (as the six parallelograms are called) the *bases* of the solid. When the other faces are at right angles to these bases, and all the faces are rectangles, the solid is called a *rectangular parallelopiped.*

When, moreover, these rectangular faces become equal squares, the solid is called a cube. (XI. Def. 25, Simson's *Euclid.*)

Euclid has proved (XI. 33) that "similar solid parallelopipeds are one to another in the triplicate ratio of their homologous sides" (or edges). Now cubes are similar solids; therefore if AB, CD be the homologous sides of two similar parallelopipeds, the solids are as the cubes with AB and CD for edges, or, as we may concisely express it, as $AB^3 : CD^3$.

Euclid has proved (VI. 23) that rectangles (for they are equiangular parallelograms) have to one another the ratio which is compounded of the ratios of their sides.

If, therefore, BC, CD and CG, CE be the adjacent sides of two rectangles,

$$BC \,.\, CD : CG \,.\, CE :: \left\{\begin{matrix} BC : CG \\ CD : CE \end{matrix}\right\}.$$

Therefore the ratio compounded of BC to CG and CD to CE is the ratio of the rectangle under the antecedents to the rectangle under the consequents.

Hence also $\left\{\begin{matrix} BC : CG \\ CD : CE \end{matrix}\right\} :: \left\{\begin{matrix} BC : CE \\ CD : CG \end{matrix}\right\}$ for each of these compound ratios is equal to the ratio of $BC \,.\, CD$ to $CG \,.\, CE$.

Dr Simson has proved (XI. D) that rectangular parallelopipeds "have to one another the ratio which is the same with the ratio compounded of the ratios of their sides" (or adjacent edges).

If then AM, AN, AO, and DL, DK, DH be the adjacent edges of two rectangular parallelopipeds,

$$\text{the solids are as } \left\{\begin{matrix} AM : DL \\ AN : DK \\ AO : DH \end{matrix}\right\}.$$

The parallelopipeds may be denoted thus:

$$AM . AN . AO \text{ and } DL . DK . DH.$$

$$\text{Therefore } \begin{Bmatrix} AM : DL \\ AN : DK \\ AO : DH \end{Bmatrix} :: AM . AN . AO : DL . DK . DH.$$

We may sometimes find it convenient to say the solid $AM . AN . AO$, instead of the longer expression the rectangular parallelopiped $AM . AN . AO$.

Hence the ratio compounded of AM to DL, AN to DK, and AO to DH, is the ratio of the solid under the antecedents to the solid under the consequents. From this enunciation it is also evident that we may interchange, in any way, the antecedents or consequents among themselves.

To find two straight lines which have the same ratio as the rectangular parallelopipeds

$$AM . AN . AO \text{ and } DL . DK . DH.$$

Find a straight line BC such that

$$AN : DK :: DL : BC$$

and a line DE such that

$$AO : DH :: BC : DE; \text{ therefore}$$

$$\begin{Bmatrix} AM : DL \\ AN : DK \\ AO : DH \end{Bmatrix} :: \begin{Bmatrix} AM : DL \\ DL : BC \\ BC : ED \end{Bmatrix} :: AM . DL . BC : DL . BC . DE,$$

but the solids $AM . DL . BC$ and $DL . BC . DE$ have equal bases $DL . BC$, and are therefore as their altitudes AM and DE.

Therefore $AM . AN . AO : DL . DK . DH :: AM : DE$.

$$\text{If } BF : FC :: \begin{Bmatrix} BD : DA \\ AE : EC \end{Bmatrix},$$

or, which is the same thing, if

$$BF : FC :: BD . AE : DA . EC,$$

then shall

$$DA . EC . BF = BD . AE . FC.$$

$$\text{For } \begin{Bmatrix} BF : FC \\ FC : BF \end{Bmatrix} :: \begin{Bmatrix} BD : DA \\ AE : EC \\ FC : BF \end{Bmatrix},$$

that is,

$$BF \,.\, FC : FC \,.\, BF :: BD \,.\, AE \,.\, FC : DA \,.\, EC \,.\, BF,$$

but $BF \,.\, FC = FC \,.\, BF$,

therefore also $BD \,.\, AE \,.\, FC = DA \,.\, EC \,.\, BF$.

$$\text{Again, } BD : CG :: \begin{Bmatrix} BD : DA \\ DA : CG \end{Bmatrix},$$

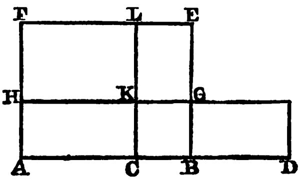
$$\text{for } \begin{Bmatrix} BD : DA \\ DA : CG \end{Bmatrix} :: BD \,.\, DA : DA \,.\, CG,$$

and these rectangles which have equal altitudes DA are as their bases BD and CG.

For the ratios $BD : DA$ and $DA : CG$ we may of course use any equivalent ratios, and interchange the antecedents or the consequents among themselves.

161. *If a straight line* AD *be divided at any two points* C *and* B, *prove that* AB . CD = AC . BD + AD . BC.

Draw BE perpendicular to AD and equal to CD, and take BG equal to BC. Complete the rectangles BF, DH, and BL.



Because CB and BG, CD and CL are equal, therefore the rectangles DK and BL are equal. To each of these equals add the rectangle CF, then the rectangles DH and HL are together equal to the rectangle BF.

But DH is the rectangle under AD and CB, for CB is equal to BG or AH, HL is the rectangle under AC and BD, for HK equals AC and KL equals BD, and BF is the rectangle under AB and CD, for BE equals CD. Q. E. D.

DEFS. (1) Three magnitudes are in *Arithmetical Progression* when the difference of the first and second is equal to the difference of the second and third.

(2) Three magnitudes are in *Geometrical Progression* when the first is to the second as the second is to the third.

(3) Three magnitudes are in *Harmonical Progression* when the first is to the third as the difference between the

first and second is to the difference between the second and third.

(4) When a series of magnitudes is such that every three consecutive magnitudes are in arithmetical, geometrical, or harmonical progression, it is called a series of magnitudes in arithmetical, geometrical, or harmonical progression.

(5) When a straight line AD (see Fig. to 161) is divided at C and B so that AC, AB, AD are in harmonical progression, the line AD is said to be divided harmonically at C and B.

(6) Any number of points lying on a straight line is called a *range*, and any number of straight lines passing through a point is called a *pencil*, and each of the straight lines is called a *ray* or *leg* of the pencil. The straight lines will generally be considered as produced indefinitely both ways through the point which is called the *vertex* of the pencil.

(7) If three magnitudes be in arithmetical, geometrical, or harmonical progression, the first and third magnitudes (or *terms*) are called extremes, and the second magnitude (or *term*) is called the mean.

(8) Any straight line cutting a system of straight lines or circles is called a *transversal.*

162. *If a straight line be divided harmonically at two points, when the three terms are measured from one end of the line, it is also divided harmonically when they are measured from the other end.*

Also, when a line is divided harmonically, the mean is divided internally and externally in the same ratio, and the rectangle under the whole line and the middle part is equal to the rectangle under the extreme parts, and conversely, when a line is divided internally and externally in the same ratio, or so that the rectangle under the whole and the middle part equals the rectangle under the extreme parts, the whole line is divided harmonically.

Let the straight line AD be divided so that AC, AB, AD are in harmonical progression, that is, so that

$$AC : AD :: CB : BD.$$

Therefore $DB : BC :: DA : AC$,

or alternately, $DB : DA :: BC : CA$, therefore DB, DC, DA are in harmonical progression.

Again, by (hyp.) and alternation, $AC : CB :: AD : DB$, therefore AB is divided internally at C and externally at D in the same ratio.

Therefore also (VI. 16) $AD . CB = AC . BD$.

The learner will find it easy to prove the converse propositions. Q. E. D.

163. *If a straight line be divided harmonically, the distance from the middle point of either mean to the two points of section and to the end of the line on the same side, are in geometrical progression, and if three distances measured in the same direction from a point along a straight line be in geometrical progression, and if a length equal to the geometric mean be measured in the opposite direction from the same point, the whole line will be divided harmonically.*

First, let AD be divided harmonically at C and B, and let O be the middle point of AB, then shall $OC . OD = OB^2$.

Since (162) $AC : CB :: AD : DB$, and AD is greater than DB, therefore AC is greater than CB, and therefore O lies between A and C.

Therefore also

$$\tfrac{1}{2}(AC - CB) : \tfrac{1}{2}(AC + CB) :: \tfrac{1}{2}(AD - DB) : \tfrac{1}{2}(AD + DB),$$

that is, $OC : OB :: OB : OD$ (1),

therefore (VI. 17) $OC . OD = OB^2$.

Next, let $OC : OB :: OB : OD$, and take OA equal to OB, then shall AD be divided harmonically at C and B.

Because $OC : OB :: OB : OD$ (hyp.),

therefore $OB + OC : OB - OC :: OD + OB : OD - OB$,

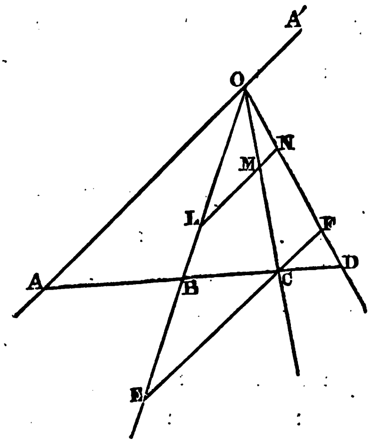
that is, $AC : CB :: AD : DB$. Therefore AB is divided internally and externally in the same ratio, and therefore (162) AD is divided harmonically at C and B.

N.B. Since $OC \cdot OD = OB^2$, if AB remain invariable, while the points C and D change their position agreeably to this relation, it is plain that if OC decrease, OD must increase, and therefore the two points C and D will move in opposite directions. Hence C and D may be called *harmonic conjugates* with respect to A and B, and similarly, A and B are harmonic conjugates to one another with respect to C and D.

This is usually expressed thus, A and B are harmonic conjugates to C and D, but the learner must distinctly understand that two alternate points, as C and D, are *conjugate the one to the other*, since if we take any point C on a finite straight line AB and produce it so that $AC : CB :: AD : DB$, the position of the point D depends upon that of C.

164. *If any transversal cut a pencil of four rays, the ratio of the rectangle under the whole transversal and its middle segment to the rectangle under its extreme segments, is constant.*

Let any transversal cut in the points, A, B, C, D, the rays of the pencil of which O is the vertex, then shall $AD \cdot BC : AB \cdot CD$ be constant.



Through C draw ECF parallel to the extreme ray AO, and draw any line LMN also parallel to AO.

Because the triangles CBE and ABO are similar, therefore $BC : AB :: EC : AO$, and because CF is parallel to AO, therefore $AD : DC :: AO : CF$.

Therefore, compounding these two pairs of ratios,

$$\left\{\begin{matrix} BC : AB \\ AD : DC \end{matrix}\right\} :: \left\{\begin{matrix} EC : AO \\ AO : CF \end{matrix}\right\},$$

that is, $AD \,.\, BC : AB \,.\, CD :: EC : CF$,

but $EC : CF :: LM : MN$, which is the same for every transversal.

Therefore $AD \,.\, BC : AB \,.\, CD$ is constant. Q. E. D.

N.B. This property may be thus enunciated—*If any transversal cut a pencil of four rays, and if a parallel be drawn to an extreme ray cutting the other three, the ratio of the rectangle under the whole transversal and its middle part to the rectangle under its extreme parts is equal to the ratio of the inner segment of the parallel to its outer segment.*

Now produce AO to any point A', then OA', OB, OC, OD form a pencil, and if we still denote the point in which any transversal cuts the *same* indefinite straight line through O by the *same* letter, the rectangle under the whole and middle part of the transversal will now be $AB \,.\, CD$, and the rectangle under the extreme parts $AD \,.\, BC$. Also LN is parallel to an extreme ray OA'. Therefore, as we have shewn, $AB \,.\, CD : AD \,.\, BC :: NM : ML$, since NM is *now* the *inner* part of the parallel to an extreme ray.

Therefore we have still $AD \,.\, BC : AB \,.\, CD :: LM : MN$, that is, if we follow the *same order of the letters* in the two rectangles their ratio remains unaltered. This is a very important principle.

Again, since $AD \,.\, BC : AB \,.\, CD :: LM : MN$ and (161) $AD \,.\, BC + AB \,.\, CD = AC \,.\, BD$, therefore

$$AD \,.\, BC : AC \,.\, BD :: LM : LN,$$

and

$$AB \,.\, CD : AC \,.\, BD :: MN : LN.$$

But $LM : LN$ and $MN : LN$ are constant ratios.

Hence the three ratios of the three pairs of rectangles which we have considered and their three reciprocals are each constant, $CD : AB$ being the reciprocal of $AB : CD$.

Any one of these six constant ratios is called, *in general*, the *anharmonic ratio* (A. R.) of the pencil. It is the same with that of the range A, B, C, D. The pencil OA, OB, OC, OD is called an *anharmonic pencil.* The pencil OA, OB, OC, OD I shall often express thus,—$O.ABCD$, where A, B, C, D are any points on its four rays.

The anharmonic ratio of the pencil is expressed thus—$\{O.ABCD\}$, and of the range thus—$[ABCD]$. These ratios, as we have seen, are equal. It will have been observed that the antecedents and consequents of any anharmonic ratio contain each the four letters differently arranged with a point between each pair of letters. If we assert that the two anharmonic ratios $[ABCD]$ and $[MRLN]$ are equal, and form the ratio $AD.BC : AB.CD$ in the first, then taking the letters in the same order in the other we have the ratio

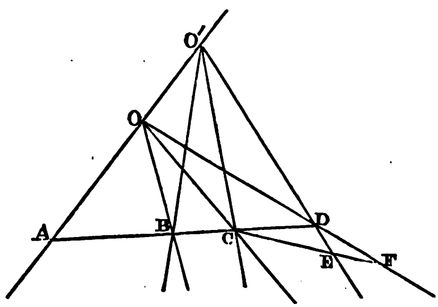
$$MN.RL : MR.LN \text{ equal to } AD.BC : AB.CD.$$

When AD is cut harmonically, then $AD.BC = AB.CD$, and therefore EC and CF are equal. In this case the pencil $O.ABCD$ is called an *harmonic pencil.* The range A, B, C, D may now be called an *harmonic range.*

Either extreme ray of an harmonic pencil is called a *fourth harmonic* to the other three, and an extreme point of an harmonic range is also called a *fourth harmonic* to the other three points of the range.

165. *If two anharmonic pencils with different vertices have a common ray and the same anharmonic ratio, the intersections of the three pairs of corresponding rays will lie in the same straight line.*

Let the pencils $O.ABCD$ and $O'.ABCD$ have the same anharmonic ratios and the common ray $O'A$, then shall B, C, D, the



intersections of the corresponding rays, lie in the same straight line.

Let CB produced meet OA in A, and, if possible, let it not pass through D, but meet OD and $O'D$ in F and E.

Because $\{O . ABCF\} = \{O' . ABCE\}$ (hyp.),

therefore $AF . BC : AB . CF :: AE . BC : AB . CE$,

or alternately, $AF . BC : AE . BC :: AB . CF : AB . CE$,

therefore $AF : AE :: CF : CE$,

and therefore $AF : AF - AE :: CF : CF - CE$,

that is, $AF : EF :: CF : EF$.

Therefore AF and CF are equal, which is impossible.

Therefore the points B, C, D must lie in the same straight line. Q. E. D.

166. *If two straight lines* AB *and* A′B′ *be similarly divided at* C, D *and* C′, D′, *then shall*

$$[ABCD] = [A'B'C'D'].$$

For $AD : AB :: A'D' : A'B'$,

and $BC : CD :: B'C' : C'D'$.

Therefore, compounding these ratios,

$AD . BC : AB . CD :: A'D' . B'C' : A'B' . C'D'$. Q. E. D.

Cor. Hence the pencils $O . ABCD$ and $O' . A'B'C'D'$, where O and O' are any assumed points, have the same anharmonic ratio.

For $\{O . ABCD\} = [ABCD]$ and $\{O' . A'B'C'D'\} = [A'B'C'D']$.

167. *If a transversal cut the sides of a triangle, the segments of any side are in a ratio compounded of the ratios of the segments of the other sides.*

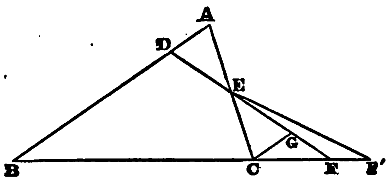


Fig. 1.

Let the transversal DEF cut the sides of the triangle ABC in the points D, E, F, so that CF, FB, BD, DA, AE, EC are the six segments taken in order.

Through C draw CG parallel to AB and meeting DF in G.

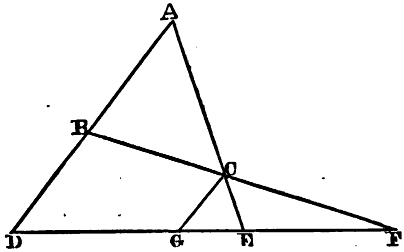


Fig. 2.

Therefore $BF : FC :: BD : CG :: \begin{Bmatrix} BD : DA \\ DA : CG \end{Bmatrix}$.

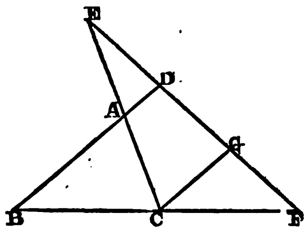


Fig. 3.

But $DA : CG :: AE : EC$, since the triangles AED and CEG are similar.

Therefore $BF : FC :: \begin{Bmatrix} BD : DA \\ AE : EC \end{Bmatrix}$,

and therefore $CF . BD . AE = FB . DA . EC$,

or the solids under the alternate segments are equal. Q. E. D.

N.B. In speaking of a transversal cutting systems of straight lines, I shall generally consider the straight lines produced indefinitely both ways.

The converse of the proposition is true, viz. If three points be taken on the sides of a triangle (*one* or *all three* lying on the sides produced) so as to satisfy the above relation, then shall the three points be in the same straight line.

For let D, E, F be three points so taken that

$$BF : FC :: \begin{Bmatrix} BD : DA \\ AE : EC \end{Bmatrix},$$

and if possible, let (Fig. 1) DE produced, not meet BC in F but in F'.

Therefore $\quad BF' : F'C :: \begin{Bmatrix} BD : DA \\ AE : EC \end{Bmatrix},$

and therefore $\quad BF' : F'C :: BF : FC.$

Therefore $\quad BF' - F'C : F'C :: BF - FC : FC,$

or $\quad BC : F'C :: BC : FC.$

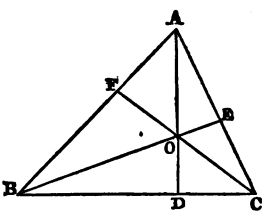
ore $F'C$ and FC are equal, which is impossible.

ore D, E and F are in the same straight line.

Any three straight lines drawn through the angles le so as to intersect in the same point, divide the des into segments, such that the segments of any a ratio compounded of the ratios of the segments of o sides.

straight lines drawn through the angles of the triangle ct in O. Then shall

$$BD : DC :: \begin{Bmatrix} BF : FA \\ AE : EC \end{Bmatrix}.$$



Because EB cuts the sides of the triangle ADC, and FC cuts the sides of the triangle ABD, therefore (167)

$$BD : BC :: \begin{Bmatrix} DO : OA \\ AE : EC \end{Bmatrix} \text{ and } BC : DC :: \begin{Bmatrix} BF : FA \\ AO : OD \end{Bmatrix}.$$

Therefore, compounding these ratios,

$$\begin{Bmatrix} BD : BC \\ BC : DC \end{Bmatrix} :: \begin{bmatrix} DO : OA \\ AO : OD \\ BF : FA \\ AE : EC \end{bmatrix}.$$

But $\begin{Bmatrix} DO : OA \\ AO : OD \end{Bmatrix}$ is a ratio of equality,

$$\text{and } \begin{Bmatrix} BD : BC \\ BC : DC \end{Bmatrix} :: BD : DC.$$

$$\text{Therefore} \qquad BD : DC :: \begin{Bmatrix} BF : FA \\ AE : EC \end{Bmatrix},$$

and therefore $AF \cdot BD \cdot CE = FB \cdot DC \cdot EA$,
that is, the solids under the alternate segments of the sides are equal. Q. E. D.

N. B. The learner should examine the cases in which the point O is without the triangle ABC.

The converse of this proposition is true, viz. If the above relation holds for three points *all* on the sides of the triangle, or *two* of them on two sides produced, then the straight lines joining these points with the opposite angles of the triangle meet in a point.

The proof is similar to the proof of the converse of (167).

Cor. Hence the lines joining the points of contact of the inscribed circle with the opposite angles of a triangle meet in the same point.

169. *If three straight lines drawn through the vertices of a triangle meet in a point, and if the points in which these lines meet the opposite sides be joined; the joining lines meet the opposite sides in three points which are in the same straight line. Also, every pencil of four lines in the figure is an harmonic pencil, and every range of four points is an harmonic range.*

Let ABC be the given triangle. Let AD, BE and CF meet in O, and let EF and BC intersect in L, FD and CA in M, DE and AB in N. Then shall L, M, N be in the same straight line, and every four-point range and four-line pencil in the figure shall be harmonic.

Because the three straight lines AD, BE, CF pass through the same point O, therefore (168)

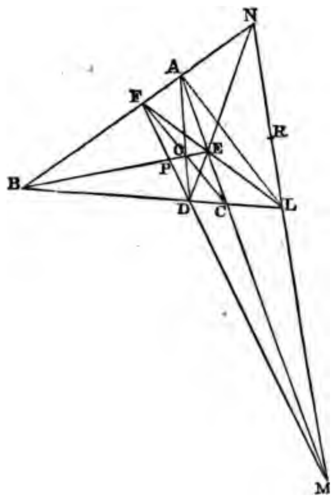
$$BD : DC :: \begin{Bmatrix} BF : FA \\ AE : EC \end{Bmatrix},$$

and because the transversal FL cuts the sides of the triangle ABC, therefore (167)

$$BL : LC :: \begin{Bmatrix} BF : FA \\ AE : EC \end{Bmatrix}.$$

Therefore $BD : DC :: BL : LC$,

and therefore BC is cut internally in D and externally in L, in the same ratio. Therefore B, D, C, L is an harmonic range, and the pencils $E.BDCL$ and $F.BDCL$ are harmonic pencils.



In the same manner it can be proved, that every other four points on the same straight line or four lines passing through the same point are harmonic ranges or pencils.

Again, join LA. Then $L.AECM$ is an harmonic pencil. Therefore if ML, LC, LE, and LA be produced to meet any transversal, they will cut it harmonically, but LC, LE, LA meet BN in B, F, A respectively, therefore ML must meet BN in the fourth harmonic to B, F, A, but N is the fourth harmonic. Therefore ML produced must pass through N. Therefore the three points L, M, N are in the same straight line. Q. E. D.

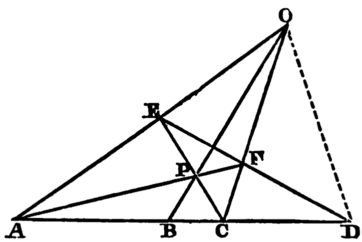
N.B. If we consider the quadrilateral $BDEF$, LN is its third diagonal. Let R be the point in which BE produced would meet LN.

Then, since $B.AECM$ is an harmonic pencil, therefore N, R, L, M form an harmonic range. That is, the third diagonal is cut harmonically by the other two. Also, P and R are harmonic conjugates to B and E, since $M.BFAN$ is an harmonic pencil. So P, M are harmonic conjugates to F, D. Hence, each diagonal of a complete quadrilateral is cut in conjugate harmonic points by the other two.

Def. In the harmonic range A, B, C, D, the alternate points B, D or A, C are called harmonic conjugate points, or simply, harmonic conjugates, and in the harmonic pencil $O.ABCD$, the alternate rays OB, OD or OA, OC are called *harmonic conjugate rays* or *harmonic conjugates.* (See 168, N.B.)

170. *Given three consecutive rays of an harmonic pencil or three consecutive points of an harmonic range; find the fourth harmonic. More generally, given any three rays of an anharmonic pencil or any three points of an anharmonic range and the anharmonic ratio; find the remaining ray or point, the position of the required ray or point with respect to the given ones being assigned.*

Let OA, OB, OC be the three given rays. Through any point P in OB draw the transversals AF and CE meeting OC and OA



in F, C and E, A. Join AC, EF, and let these lines meet in D. Join OD. OD is the required fourth harmonic. This is evident from (169), since the three straight lines AF, CE and OB drawn through the angles of the triangle AOC meet in the same point P.

If three points A, B, C of an harmonic range be given, to find the fourth. Join A, B, C with any point O. Draw AF and CP through the same point P on OB. EF and AC produced will obviously meet in D the required fourth harmonic.

Again (see Fig. to 164). Let OA, OB be two of the given rays, draw any line LN parallel to OA, then if OC be the third given ray, take MN so that $LM : MN$ is the same with the given anharmonic ratio, and join ON, this is the remaining ray. But if OA, OB and OD be the given rays, then divide LN in M, so that

$LM : MN$ is the same with the given ratio. Join OM, which is the required ray. If three of the four points of an anharmonic range be given, join them with any point O and find, as above, the remaining ray of the pencil with the given anharmonic ratio. This ray will cut the line joining the three given points in the required fourth point.

171. *The internal and external bisectors of any angle of a triangle are harmonic conjugates with respect to the sides containing that angle, and the three points in which the three external bisectors of the angles of a triangle meet the opposite sides are in the same straight line. Also, the points in which the external bisector of any angle and the internal bisectors of the other two angles intersect the sides respectively opposite to them lie in the same straight line.*

Fig. to 169. Let ABC be the given triangle, and suppose AD, BE and CF to be the bisectors of its angles, and AL to be the bisector of the external angle CAN.

The three internal bisectors pass through the same point O (73).

Also $BD : DC :: AB : AC$ (VI. 3),

and $BL : LC :: AB : AC$ (VI. A),

Therefore $BD : DC :: BL : LC$,

and therefore B, D, C, L form an harmonic range.

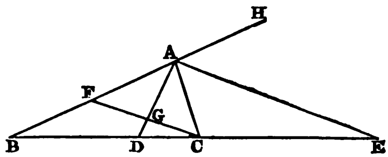
Therefore also $A . BDCL$ is an harmonic pencil.

Again (169), FE produced must meet BC in the fourth harmonic conjugate to B, D, C. Therefore FE passes through L. In like manner it can be proved that FD passes through M, and DE through N, where M and N are the other two points in which the external bisectors of the angles ABC and ACB meet the opposite sides. Hence it can be proved, exactly as in (169), that the three points L, M, N are in the same straight line. In fact, this proposition is only a particular case of (169).

172. *If two alternate rays of an harmonic pencil contain a right angle, they bisect the angles contained by the other two rays.*

Let $A . BDCE$ be the given harmonic pencil, and DAE the right angle. Then shall AD bisect the angle BAC, and AE bisect its supplement CAH.

Draw CF parallel to AE, and meeting AD in G and AB in F.



Therefore (164, N. B.) CG and GF are equal, and the angle AGF is equal to the alternate angle GAE, which is a right angle (hyp.). Therefore (I. 4) the angles FAG and CAG are equal, and therefore their complements EAH and EAC are also equal. Q. E. D.

173. *Given the base and the ratio of the sides of a triangle; find the locus of its vertex.*

Let BC be the given base, and divide it internally in D and

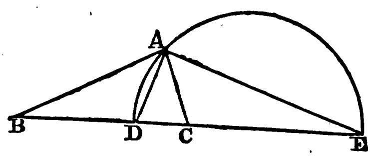


Fig. I.

externally in E in the ratio of the sides so that B, D, C, E is an harmonic range.

On DE describe the semicircle DAE. This is the required locus. For join any point A in the arc of the semicircle to the points of the range. Therefore $A \,.\, BDCE$ is an harmonic pencil, and the two alternate rays AD, AE contain a right angle.

Therefore (172) AD bisects the angle BAC, and therefore (VI. 3) $AB : AC :: BD : DC$, that is, in the given ratio.

Cor. *Hence given the base, the vertical angle and the ratio of the sides, we can construct the triangle.* For when the base and vertical angle are given, the locus of the vertex is a known segment upon the base (53). Also when the base and the ratio of the sides are given, the locus of the vertex is, by the proposition, a known semicircle.

A point in which these two circles intersect will determine the vertex.

Otherwise thus. On the given base BC describe a segment BAC containing an angle equal to the given vertical angle, and bisect the remaining segment BEC in E. Divide BC in D in the given ratio of sides. Join ED, and produce ED to meet the opposite circumference in A. BAC is the required triangle. Because the arcs BE and CE are equal, therefore AD bisects the vertical angle BAC, and therefore (VI. 3)

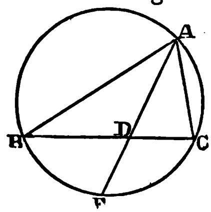
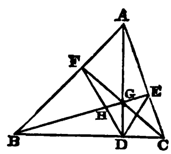


Fig. 2.

$$BA : AC :: BD : DC.$$

174. *If two straight lines drawn from the ends of the base of a triangle to meet the opposite sides intersect on the perpendicular to the base, the straight lines joining their points of intersection with the sides to the foot of the perpendicular, are equally inclined to the perpendicular.*

Let ABC be the given triangle, AD the perpendicular on the base, and BE, CF the straight lines drawn from the ends of the base, intersecting in any point G on AD. Join ED, FD. These lines are equally inclined to AD.



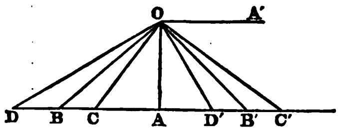
Because three straight lines drawn from the angles of the triangle ABC intersect in the same point G, therefore (169) $D.BFAE$ (or $D.BHGE$) is an harmonic pencil, but BDG is a right angle, therefore (172) GD bisects the angle HDE, and BD bisects its supplement. Q. E. D.

N.B. That the perpendiculars of a triangle bisect the angles of the triangle formed by joining their feet is only a particular case of this theorem. It can, however, be easily proved independently.

175. *The reciprocals of lines in harmonical progression are in arithmetical progression, and conversely, the reciprocals of lines in arithmetical progression are in harmonical progression.* (See 150, Def.)

It will be sufficient to consider three consecutive terms of the progression, since every other three consecutive terms can be similarly treated.

Let then AC, AB, AD be in harmonical progression. Draw AO perpendicular to AD and equal to the linear unit.



Draw OD', OB', OC', OA' respectively perpendicular to OD, OB, OC, OA. Therefore OA' is parallel to AD, and the angles DOD', BOB' are right angles; therefore, taking away the common angle BOD', the angles DOB and $D'OB'$ are equal. Similarly, the angles BOC and $B'OC'$, COA and $C'OA'$ are equal. Therefore the pencils $O.DBCA$ and $O.D'B'C'A'$ have their angles equal, each to each, and therefore they admit of superposition. Hence

$$\{O.DBCA\}=\{O.D'B'C'A'\}.$$

Therefore $O.D'B'C'A'$ is an harmonic pencil, and since AC' is parallel to its extreme ray OA', therefore $D'B'$ and $B'C'$ are equal, and therefore AD', AB', AC' are in arithmetical progression, but those lines are the reciprocals of AD, AB, AC respectively. For $AD.AD'=AO^2$, since DOD' is a right angle and AO perpendicular to DD', that is,

$$AD : AO :: AO : AD',$$

or AD' is a third proportional to AD and the linear unit AO. In like manner it can be shewn that AB' is the reciprocal of AB and AC' of AC.

Therefore the reciprocals of lines in harmonical progression are in arithmetical progression.

The converse follows at once from the same construction.

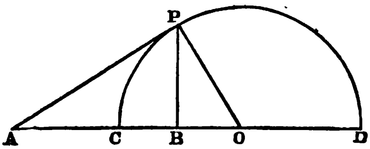
N.B. Since twice the arithmetic mean between two given lines is equal to the sum of the extremes, if we represent the reciprocal of AB by $\frac{1}{AB}$, which is to be read, *the reciprocal of AB*, the first part of the above theorem may be expressed thus:—

$$2.\frac{1}{AB}=\frac{1}{AC}+\frac{1}{AD}, \text{ or } \frac{2}{AB}=\frac{1}{AC}+\frac{1}{AD},$$

which is to be read, *twice the reciprocal of the harmonic mean between two lines is equal to the sum of the reciprocals of the extremes.*

176. *Given two unequal straight lines; find the arithmetic, geometric and harmonic means between them, and prove that the geometric mean is a mean proportional between the other two means.*

Let AC and AD be the given extremes. On CD describe the



semicircle with centre O. Draw the tangent AP and PB perpendicular to AD. Join PO.

AO is evidently the arithmetic mean, for the difference between AC and AO is equal to the difference between AO and AD.

AP is the geometric mean, for $AP^2 = AC \cdot AD$ (III. 36).

Also AB is the harmonic mean, for PB is perpendicular to the hypotenuse AO of the right-angled triangle APO, and therefore

$$OB \cdot OA = OP^2 = OC^2.$$

Therefore (163) AD is cut harmonically in C and B, that is, AB is the harmonic mean between AC and AD.

Again, from the right-angled triangle APO, $AP^2 = AO \cdot AB$, that is, the geometric mean is a mean proportional between the other two means. It is also obvious from the figure, that the arithmetic mean is the greatest and the harmonic the least.

N.B. If we take PB as the linear unit, then BO is the reciprocal of AB, for $AB \cdot BO = PB^2$. Also, BD is the reciprocal of CB, and CB of BD, for $CB \cdot BD = PB^2$, but the difference of CB and BD is twice BO; therefore *twice the reciprocal of the mean* AB *is equal to the difference of the reciprocals of the two differences* CB *and* BD, or, as we may concisely express it,

$$\frac{2}{AB} = \frac{1}{CB} - \frac{1}{BD}.$$

Again, if we take AP as the linear unit, the reciprocal of AC is AD, of AB, AO and of AD, AC, and these reciprocals AD, AO, AC are obviously in arithmetical progression.

This property has been already proved in (175), where it is expressed by $\frac{2}{AB} = \frac{1}{AC} + \frac{1}{AD}$.

177. *If through a fixed point two transversals be drawn, intersecting two given straight lines, and if the points of intersection be joined transversely; find the locus of the point of intersection of the joining lines.*

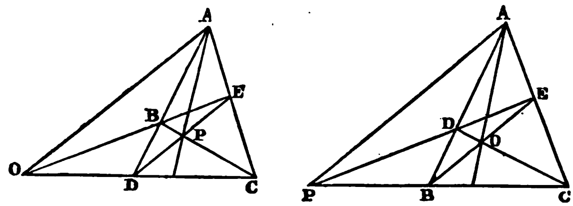


Fig. 1. Fig. 2.

First, let the two given lines AB, AC intersect in A.

Let P be the fixed point. Through P draw the transversals PDE, PBC, and let the joining lines CD, EB meet in O. It is required to find the locus of O.

Join AP, AO. Then $A \,.\, PBOC$ is an harmonic pencil (169), and its three rays AP, AB, AC are fixed, therefore (170) the remaining ray AO is determined, which is the required locus.

Next, let the two given lines AB, CD be parallel. Through the fixed point P draw the transversals PAC, PBD. Join PO, meeting the given lines in E and F. Because the triangles DOC and AOB are similar,

therefore $DO : OA :: CD : AB$,

but the triangles DOF and AOE are similar,

therefore $DO : OA :: FO : OE$.

Therefore $FO : OE :: CD : AB$.

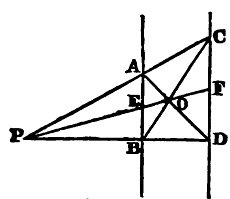


Fig. 3.

Also, because CD and AB are parallel,

$$CD : AB :: DP : PB :: FP : PE.$$

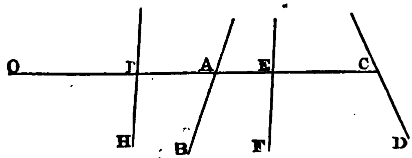
Therefore $FO : OE :: FP : PE$. Therefore FP is cut harmonically in E and O. But P, E and F are given points. Therefore O is known, and a parallel to the given lines through O is the required locus.

N.B. In finding the locus of O, when the two given lines are parallel, we have proved the case of (169), in which one of the lines joining the points of intersection of two sides is parallel to the third side of the triangle.

For the three lines drawn through the angles of the triangle PCD intersect in O, and meet the opposite sides in A, B, F, and AB is parallel to the third side CD.

178. *Through a given point without two given straight lines any transversal is drawn and a point taken on it, such that the reciprocal of its distance from the given point is equal to the sum of the reciprocals of the intercepts between the given point and the given lines; find the locus of the point of section.*

Let O be the given point, AB and CD the given lines.



Draw any transversal through O, meeting the given lines in A and C, and (170) find the point E such that O, A, E, C may be an harmonic range. Then the locus of E is a fixed line EF passing through the intersection of AB and CD, or parallel to them, if the given lines be parallel (177). Bisect OE in P, and draw PH parallel to EF. PH is the required locus. Because O, A, E, C is an harmonic range, therefore (175)

$$\frac{2}{OE} = \frac{1}{OA} + \frac{1}{OC}, \quad \text{but } \frac{2}{OE} = \frac{1}{OP},$$

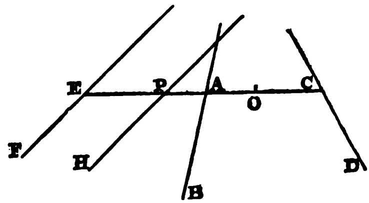
since the reciprocal of half a line is evidently twice the reciprocal of the whole line.

$$\text{Therefore } \frac{1}{OP} = \frac{1}{OA} + \frac{1}{OC},$$

and therefore PH is the required locus.

179. *Through a given point within two given straight lines any transversal is drawn, and a point taken on it such that the reciprocal of its distance from the given point is equal* *to the difference of the reciprocals of the intercepts between the given point and the given lines; find the locus of the point of section.*

Let O be the given point, and AB, CD the two given lines.



Draw any transversal CE through O, and find the locus of a point E such that E, A, O, C is an harmonic range.

This locus EF will either pass through the intersection of AB and CD, or be parallel to them, if the given lines be parallel (177).

Bisect OE in P, and draw PH parallel to EF. PH is the required locus. Because O and E are harmonic conjugates to C and A, therefore (176, N. B.)

$$\frac{2}{OE} = \frac{1}{AO} - \frac{1}{OC};$$

but $\frac{1}{OP} = \frac{2}{OE}$, therefore $\frac{1}{OP} = \frac{1}{AO} - \frac{1}{OC}$.

Therefore PH is the required locus.

180. *If through a given point without any number of given straight lines a transversal be drawn and a point taken on it, such that the reciprocal of its distance from the given point is equal to the sum of the reciprocals of the intercepts between the given point and given lines; find the locus of the point of section.*

Let O be the given point, and AB, CD, EF, ... the given lines.

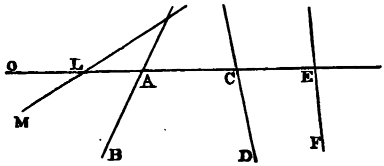
Draw any transversal through O, cutting the given lines in A, C, E, ...

Find (178) LM the locus of a point L such that

$$\frac{1}{OL} = \frac{1}{OA} + \frac{1}{OC},$$

we can now replace AB and CD by LM, since

$$\frac{1}{OL}+\frac{1}{EF}+\ldots=\frac{1}{OA}+\frac{1}{OC}+\frac{1}{OE}\cdots$$

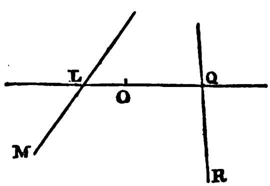


In this way we finally reduce the problem to (178). Hence the required locus is a fixed straight line.

181. *If through a given point within any number of straight lines a transversal be drawn, and a point taken on it, such that the reciprocal of its distance from the given point is equal to the excess of the sum of the reciprocals of the intercepts between the given point and the lines on one side of it over the sum of the reciprocals of the intercepts on the other side of it; find the locus of the point of section.*

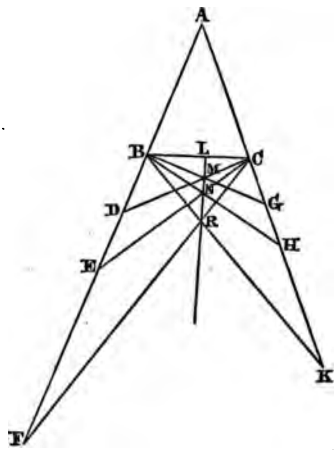
Let O be the given point, and LQ any transversal through it.

Find by (180) a line LM such that the reciprocal of OL is equal to the sum of the reciprocals of the intercepts between O



and the lines to the left of O and a line QR, such that the reciprocal of OQ is equal to the sum of the reciprocals of the intercepts to the right of O. The problem is now reduced to finding the locus of a point on LQ, such that the reciprocal of its distance from O is equal to the difference of the reciprocals of OL and OQ. The locus is therefore (179) a fixed straight line.

182. *On the sides of a triangle produced through the ends of the base, parts are taken in a given ratio, and their* *extremities joined to the remote ends of the base; find the locus of the intersection of the joining lines.*



Let ABC be the given triangle, and let $BD : CG$, $BE : CH$, and $BF : CK$ be taken in the given ratio. Join D, E, F to C, and G, H, K to B.

Because the straight lines BF and CK are similarly divided, therefore (166, and Cor.)

$$\{B \,.\, CGHK\} = \{C \,.\, BDEF\},$$

and the pencils have different vertices B and C, and the ray BC common, therefore (165) the three points M, N, R, in which the corresponding rays intersect, lie in the same straight line; therefore if we suppose the points M, N fixed, the straight line MN passes through any variable point R on the locus. Therefore MN produced indefinitely is the required locus.

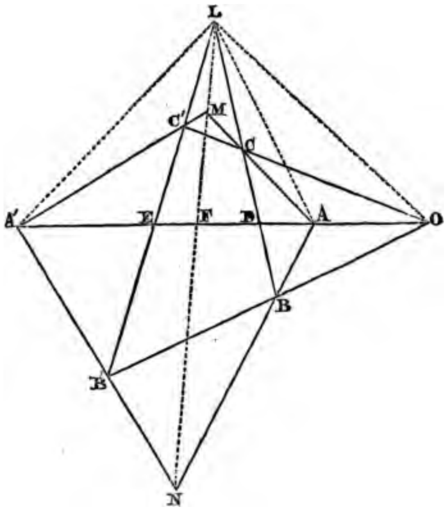
DEF. Two triangles are called *co-polar* when the lines joining corresponding vertices meet in a point, and this point is called the *pole* of the triangles. Two triangles are called *co-axial* when the intersections of the corresponding sides lie on the same straight line, and this line is called the *axis* of the triangles.

183. *If two triangles be co-polar, they shall also be co-axial; and conversely, if two triangles be co-axial they shall also be co-polar.*

First, let ABC, $A'B'C'$ be the two co-polar triangles, O being the pole, so that A and A', B and B', C and C' are the corresponding vertices, and the sides BC and $B'C'$, CA and $C'A'$, AB and $A'B'$, opposite to these vertices, the corresponding sides.

Let the corresponding sides meet in L, M, N, and join LA', LM, LA, LO and MN.

Because the transversals LB and LB' cut the same pencil with



vertex O, therefore $[LCDB] = [LC'EB']$, and therefore

$$\{A \,.\, LCDB\} = \{A' \,.\, LC'EB'\}.$$

Therefore the pencils $A \,.\, LCDB$ and $A' \,.\, LC'EB'$ have a common ray AA' and the same anharmonic ratio, therefore (165) the intersections of the corresponding rays, AL and $A'L$, AC and $A'C'$, AB and $A'B'$ lie on the same straight line, that is, the points L, M, N are in the same straight line. Therefore two co-polar triangles are also co-axial.

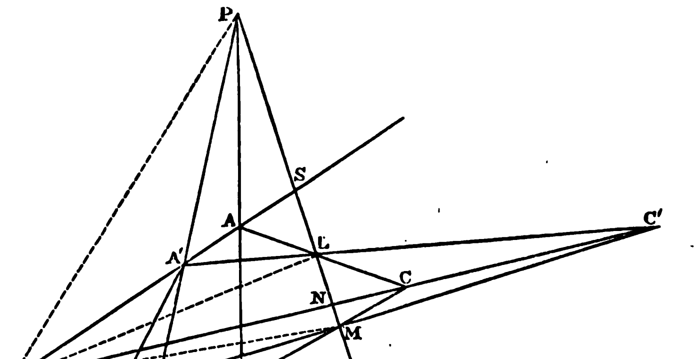
Next, let the two triangles ABC, $A'B'C'$ be co-axial, so that the points L, M, N are now given in the same straight line. Then shall CC', BB', and AA' meet in the same point.

Let $C'C$ and $B'B$ be produced to meet in O.

Now consider the two triangles $MC'C$ and $NB'B$. The lines joining the corresponding vertices M and N, C' and B', C and B meet in L, since N, M, L are in the same straight line (hyp.). Therefore, by the proposition just proved, the intersections of the corresponding sides must lie on the same straight line, that is, the intersections of MC' and NB', $C'C$ and $B'B$, CM and BN, but these intersections are the points A', O and A. Therefore $A'A$ produced passes through O. Therefore two co-axial triangles are also co-polar. Q. E. D.

184. *Given three fixed straight lines meeting in a point, if the three vertices of a triangle move one on each of these lines, and two sides of the triangle pass through fixed points; prove that the remaining side passes through a fixed point on the line joining the two given points.*

Let OA, OB, OC be the three fixed lines, L, M the two fixed points. Take two positions ABC, $A'B'C'$ of the triangles, and pro-



duce AB, LM to meet in P. Join PO, PA'. PA' produced will pass through B'. For, if possible, let PA' meet OB in β, and join OL, OM,

$$[ONCC'] = \{L.ONCC'\} = \{M.ONCC'\},$$

but obviously $\{L.ONCC'\} = \{L.OSAA'\}$.

In fact, these are anharmonic ratios of the same pencil.

Similarly $\{M.ONCC'\} = \{M.OTBB'\}$.

Therefore $[OTBB'] = [OSAA']$,

and therefore

$$OT.B'B : OB'.BT :: OS.A'A : OA'.AS,$$

but $$OS.A'A : OA'.AS :: OT.\beta B : O\beta.BT,$$

since each ratio is the anharmonic ratio of the pencil with vertex P.

Therefore

$$OT.\beta B : O\beta.BT :: OT.B'B : OB'.BT,$$

or, alternately,

$$OT.\beta B : OT.B'B :: O\beta.BT : OB'.BT.$$

Therefore (VI. 1) $\beta B : B'B :: O\beta : OB'$.

But βB is greater than $B'\beta$, therefore $O\beta$ is also greater than OB', which is impossible. Therefore P, A' and B' must lie on the same straight line.

Therefore $A'B'$ always passes through the intersection of LM and AB.

If therefore we consider ABC as a fixed triangle, the intersection of AB with the fixed line LM will determine the fixed point through which the base $A'B'$ of any variable triangle $A'B'C'$, satisfying the conditions of the proposition, always passes. Q. E. D.

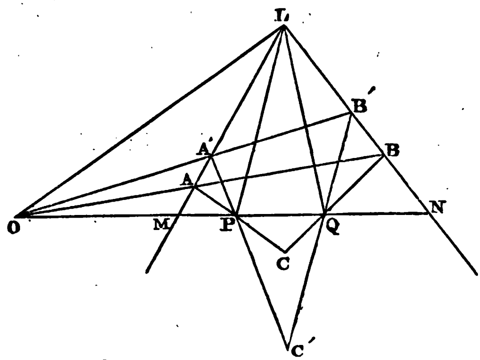
N.B. In this proposition we have incidentally proved an important principle, viz. *If two anharmonic ranges not formed on the same straight line have a common point and the same anharmonic ratio, the straight lines joining the three pairs of corresponding points meet in a point.* (Compare 165.)

For $[OB'BT] = [OA'AS]$, and we have proved that $A'B'$, AB and ST meet in the same point P.

COR. Hence we can describe a triangle having each of its vertices on one of three fixed straight lines meeting in a point, and each of its sides passing through a fixed point.

Let OA, OB, OC be the fixed lines, L, M two of the fixed points through which the two sides of the required triangle are to pass. Omitting the third point, find the point P through which the base always passes. Join P with the omitted point, and let the joining line meet OA, OB in the points A, B. AB is the base of the required triangle. Join AL and BM meeting in C. ABC is the required triangle.

185. *Two vertices of a triangle move on fixed straight lines, and the three sides pass through three fixed points, which lie on a straight line; find the locus of the third vertex.*



Let LM, LN be the two fixed straight lines, O, P, Q the three fixed points. Take two positions of the triangle, ABC, $A'B'C'$, and join LO, LP, LQ.

$[MAA'L] = [NBB'L]$, since each is the anharmonic ratio of the pencil with vertex O.

Therefore $\{P \,.\, MAA'L\} = \{Q \,.\, NBB'L\}$.

Therefore the pencils $P \,.\, MAA'L$ and $Q \,.\, NBB'L$ with different vertices and a common ray PQ have the same anharmonic ratio, therefore (165) the intersections of the corresponding rays PA and QB, PA' and QB', PL and QL lie on the same straight line. But these intersections are the points C, C', L. Therefore if the triangle ABC be considered fixed, the fixed straight line LC will always pass through the vertex C' of any variable triangle of the system. Therefore the required locus is a fixed straight line passing through the intersection of the straight lines on which the two vertices move.

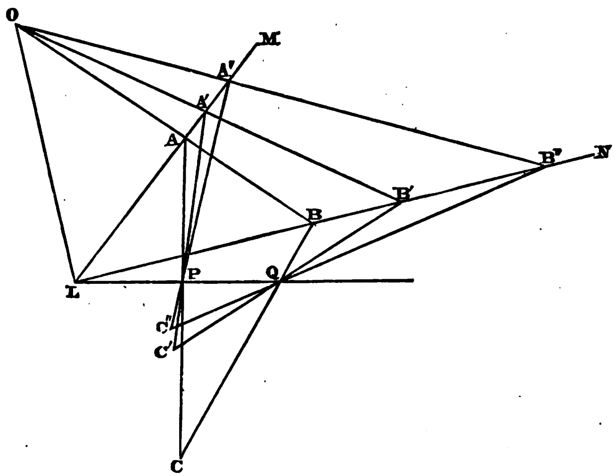
Cor. Hence we can inscribe in a given triangle another, so that its sides shall pass through three given points in a straight line.

For, let LM, LN be the two sides of the given triangle, O, P, Q the three given points. Omitting the third side of the given triangle, we can find the locus of the vertex of the required triangle. Let this locus meet the omitted side in C. Join CP,

CQ meeting LM, LN in A, B respectively. ABC is the required triangle.

186. *The base of a triangle passes through a fixed point, the base angles move on two fixed straight lines, and the sides pass through two fixed points which lie on a straight line passing through the intersection of the two fixed lines; find the locus of the vertex.*

Let O be the point through which the base passes, LM, LN the two given lines on which the base angles move, P, Q the given points through which the sides pass, L, P and Q being in a straight line.



Take three positions of the triangle, ABC, $A'B'C'$ and $A''B''C''$, and join OL.

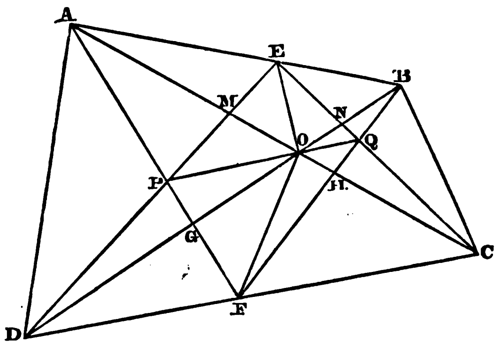
$[LAA'A''] = [LBB'B'']$, since each of these ratios is the anharmonic ratio of the pencil with vertex O.

Therefore $\{P . LAA'A''\} = \{Q . LBB'B''\}$.

Therefore the pencils $P . LAA'A''$ and $Q . LBB'B''$ have a common ray PQ and the same anharmonic ratio, therefore (165) the intersections of PA and QB, PA' and QB', PA'' and QB'', that is, the points C, C', C'' lie on the same straight line. Therefore if the two triangles ABC, $A'B'C'$ be considered fixed, the fixed straight line CC' will always pass through the vertex C'' of any variable triangle of the system. This fixed straight line is therefore the required locus.

187. *Any quadrilateral is divided by a straight line into two others; prove that the intersections of the diagonals of the three lie in a straight line.*

Let $ABCD$ be the given quadrilateral, and E, F the points in which the straight line dividing it into two others meets the



opposite sides. (This straight line is not drawn, in order not to complicate the figure.)

Let the diagonals of the three quadrilaterals intersect in P, O, Q; then shall P, O, Q be in a straight line. Join PO, OQ, EO, OF.

The transversals DB and CA cut the rays of the pencil with vertex F in the points D, G, O, B and C, A, O, H respectively, taken in order.

Therefore $\{A \,.\, DGOB\} = \{B \,.\, CAOH\}$.

Again, DE cuts the pencil $A \,.\, DGOB$, and CE cuts the pencil $B \,.\, CAOH$. Therefore $[DPME] = [CENQ]$.

Therefore $\{O \,.\, DPME\} = \{O \,.\, CENQ\}$.

Now the two pencils $O \,.\, DPME$ and $O \,.\, CENQ$ have the same anharmonic ratio, the ray OE common, the rays OD and ON in the same straight line, and OM and OC in the same straight line; therefore the remaining rays OP and OQ must also form one straight line. Q.E.D.

N.B. That PO and OQ form one straight line may be thus proved. (See also 170, which implies that PO and OQ form one determinate straight line.)

If possible, let PO produced meet CE in a point Q' different from Q. Then $\{O\,.\,DPME\} = \{O\,.\,NQ'CE\}$. The points D and N correspond, for the transversals DE and EC cut the *same ray* OD in the points D and N. So P and Q' are points on the *same ray*, and M and C.

But $\{O\,.\,DPME\} = \{O\,.\,CENQ\}$, as has been proved above.

Therefore $\quad [CENQ] = [NQ'CE],$

and therefore $\quad CE\,.\,NQ : CQ\,.\,NE :: CE\,.\,NQ' : NE\,.\,CQ',$

or, alternately, $\quad CE\,.\,NQ : CE\,.\,NQ' :: CQ\,.\,NE : NE\,.\,CQ'.$

Therefore $\quad NQ : NQ' :: CQ : CQ'.$

If Q' lie between N and Q, then NQ' is less than NQ, from the figure, and consequently CQ' greater than CQ, but since NQ' is less than NQ, and $NQ : NQ' :: CQ : CQ'$, therefore CQ' is also less than CQ, but it has been shewn to be greater than it, which is impossible.

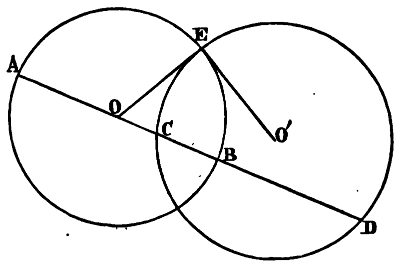
In like manner, it can be proved that Q' cannot lie between Q and C. Therefore PO produced must pass through Q. The principle proved in this note is often useful.

The proposition (187) may be enunciated in several other ways. Thus, (α) *Given in magnitude and position the bases* AB, CD *of two triangles* AFB, CED, *and that the vertex of each triangle moves on the base of the other; prove that the line* PQ *joining the intersections of their sides* AF, DE, *and* BF, CE *always passes through the fixed point* O *in which the lines joining the ends of the bases intersect.*

Or thus,—(β). *If the angles of a variable triangle* PFQ *move on three given straight lines* ED, DC *and* CE, *whilst the two sides* PF, FQ *pass respectively through two fixed points* A, B *lying on a straight line passing through* E, *the intersection of the lines on which the base angle moves, the base* PQ *always passes through another fixed point.*

188. *If two circles cut one another orthogonally, any straight line drawn through the centre of either and meeting both circles is cut harmonically by the two circumferences.*

Let the two circles whose centres are O and O' cut orthogonally at E. Then shall any straight line AD drawn through the centre of either be cut harmonically at C and B.



Because OEO' is a right angle (*hyp.*), therefore (III. 16) OE is a tangent to the circle CED, and therefore (III. 36) $OC.OD = OE^2 = OB^2$. Therefore (163) A, C, B, D form an harmonic range. Q. E. D.

N.B. The converse is also true, viz. *If a line be cut harmonically, any circle through one pair of conjugate points is cut orthogonally by the circle on the distance between the other pair of conjugates as diameter.*

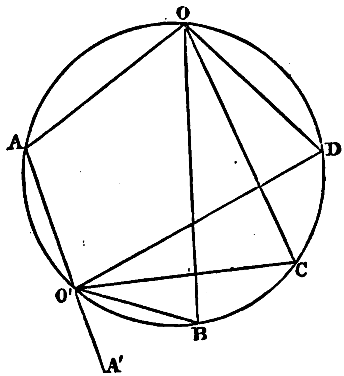
For, let AD be cut harmonically in C and D. Describe any circle passing through C and D, and describe a circle on AB as diameter intersecting the former in E. Join OE. Because AD is cut harmonically and AB is bisected in O, therefore (163) $OC.OD = OB^2 = OE^2$. Therefore (III. 37) OE touches the circle CED at E. Therefore the two circles cut one another orthogonally.

DEF. If four fixed points on a circle be joined to any variable point on the circle, the anharmonic ratio of the pencil thus formed is called the *anharmonic ratio of the four points*, and if a variable tangent meet four fixed tangents, the anharmonic ratio of the four points of intersection is called *the anharmonic ratio of the four tangents.*

189. *The anharmonic ratio of four fixed points on a circle is constant.*

Let $ABCD$ be the four fixed points on the circumference, and O or O' any variable point on the circumference.

While O remains on the arc AOD it is evident that the anharmonic ratio of the pencil $O.ABCD$ continues the same, since its angles continue the same.



Let O pass to O' between A and B, and produce AO' to A'.

The angles $A'O'B$ and AOB are equal (III. 22), because $AOBO'$ is a quadrilateral in a circle. Therefore the two pencils $O.ABCD$ and $O'.A'BCD$ have their angles respectively equal. Therefore $\{O.ABCD\}=\{O'.A'BCD\}$, but the pencils $O'.ABCD$ and $O'.A'BCD$ are really the same (164, N. B.):

Therefore $\{O.ABCD\}=\{O'.ABCD\}$. Therefore the anharmonic ratio of four fixed points on a circle is constant.

190. *If a variable tangent meet two fixed tangents, the intercept on it subtends a constant angle at the centre of the circle.*

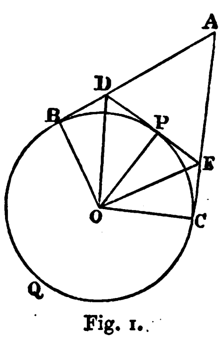


Fig. 1.

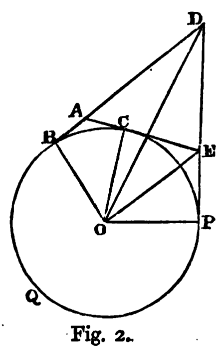


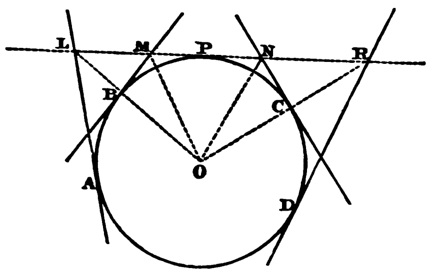
Fig. 2.

Let O be the centre of the given circle, B, C the points of contact of the fixed tangents which meet in A, and P the point of contact of the variable tangent DE. Then shall the intercept DE subtend at the centre an angle equal to half the angle subtended at the centre by the points of contact B and C.

From the figures it is clear that the angle DOP is half the angle BOP, and the angle EOP half the angle COP. Therefore the angle DOE is half the angle BOC. Q. E. D.

191. *The anharmonic ratio of four fixed tangents is constant.*

Let O be the centre of the circle, A, B, C, D the points of contact of the fixed tangents, and P the point of contact of any variable tangent. Let the variable tangent meet the fixed tangents in L, M, N, R respectively, and join these points to the centre O by the dotted lines, which do not necessarily pass through any of the points of contact A, B, C, D.



The pencil $O \, . \, LMNR$ has all its angles constant by (190).

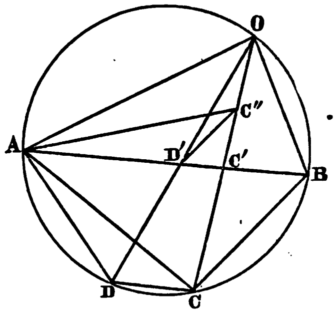
Therefore $\{O \, . \, LMNR\}$ is always constant. Q. E. D.

N. B. By comparing (189) and (191) it is obvious that the anharmonic ratio of any four tangents is equal to that of their four points of contact, since the angles which the intercepts on the tangents subtend at the centre are respectively equal to the angles subtended by the points of contact at a point on the circumference.

192. *Prove that the anharmonic ratio of four points on a circle is the same with the ratio of the rectangles under the opposite sides of the quadrilateral formed by joining the four points.*

Let A, B, C, D be the four fixed points on the circle.

Take AD' equal to AD, join DD' and produce it to meet the circle again in O.



Join OA, OC, OB, and draw $D'C''$ parallel to CB. Join AC'', AC.

Because $C''C$ meets the two parallels $D'C''$, CB, therefore the angles $D'C''C$ and OCB are equal, but OCB and OAB in the same segment are equal, therefore $D'C''C$ equals OAB, and therefore the quadrilateral $AOC''D'$ is circumscribable by a circle. Therefore the angles $AC''D'$ and AOD', $D'AC''$ and $D'OC''$ are equal, but AOD' and ACD, $D'OC''$ and CAD are equal. Also AD' and AD are equal (constr.). Therefore (I. 26) the triangles $AD'C''$ and ADC are equal in all respects; therefore $D'C''$ and DC are equal.

From the similar triangles $C'C''D'$ and $C'CB$,

$$D'C' : C'B :: (D'C''=) DC : CB,$$

also $$AB : AD' :: AB : AD,$$

since AD' and AD are equal.

Therefore compounding these ratios,

$$AB . D'C' : AD' . C'B :: AB . CD : AD . CB. \quad \text{Q. E. D.}$$

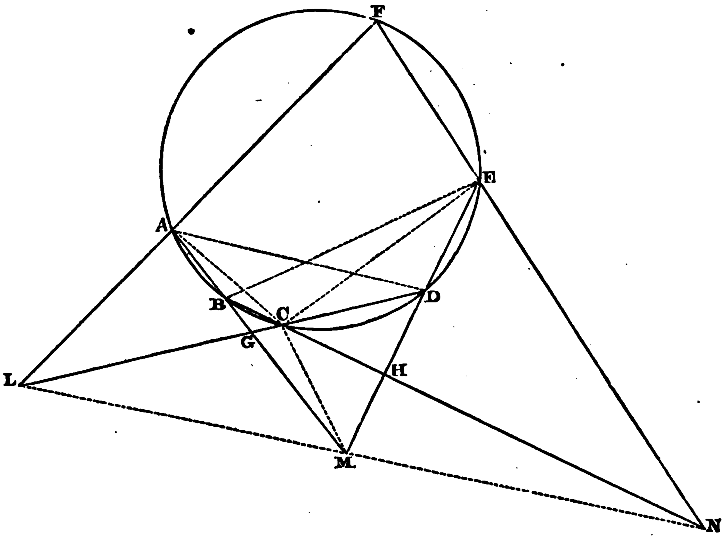
193. *If any hexagon be inscribed in a circle, the intersections of the three pairs of opposite sides lie on the same straight line.* (Pascal's Theorem.)

Let $ABCDEF$ be any hexagon inscribed in a circle, and L, M, N the points in which its opposite sides meet. L, M, N are in the same straight line. Join LM, MN, AC, AD, EC, EB and MC.

By (189) $\{A\,.\,FBCD\} = \{E\,.\,FBCD\}$.

But the pencils $A\,.\,FBCD$ and $A\,.\,LGCD$ are identical, and so are $E\,.\,FBCD$ and $E\,.\,NBCH$.

Therefore $\{A\,.\,LGCD\} = \{E\,.\,NBCH\}$.



Also, clearly, $\{A\,.\,LGCD\} = \{M\,.\,LGCD\}$,

and $\{E\,.\,NBCH\} = \{M\,.\,NBCH\}$.

Therefore $\{M\,.\,LGCD\} = \{M\,.\,NBCH\}$.

Therefore (see 187, N.B.), LM and MN form one straight line. Q. E. D.

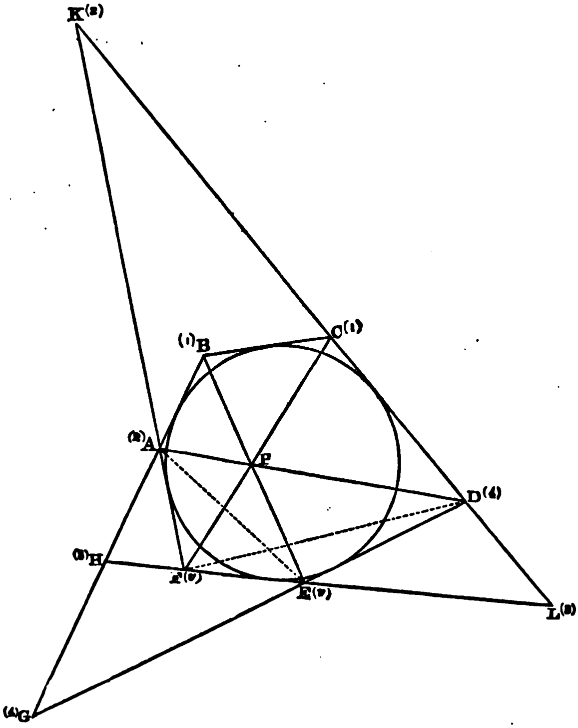
194. *The straight lines joining the opposite angles of any hexagon described about a circle pass through the same point.* (Brianchon's Theorem.)

Let $ABCDEF$ be the hexagon circumscribing the circle, and let its two *alternate* sides AB and CD meet the other four sides in B, A, H, G and C, K, L, D, respectively. Join EA, FD.

Therefore (191), $[BAHG] = [CKLD]$.

And therefore $\{E\,.\,BAHG\} = \{F\,.\,CKLD\}$.

Also the ray EF is common to the two pencils $E.BAHG$ and $F.CKLD$, therefore (165) the intersections of the corresponding rays EB and FC, EA and FK, EG and FD lie on the same straight line, but these intersections are the intersections of the two diagonals EB, FC and the two points A, D respectively; therefore the diagonal AD passes through the intersection of the other two diagonals EB and FC. Q. E. D.

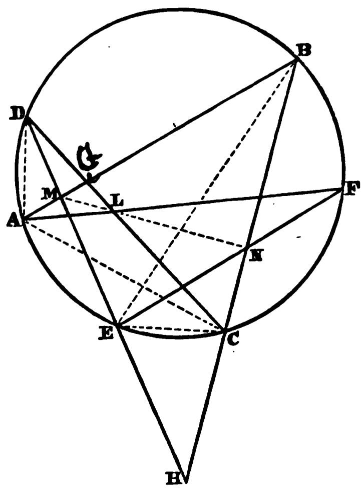


N.B. It is worthy of remark, that in (194) we have cut the same four sides of the hexagon by any two sides AB, CD, which have only *one* side BC between them, and that the vertices of the pencils are at the extremities of the side opposite BC.

Again, in (193) the vertices of the pencils are at any two vertices A, E of the hexagon separated by only *one* vertex F between them, and the four other vertices of the hexagon correspond to the four sides which are cut by the other two in (194).

In propositions like the present one, the learner is recommended to place small numbers and small letters at the corresponding points or lines, as has been done in (194). Thus the points (1) are both on BC, (2) on AF, (3) on FE, and (4) on ED, and (v) is placed at the vertex of each pencil. If it conduce to clearness, (v') may be used at one vertex for (v), and one set of numbers may also carry accents (1′), &c. We have then (191) $\{v\,.\,1234\} = \{v'\,.\,1'2'3'4'\}$, and the theorem is now manifest by (165).

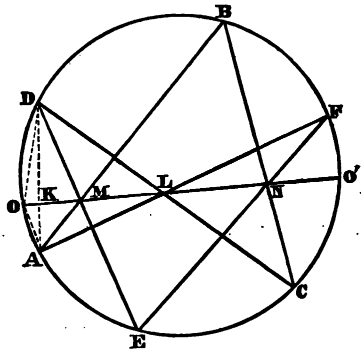
195. *If we assume six points on the circumference of a circle, and join them in the order* ABCDEFA (*forming a figure which we may call, by an extension of the term, an inscribed hexagon or* hexagram), *the intersections of the opposite sides* FA *and* CD, AB *and* DE, BC *and* EF *lie on the same straight line.*



This is proved by applying to the figure here the words of the proof of (193).

In like manner, if we join the six points in any other order, the points corresponding to L, M, N will still be in a straight line.

196. *Given six points on the circumference of a circle; find a seventh point on the circumference, such that the anharmonic ratio of it and three of the points taken in an assigned order, shall be equal to the anharmonic ratio of it and the other three points taken in an assigned order.*



Let it be required to find a point O on the circle, such that the anharmonic ratios of O, A, E, C and O, D, B, F, the points being taken in this order, shall be equal, where A, E, C, D, B, F are the six given points.

Write down the given points in the above order, thus

$$\begin{pmatrix} AEC \\ DBF \end{pmatrix}.$$

Now interchange the middle points, thus

$$\begin{pmatrix} ABC \\ DEF \end{pmatrix},$$

and form a hexagon by taking the points in this order, $ABCDEF$.

Let the opposite sides of this hexagon meet in L, M, N. These points will be in a straight line (195), and this straight line produced will meet the circle in two points O and O', either of which will be the required seventh point.

Join DO, OA, and AD cutting OO' in K.

Then,

$$\{D \,.\, OAEC\} = \{D \,.\, OKML\}, \text{ and } \{A \,.\, ODBF\} = \{A \,.\, OKML\},$$

$$\text{but} \qquad \{D \,.\, OKML\} = \{A \,.\, OKML\}.$$

Therefore $[OAEC] = [ODBF]$.

Therefore O is the required seventh point. In like manner it can be proved that

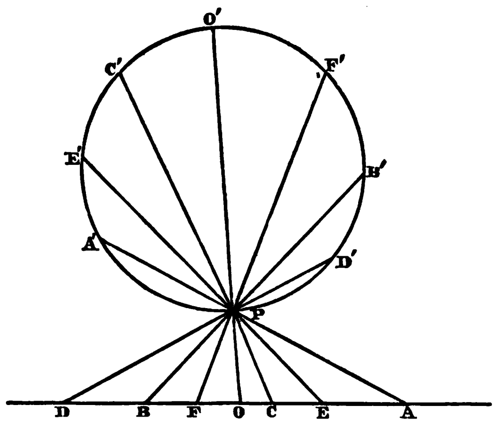
$$[O'AEC] = [O'DBF],$$

so that there are two points satisfying the conditions of the problem.

N.B. It will have been observed that we have used $[OAEC]$ to denote the anharmonic ratio of four points on a circle, as well as that of four points on a straight line, but no confusion can arise from this extension of the meaning of $[OAEC]$, since the figure will always indicate whether a range, or four points on a circle, are meant.

197. *Given six points on a straight line; find a seventh point on the given line, such that the anharmonic ratio of it and three of the points taken in an assigned order, shall be equal to the anharmonic ratio of it and the other three points taken in an assigned order.*

Let A, E, C, D, B, F be the given points. Describe any circle, and join any point P on its circumference to the six given



points. Let these joining lines meet the circumference again in A', E', C', D', B', F', and find (196) a point O' such that

$$[O'A'E'C'] = [O'D'B'F'].$$

Join $O'P$, and produce it to meet the given line in O. Then O is the required point.

For, obviously, $\{P \,.\, OAEC\} = \{P \,.\, O'A'E'C'\}$,

and $\{P \,.\, ODBF\} = \{P \,.\, O'D'B'F'\}$.

But $\{P \,.\, O'A'E'C'\} = \{P \,.\, O'D'B'F'\}$, (constr.)

therefore $\{P \,.\, OAEC\} = \{P \,.\, ODBF\}$,

or $[OAEC] = [ODBF]$.

Therefore O is the required point.

Another point may be found satisfying the conditions of the problem, since another point besides O' may be found on the circle.

198. *Inscribe in a given polygon another of the same number of sides, so that each of its sides shall pass through a given point.*

All that is meant here is that there is a number of given points, and the same number of given straight lines indefinitely produced both ways, and that it is required to describe a polygon so that each of its sides shall pass through one of the given points, and each of its vertices be upon one of the given straight lines.

Let $ABCD$ be the given polygon, L, M, N, R the given points.

On any side BC of the polygon take three points G_1, G_2, G_3 as trial positions of one vertex (or angular point) of the required polygon.

Join G_1M meeting BA in F_1; join F_1N meeting AD in E_1; join E_1R meeting DC (*the side adjacent to* CB *with which we began*) in H_1, and join G_1L, also meeting DC in K_1.

Proceed in the same way with G_2 and G_3.

Thus the points H_1, H_2, H_3 and K_1, K_2, K_3 are determined on the side DC adjacent to BC, on which the three points were assumed at first.

Now suppose $EFGH$ to be the required inscribed polygon.

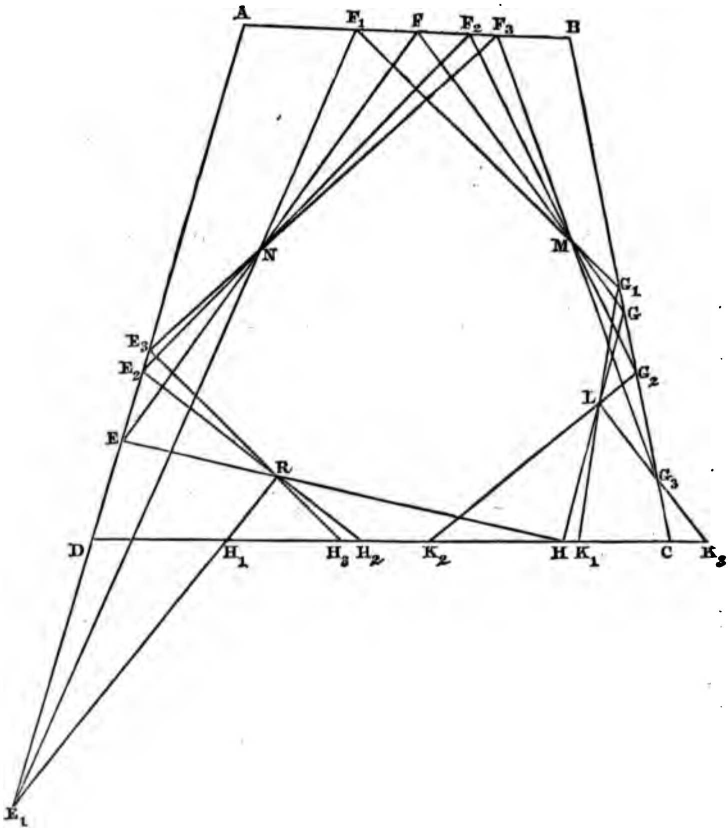
Then clearly

$$\{L \,.\, HK_1K_2K_3\} = \{L \,.\, GG_1G_2G_3\} = \{M \,.\, GG_1G_2G_3\} = \{M \,.\, FF_1F_2F_3\}$$
$$= \{N \,.\, FF_1F_2F_3\} = \{N \,.\, EE_1E_2E_3\} = \{R \,.\, EE_1E_2E_3\} = \{R \,.\, HH_1H_2H_3\}.$$

Therefore $\{L . HK_1K_2K_3\} = \{R . HH_1H_2H_3\}$,

or $[HK_1K_2K_3] = [HH_1H_2H_3]$.

Hence the problem is reduced to (197), since H is the required point, and the other six points are given.



N.B. We have supposed the successive order of the angles and sides of the required polygon with respect to the sides of the given polygon and the given points to be assigned. Thus there will be, *in general, two solutions*, since another point h can be found such that $[hK_1K_2K_3] = [hH_1H_2H_3]$.

I will not stop to discuss the particular cases of the problem, such as, when all the given points are in a straight line, or when all the given straight lines pass through the same point. In either case there will be only *one effective solution.* (See 185 and 184.)

The elegant applications of anharmonics in (196), (197), and (198) are due to the Rev. R. Townsend, F.R.S., Fellow of Trinity College, Dublin, and Professor of Natural Philosophy in the University.

199. *Find the locus of the intersection of equal tangents to two given circles, and prove that it is a straight line, perpendicular to the line joining the centres of the circles and dividing this line, so that the difference of the squares on its segments is equal to the difference of the squares on the corresponding radii.*

Let A and B be the centres of the two given circles, CD, CE equal tangents from a point C in the required locus.

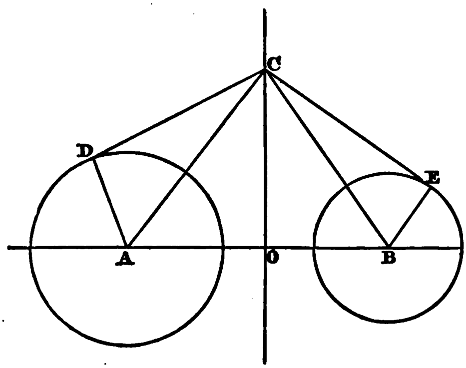


Fig. 1.

Draw (Fig. 1) CO perpendicular to AB, and join AD, AC, BE, BC.

The difference of the squares on AO and OB is equal to the difference of the squares on AC and CB, and this difference is equal to the difference of the squares on AD and BE, since CD and CE are equal (hyp.).

Therefore CO always divides AB, so that the difference of the squares on its segments is equal to the difference of the squares on the radii.

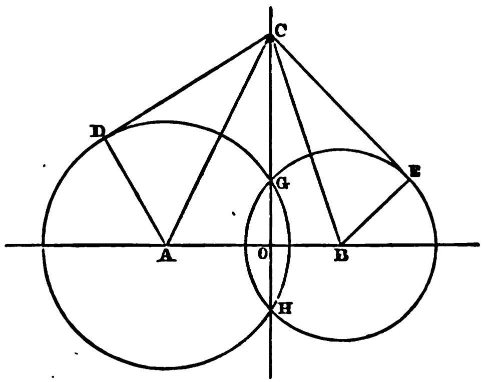


Fig. 2.

Therefore CO is a fixed straight line, and is therefore the required locus.

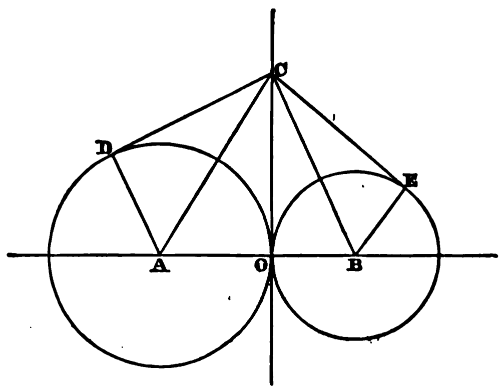


Fig. 3.

If the circles *intersect* as in (Fig. 2), it is evident that the chord of intersection is the required locus, for if C be a point on the chord of intersection, $CD^2 = CH . CG = CE^2$. If the circles touch at O as in (Fig. 3), the common tangent CO is the required locus, for $CD = CO = CE$.

In all cases $AO^2 \sim BO^2 = AD^2 \sim BE^2$.

If one circle be entirely within the other, the required locus will cut the line joining the centres *externally outside both circles, and on the side of the centre of the less circle.* This at once appears from the fact that the line joining the centres of the two circles is *less than the difference of the radii.*

It is also clear from the three figures, that if from any point on CO produced indefinitely both ways we draw two straight lines, one cutting each circle, the rectangle under the segments into which the chord of one circle is divided at the assumed point on CO is equal to the rectangle under the segments of the chord of the other circle, made by the same point.

DEF. The locus of the intersection of equal tangents to two circles is called the *radical axis* of the two circles, or, which is the same thing, the locus of a point such that if straight lines be drawn through it, cutting both circles, the rectangles under the segments of the chords made by the point are equal, is called the *radical axis* of the two circles.

200. *The radical axes of each pair of a system of three circles meet in a point.*

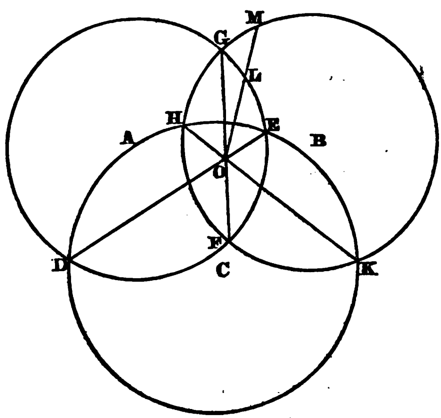


Fig. 1.

First, let the three given circles all intersect one another, then shall the three chords of intersection (that is, the three radical axes) pass through the same point *within all the circles.*

Let A, B, C be the centres of the three circles, and let the chords DE and HK intersect in O. Join FO. FO, if produced, will pass through G, for, if possible, let it not pass through G, but meet the two circles (A) and (B) in L and M.

Therefore (III. 35) $DO \cdot OE = HO \cdot OK$, because DE, HK are chords of the circle (C). But $DO \cdot OE = FO \cdot OL$, because DE, FL are chords of (A), and $HO \cdot OK = FO \cdot OM$, because HK, FM are chords of (B).

Therefore $FO \cdot OL = FO \cdot OM$, and therefore $OL = OM$, which is impossible; therefore FO produced must pass through G, that is, the three chords of intersection meet in the same point O within all the circles.

Next, let the circles not all intersect one another.

Let DO, the radical axis of (A) and (B), and EO, the radical axis of (A) and (C), meet in O, and from O draw the three tangents OF, OG, OH to the three circles. Because O is a point in the radical axis of (A) and (B) the tangents OF and OG are

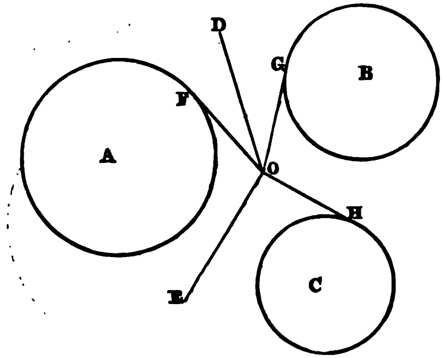


Fig. 2.

equal, and because O is a point in the radical axis of (A) and (C) OF and OH are equal; therefore the tangents OG and OH to the two circles (B) and (C) are equal. Therefore O is a point in the radical axis of (B) and (C). Therefore the three radical axes of a system of three circles pass through the same point. Q. E. D.

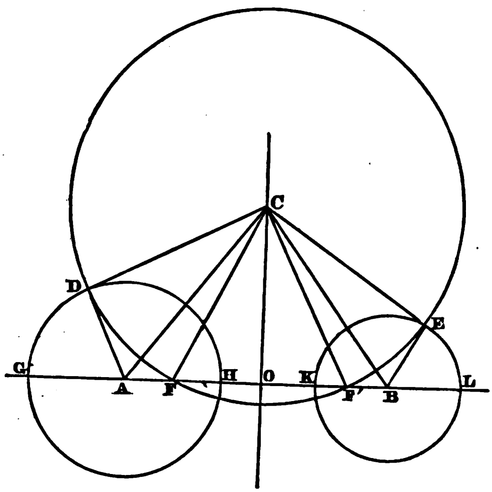
DEF. The point in which the three radical axes of a system of three circles meet is called the *radical centre* of the system.

COR. Hence, when the radical centre is without the circles, we can describe a circle to cut the three orthogonally, for (Fig. 2) the circle with centre O and radius OF cuts all three circles orthogonally (110).

It is also obvious that a circle described with any point on the radical axis of two circles as centre and one of the equal tangents from this point to the circles as radius cuts both circles orthogonally.

201. *If two circles do not meet one another, any system of circles cutting them orthogonally always passes through two fixed points on the line joining the centres of the two given circles.*

Let A and B be the centres of two circles which do not meet one another, and which are consequently not met by their radical



axis. From any point C in their radical axis CO draw the tangents CD and CE, and from the centre C at the distance CD or CE describe the circle $DFF'E$ cutting AB in F and F'. Then shall F and F' be fixed points. For, in the triangle ACF the difference of the squares of the sides AC, CF is equal to the difference of the squares on the segments of the base AO, OF made by the perpendicular CO.

But CF is equal to CD, therefore the difference of the squares on AO and OF is equal to the square on the radius AD, and therefore OF is of a known length, and since the point O is fixed, therefore F is a fixed point, but OF' equals OF. Therefore F' is also a fixed point.

In fact OF is equal to the tangent from O to the circle (A), since the squares on OF and AD are together equal to the square on AO.

Hence any circle with its centre on the radical axis of (A) and (B), and its radius equal to the tangent drawn from its centre to (A) or (B), always cuts (A) and (B) orthogonally, and passes through two fixed points F and F' on the line joining the centres A and B.

N.B. Further, (α) if from any point on AB not situated between F and F' we draw a tangent to the circle DFE, and from the point as centre with this tangent as radius describe a circle, it will cut DEF orthogonally (110). Now since $F'A \,.\, AF = AD^2$, it is clear that as the centre approaches F from A, the radius AD diminishes, and when it reaches F it vanishes, and no circle with its centre on FF' can cut DFE orthogonally. Therefore the system of circles with centres on AB, which cut the system with centres on CO orthogonally, have their centres on AB produced indefinitely both ways, but no centre lies on FF'. Also the radii increase as A moves off towards the left of F, or B towards the right of F'. At F and F' the radii vanish. Hence these two points have been called the *limiting points* of the system with centres on AB.

(β) Since FF' is the radical axis of the system with centres on CO, we see that *if two systems of circles cut one another orthogonally the centres of each system are on the radical axis of the other.* One system in which the circles do not meet has *limiting points* (F and F') through which the other system always passes, but the system of circles which intersect has no limiting points: in other words, a system of circles cutting orthogonally another system which passes through two fixed points never intersects.

This is also apparent from (199, Fig. 2).

Since (176) the arithmetic mean between two unequal lines is greater than the geometric mean, therefore CO the arithmetic mean between CH and CG is greater than CD the geometric mean between CH and CG, and therefore the circle with centre C and radius CD cannot meet AB. Therefore the system of circles cutting orthogonally the system passing through G and H has no *real* limiting points.

If, however, the circles (A) and (B) touch at O as in (199, Fig. 3), then the system cutting orthogonally the system of which the circles (A) and (B) are two always passes through O, and touches AB at O. Hence both systems pass through O, so that the two limiting points of each system coincide at O.

Again, $F'A \,.\, AF = AD^2 = AH^2$.

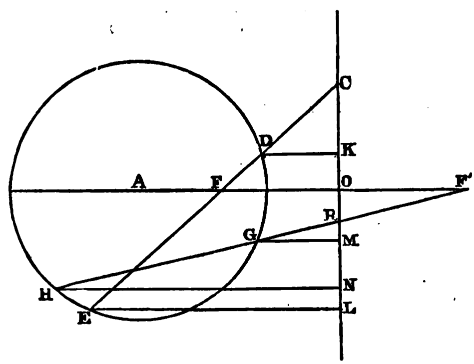
Therefore (163) F, F' are harmonic conjugates to G, H.

Similarly F, F' are also harmonic conjugates to K, L.

(γ) Therefore, *the limiting points are harmonic conjugates to every two points in which a circle of their system cuts the straight line passing through them.*

202. *If through either of the limiting points of a system of circles having a common radical axis, a straight line be drawn intersecting any circle of the system, and if perpendiculars be drawn from the points of intersection to the radical axis, the rectangle under the perpendiculars is constant.* (See 150.)

Let A be the centre of any circle of the system, CO the radical



axis, and F, F' the two limiting points. Through F, F' draw CE, HF', and draw DK, EL, GM, HN perpendicular to CO.

Because F and F' are limiting points, CF is equal to the tangent from C, and RF' (which is equal to RF) is equal to the tangent from R.

Therefore, $CD : CF :: CF : CE,$

but $CD : CF :: DK : FO,$ since DK and FO are parallel, and $CF : CE :: FO : EL.$

Therefore, $DK : FO :: FO : EL,$

and therefore $DK . EL = FO^2$, which is constant.

Also, $GR : RF' :: RF' : RH,$

but $GR : RF' :: GM : OF',$

and $RF' : RH :: OF' : HN.$

Therefore $GM : F'O :: F'O : HN,$

and therefore $GM . HN = F'O^2 = FO^2$, which is constant.

Therefore $DK . EL = GM . HN = FO^2.$ Q. E. D.

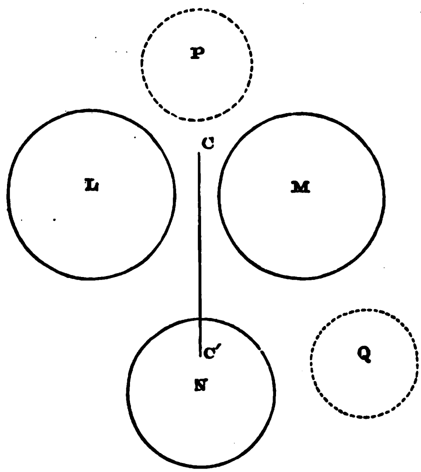
203. *Given three circles. Describe any circle, and form a triangle* ABC *with the three radical axes of this circle and each of the given circles. Describe any other circle, and similarly form a triangle* A′B′C′. *The straight lines joining corresponding vertices of these two triangles will meet in a point, and the points of intersection of the corresponding sides will lie on the same straight line.*

Let L, M, N be the three given circles, P and Q any two circles.

Let the radical axis of P and L form the side BC of the triangle AB, that of P and M the side CA, and that of P and N the side AB.

Similarly let the radical axis of Q and L form the side $B'C'$ of the triangle $A'B'C'$, that of Q and M the side $C'A'$, and that of Q and N the side $A'B'$.

Since the radical axes of the three circles P, L, M meet in their radical centre (200), therefore BC and CA meet on the radical axis of L and M, that is, C lies on the radical axis of L and M. In like manner it can be proved that C' lies on the radical axis of L and M. Therefore the straight line CC' is the radical axis of L and M, and therefore CC' passes through the radical centre of the three given circles L, M, N (200). In like manner it can be proved that BB' and AA' pass each through the radical centre of the three given circles. Therefore the triangles ABC,



$A'B'C'$ are co-polar, and therefore (183) they are also co-axial. Q. E. D.

204. *Describe a circle, such that the radical axes of it and each of three given circles shall pass respectively through three given points.*

Let (Fig. 203) L, M, N be the given circles, P the required circle.

Then, whatever circle P may be, we have proved in (203) that the three radical axes of P and L, P and M, P and N will form a triangle ABC, having its vertices on the radical axes of the three given circles. Now these radical axes meet in a point (the radical centre of the three given circles). Hence the problem is reduced to (184, Cor.). Having constructed the triangle ABC by this Cor., we have still to find the centre and radius of the required circle.

Since the straight line joining the centres of two circles is perpendicular to their radical axis, therefore the perpendicular from the centre of L on BC, the radical axis of P and L, must pass through the centre of P.

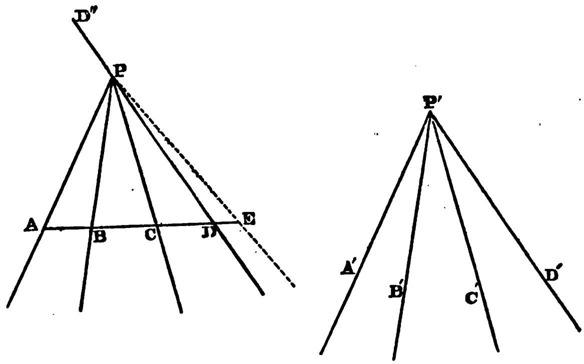
Similarly, the perpendicular from M on CA must also pass through P.

These perpendiculars therefore intersect at the centre of the required circle.

Now LP is known, and the point (D suppose) where BC meets LP. But the difference of the squares of LD and DP is equal to the difference of the squares of the radii of L and P. Therefore the radius of P is determined.

205. *If two pencils have the same anharmonic ratio, and if two angles of the one be respectively equal to two angles of the other, the two remaining angles shall also be equal or one of them the supplement of the whole angle of the other pencil.*

Let $P.ABCD$ and $P'.A'B'C'D'$ be two pencils with the same anharmonic ratio, and having the angles APB and $A'P'B'$, BPC and $B'P'C'$ equal, then shall the angles CPD and $C'P'D'$ be also equal, or one of them the supplement of the whole angle of the other pencil.



For if P' be placed upon P so that $P'A'$ shall fall upon PA, $P'B'$ upon PB, and $P'C'$ upon PC, then $P'D'$ must fall upon PD or its production PD''; for if possible, let $P'D'$ take the position PE, and draw the transversal AE.

Since $\{P.ABCD\} = \{P.ABCE\}$ (hyp.),

therefore $AD.BC : AB.CD :: AE.BC : AB.CE$,

or, alternately,

$$AD.BC : AE.BC :: AB.CD : AB.CE,$$

therefore (VI. 1) $AD : AE :: CD : CE$,

but AD is less than AE, therefore CD is less than CE, which is absurd.

Therefore the point E must lie upon PD or its production through P. Q. E. D.

206. *Given two pairs of points in a straight line; find a point in the line such that the rectangle under its distances from one pair of the points shall be equal to the rectangle under its distances from the other pair.*

Let A, A' and B, B' be the given points. Through AA' describe any circle PAA', and through B, B' and any point P on

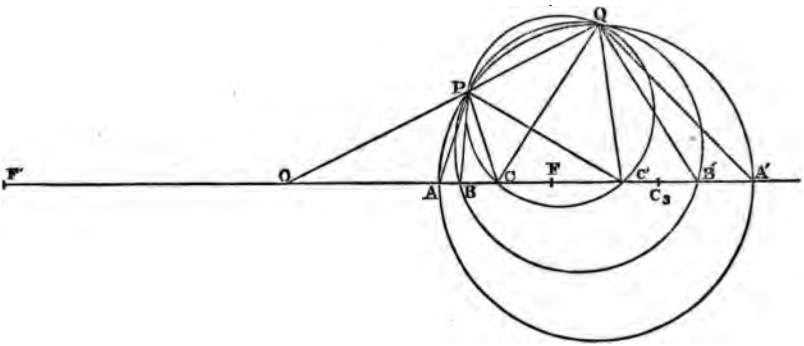


Fig. 1.

this circle, describe the circle PBB', meeting the first circle again in Q. Join PQ, and produce it (if necessary) to meet AA' in O. O is the required point.

For (III. 35, 36) $OA \cdot OA' = OP \cdot OQ = OB \cdot OB'$.

Cor. 1. Hence, given a fifth point C, we can find a sixth C', such that $OC \cdot OC' = OA \cdot OA' = OB \cdot OB'$. For the circle described through P, Q, C will obviously determine the required sixth point C'.

Cor. 2. Again, it is evident that when two pairs of points A, A' and B, B' are given, we can determine an indefinite number of other pairs of points, C and C', D and D', &c. such that

$$OA \cdot OA' = OB \cdot OB' = OC \cdot OC' = OD \cdot OD' = \&c.$$

For we have only to describe circles through the points P, Q, intersecting the indefinite straight line AA'. These points of intersection will be the required points.

N.B. When the point Q coincides with P, that is, when the second circle PBB' touches the first circle PAA' at P (which is evidently only possible in figure 1), we must draw PO a tangent to both circles at P. The point O in which this tangent meets AA' will be the required point, but it will not often be necessary to point out so obvious modifications in the construction.

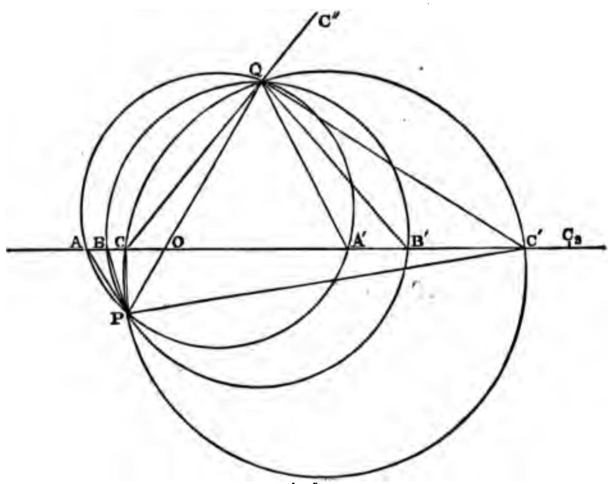


Fig. 2.

It is hardly necessary to remark that only one such point as O exists, for which $OA \,.\, OA' = OB \,.\, OB'$, for O is clearly the point where the radical axis of the two circles PAA', PBB' meets AA'.

DEF. 1. Three pairs of points A, A'; B, B'; C, C', so taken on a straight line that

$$OA \,.\, OA' = OB \,.\, OB' = OC \,.\, OC',$$

are said to form a system in *involution*. Some writers extend this term to any number of pairs of points taken as above, and I may occasionally adopt this extension of the term.

The points A, A'; B, B'; or C, C' are said to be *conjugate* to one another.

In both Figures 1 and 2, the conjugate points A and A' move in opposite directions with respect to O. Thus, when A moves to B, A' moves to B'.

In Fig. 2, one set of points lies on one side of O, and their conjugates on the other; but in Fig. 1, all the points lie on the same side of O. It is clear that in Fig. 2, a point cannot coincide with its conjugate, but in Fig. 1 a point can coincide with its conjugate. Let F be such a point, so that

$$OF^2 = OA \, . \, OA' = OB \, . \, OB' = \&c. \quad \text{Take } OF' = OF.$$

Since OF^2 or $OF'^2 = OA \, . \, OA' = OP \, . \, OQ$, therefore the circle through P, Q and F or F' will touch AA' at F or F'.

DEF. 2. The points F, F' are called the *foci* or *double points* of the system, and O is called the *centre*.

We may further remark (α) *that any two conjugate points of the system, together with the two foci, form an harmonic range.*

For $OA \, . \, OA' = OF^2$ and $OF' = OF$; therefore (163) F', A, F, A' form an harmonic range. The converse follows immediately also from (163), viz. (β). *If there be a system of pairs of points in a straight line, such that each pair forms, with two given points, an harmonic range, the assemblage of the pairs of points will form a system in involution, of which the two given points are the foci.*

In Fig. 2, the *foci* are usually said to be *imaginary*, but the discussion of imaginary points and lines, and of infinite magnitudes, is necessarily excluded from so very elementary a work as the present.

207. *In a system of points in involution, the anharmonic ratio of any four points is equal to that of their four conjugates.*

Figs. to (206). Let A, A'; B, B'; C, C', be a system of points in involution determined as in (206), and let O be the centre.

Join P to A, B, C, C' and Q to their conjugates A', B', C', C, CQ being produced to C'' in Fig. 2.

In Fig. 1, since $APQA'$ is a quadrilateral in a circle, therefore (III. 22) the angles OPA and $QA'A$ are equal. Similarly OPB and $QB'B$ are equal.

Therefore the difference of OPA and OPB is equal to the difference of $QA'A$ and $QB'B$, that is, the angles APB and $A'QB'$ are equal. In like manner it can be proved, that the angles BPC and $B'QC'$ are equal. And the angles CPC' and CQC' are equal, since they are in the same segment.

Therefore the pencils $P.ABCC'$ and $Q.A'B'C'C$ admit of superposition, since their angles are equal, each to each, therefore

$$\{P.ABCC'\} = \{Q.A'B'C'C\},$$

and therefore $$[ABCC'] = [A'B'C'C].$$

Again, in Fig. 2, the angles QPA and $QA'A$ in the same segment are equal, and QPB and $QB'B$ are equal. Therefore the difference of QPA and QPB is equal to the difference of $QA'A$ and $QB'B$, that is, the angles APB and $A'QB'$ are equal. Similarly, the angles BPC and $B'QC'$ are equal; and because the quadrilateral $PCQC'$ is in a circle, the external angle $C'QC''$ is equal to the internal and opposite angle CPC'.

Therefore $$\{P.ABCC'\} = \{Q.A'B'C'C''\}.$$

But the pencils $Q.A'B'C'C''$ and $Q.A'B'C'C$ are identical.

Therefore $$\{P.ABCC'\} = \{Q.A'B'C'C\},$$

and therefore $$[ABCC'] = [A'B'C'C].$$

In like manner it may be proved, that the anharmonic ratio of any other four points is equal to that of their four conjugates. Q. E. D.

N.B. The learner should bear in mind that if two anharmonic pencils have two angles of the one respectively equal to two angles of the other, but the *remaining angle* of the one equal to the supplement of the *whole angle* of the other, the anharmonic ratios of the pencils are equal. For we have just seen that the two pencils (Fig. 2), $P.ABCC'$ and $Q.A'B'C'C$ have the angles APB, BPC respectively equal to $A'QB'$, $B'QC'$, but the angle CPC' the supplement of the whole angle $C'QC$, and that the anharmonic ratios are equal.

208. *Given three pairs of points in a straight line, such that the anharmonic ratio of four of the points is equal to that* *of their four conjugates; prove that the points form a system in involution.* (See 209, Cor. 1.)

Figs. to (206). Let A, A'; B, B'; C, C', be the given points, and let $[ABCC'] = [A'B'C'C]$. Then shall the system be in involution.

Through two pairs of conjugate points A, A' and B, B' describe two circles intersecting in P and Q, and let PQ meet AA' in O. The circle described through P, Q and C must pass through C', for, if possible, let it meet AA' in C_3. Therefore

$$OA . OA' = OB . OB' = OC . OC_3,$$

and therefore A, A'; B, B'; C, C_3 are in involution. Therefore (207)

$$[ABCC_3] = [A'B'C_3C].$$

Therefore $AB . CC_3 : AC_3 . BC :: A'B' . C_3C : A'C . B'C_3$,

or alternately, $AB . CC_3 : A'B' . C_3C :: AC_3 . BC : A'C . B'C_3$,

therefore $AB : A'B' :: AC_3 . BC : A'C . B'C_3$.

But $[ABCC'] = [A'B'C'C]$, by hypothesis;

therefore $AB . CC' : AC' . BC :: A'B' . C'C : A'C . B'C'$,

or alternately $AB . CC' : A'B' . C'C :: AC' . BC : A'C . B'C'$;

therefore $AB : A'B' :: AC' . BC : A'C . B'C'$.

Therefore $AC_3 . BC : A'C . B'C_3 :: AC' . BC : A'C . B'C'$,

or alternately, $AC_3 . BC : AC' . BC :: A'C . B'C_3 : A'C . B'C'$,

therefore $AC_3 : AC' :: B'C_3 : B'C'$.................. ($a$).

Now in Fig. 1, AC_3 is greater than AC', therefore $B'C_3$ is greater than $B'C'$, which is impossible.

Again, for Fig. 2, we have from (a),

$$AC_3 - AC' : AC' :: B'C_3 - B'C' : B'C',$$

or

$$C'C_3 : AC' :: C'C_3 : B'C';$$

therefore $AC' = B'C'$, which is impossible.

The proof would be exactly similar, if C_3 had been assumed on the left of C' instead of on the right. Therefore the circle described through P, Q, C passes through C'. Consequently the six points form a system in involution, and therefore *when three pairs of points are given* (in either of the orders assigned in Figs. 1 and 2, 206) *in a straight line such that the anharmonic ratio of four of* *them is equal to that of their four conjugates, then the anharmonic ratio of any other set of four points is equal to that of their four conjugates.* Some writers give this as the definition of involution.

209. *Given four points* A, B, B′, A′ *in a straight line; find the locus of a point at which* AB *and* B′A′ *shall subtend equal angles.*

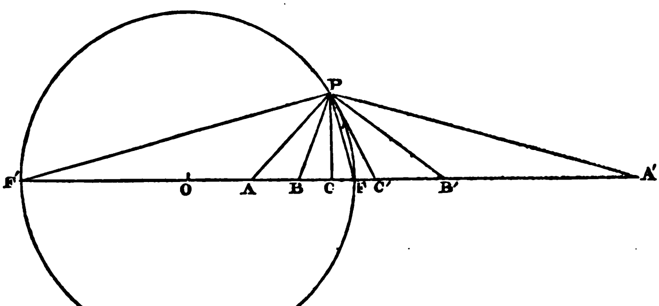
Find (206) a point O on the straight line AA', such that $OA \,.\, OA' = OB \,.\, OB'$, and take F and F' such that

$$OF^2 = OF'^2 = OA \,.\, OA'.$$

{In other words, find the centre O and the foci F, F' of the system of points in involution determined by the given pairs A, A' and B, B'}.

On $F'F$ as diameter describe a circle. This circle is the required locus. For join any point P on its circumference to the four given points. Since F', A, F, A' form a harmonic range (206, *a*), and the angle FPF' is right, therefore (172) PF bisects the angle APA'; similarly, PF bisects the angle BPB'.

Therefore the angles APB and $A'PB'$ are equal, and therefore the circle on the diameter $F'F$ is the required locus. Q. E. F.



N.B. Since, as we have proved, PF bisects the angle subtended at P by every pair of conjugate points of a system in involution, being given any fifth point C we can very simply find its conjugate, for it is only necessary to draw PC' making the angle FPC' equal to FPC; or again, we can determine as many pairs of conjugate points as we please by fixing upon one point arbitrarily as C, and then determining C' as above.

COR. 1. In a system of three pairs of points A, A'; B, B'; C, C', arranged as in Fig. 1 (206), the present construction furnishes a very simple proof of (208), viz. *If the anharmonic ratio of four of the six points be equal to that of their four conjugates, the anharmonic ratio of any other set of four points is equal to that of their conjugates.* For the angles APB and $A'PB'$ are equal (constr.), and $\{P.ABCC'\}=\{P.A'B'C'C\}$, (hyp.). Also, the angle CPC' is common; therefore (205) the remaining angles BPC, $B'PC'$ of the two pencils are equal, since they are obviously not supplemental.

Consequently the angle subtended at P by any two points is equal to that subtended at P by their conjugates. Q. E. D.

COR. 2. *If a system of circles have a common radical axis, and any circle be described cutting them orthogonally, and if any transversal be drawn through its centre* O, *meeting the circles in the points* A, A′; B, B′; C, C′, *&c., the points* A, A′, *&c. will form a system in involution, and the points* F, F′ *where the transversal meets the circle* (O) *will be the foci of the system, and* O *will be its centre. Also, any two points, as* A, B, *will subtend at any point on the circle* (O) *the same angle as their conjugates* A′, B′.

For let O be the centre of the circle cutting the given circles orthogonally so that O is on the radical axis, and let the transversal through O meet the given circles in A, A'; B, B'; C, C', &c. and the other circle in F, F'. Then, since O is on the radical axis $OF^2=OA.OA'=OB.OB'=OC.OC'=$ &c. Q. E. D.

DEF. A system of circles which have a common radical axis is called a *co-axal* system.

COR. 3. *If any transversal cut a co-axal system in* A, A′; B, B′; C, C′, *&c., and the radical axis in* O, *the points* A, A′, *&c. form a system in involution of which* O *is the centre.* For $OA.OA'=OB.OB'=OC.OC'=$&c., since O is a point on the radical axis. If the conjugates A and A', B and B', &c. lie on the same side of O, the system in involution has real foci, but if A and A', &c. lie on different sides of O, the foci have no geometrical existence.

210. *Prove that if a system of six points* A, A′; B, B′; C, C′, *be in involution,*

$$AB'.BC'.CA'=A'B.B'C.C'A.$$

By (207) $[AB'BC'] = [A'BB'C]$.

Therefore $AB'.BC' : AC'.B'B :: A'B.B'C : A'C.BB'$,

or, alternately,

$$AB'.BC' : A'B.B'C :: AC'.B'B : A'C.BB'$$
$$:: AC' \quad : A'C.$$

Therefore $AB'.BC'.CA' = A'B.B'C.C'A$. Q. E. D.

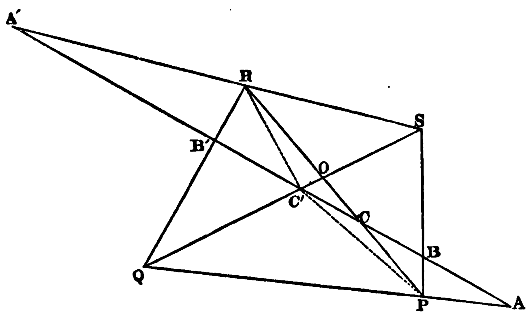
211. *If a straight line intersect three given circles in a system of points in involution, it will pass through a fixed point (the radical centre of the three circles).*

Let L, M, N be the given circles, and let the transversal meet them in the points A, A'; B, B'; C, C' respectively, and let O be the centre of the system of points in involution.

Therefore $OA.OA' = OB.OB' = OC.OC'$ (hyp.), but since $OA.OA' = OB.OB'$, therefore O is a point on the radical axis of the circles L and M. Similarly O lies on the radical axes of M and N and of N and L; therefore O is the radical centre of the three given circles. Q. E. D.

212. *A straight line meeting the sides and diagonals of any quadrilateral is divided in six points in involution.*

Let $PQRS$ be the given quadrilateral, and let any transversal



meet its sides and diagonals in A, A', B, B', C, C'. Then shall these six points be in involution. Join C', the point of section on either diagonal, with the vertices of the opposite angles, P and R.

$\{P . ABCC'\} = \{P . QSOC'\} = \{R . QSOC'\} = \{R . B'A'CC'\}$.

But $\{R . B'A'CC'\} = \{R . A'B'C'C\}$, for these ratios are $B'C' . A'C : B'A' . CC'$ and $A'C . B'C' : A'B' . C'C$, which are identical.

Therefore $\{P . ABCC'\} = \{P . A'B'C'C\}$,

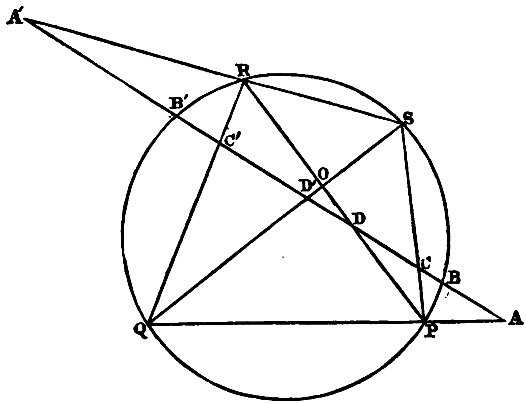
and therefore $[ABCC'] = [A'B'C'C]$,

hence A and A', B and B', C and C' are conjugate pairs of points in involution (208).

N.B. If the transversal pass through O, the intersection of the diagonals, O is a focus of the system, since the two conjugate points C and C' then coincide at O.

213. *Any straight line meeting a circle and the sides of any inscribed quadrilateral is cut in involution.*

Let the transversal meet the circle in B, B' and the opposite



sides in A, A' and C, C'. Then shall these three pairs of points be in involution.

For

$$\{P . ABCB'\} = \{P . QBSB'\} = \{R . QBSB'\} = \{R . C'BA'B'\}$$

$$= \{R . A'B'C'B\}, \text{ since } \{R . C'BA'B'\} = C'B' . BA' : C'B . A'B',$$

$$\text{and } \{R . A'B'C'B\} = A'B . B'C' : A'B' . C'B,$$

which two ratios are identical, or if the rays RC', RB be produced the pencil will become $R . A'B'C'B$.

Therefore $[ABCB'] = [A'B'C'B]$, and therefore (208) A and A', B and B', C and C' are three conjugate pairs of points in involution. Q. E. D.

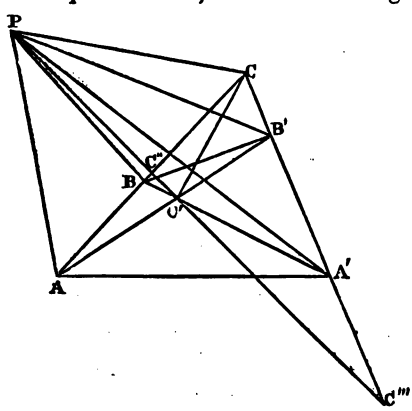
N.B. If the transversal meet the diagonals of the quadrilateral in D and D', then (212) D and D' are in involution with A, A' and C, C', but B, B' are also in involution with A, A' and C, C'. Therefore any three of the four pairs of points A, A'; B, B'; C, C'; D, D', are in involution.

DEF. If six points in involution be joined to any seventh point outside the points, the pencil thus formed is called a *pencil in involution.*

Since the anharmonic ratio of any four of six points in involution is equal to that of their four conjugates, and the anharmonic ratio of a pencil of four rays is the same as that of the range formed upon any transversal meeting the pencil, it is plain that pencils in involution may be treated similarly to ranges in involution, and that they possess kindred properties. If therefore three pairs of points be in involution, the pencil joining them to any point will be in involution, and conversely, any transversal meeting a pencil in involution is cut in involution.

214. *The straight lines drawn from any point to the* six *angular points of a* complete *quadrilateral form a system in involution.*

Let $CBC'B'$ be a quadrilateral, AA' its third diagonal.



Join any point P with the extremities of the three diagonals. Then the pencil $P.ABCC'B'A'$ will be in involution.

For, let PC' meet AC in C'' and CA' in C'''.

Then, evidently,

$$\{P.ABCC'\} = \{P.ABCC''\} = \{C'.ABCC''\} = \{C'.B'A'CC'''\}$$
$$= \{P.B'A'CC'''\} = \{P.B'A'CC'\}.$$

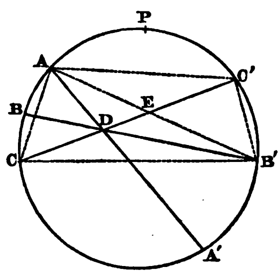
But $\{P.B'A'CC'\} = \{P.A'B'C'C\}$.

Therefore $\{P.ABCC'\} = \{P.A'B'C'C\}$,

and therefore (208) the pencil $P.ABCC'B'A'$ is in involution. Q. E. D.

215. *If three chords of a circle meet in a point within or without the circle, the six straight lines joining any point on the circumference to the extremities of the chords form a pencil in involution.*

Let AA', BB', CC' be the three chords intersecting in D. Join



AC, AC', $B'C$, $B'C'$, and AB' meeting CC' in E. Let P be any point on the circumference.

Now $\{A.CA'B'C'\} = \{A.CDEC'\} = \{B'.CDEC'\}$
$$= \{B'.CBAC'\} = \{B'.C'ABC\}.$$

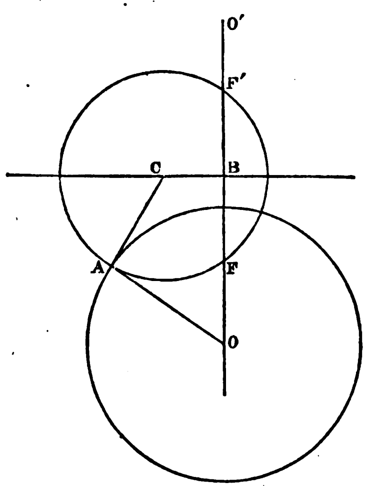
Therefore (189), $\{P.CA'B'C'\} = \{P.C'ABC\}$, and therefore (208) $P.ABCA'B'C'$ is a pencil in involution. Q. E. D.

216. *If a system of circles be described cutting a given circle orthogonally, and having their centres in a given* *straight line, the radical axis of the system will be the perpendicular from the centre of the given circle on the given line.*

Let O be the centre of the given circle, and CB the given line.

Draw OBO' perpendicular to CB, and from C draw the tangent CA to the given circle. Describe the circle AFF' with any point C on CB as centre and radius CA. This circle cuts the given circle orthogonally (110), and OA is a tangent to it from O.

Similarly, the tangents from O to all the circles of the system are each equal to OA. Therefore O is a point on their radical

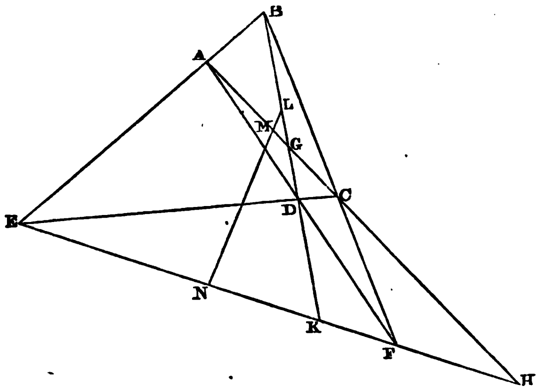


axis, but the radical axis of two circles is perpendicular to the straight line joining their centres; therefore the perpendicular OB is the radical axis of the system. Q. E. D.

N.B. It is easy to see if a system of circles be described with their centres in OB, and cutting orthogonally the system of which AFF' is one, that CB will be their radical axis. For, take BO' equal to BO, and with centre O' and radius equal to OA describe a circle. This circle will evidently cut AFF' orthogonally, and CB will be the radical axis of the circles (O) and (O'). Also, when CB does not meet the circle (O), F and F' will be the limiting points of the system with centres in OB, but when CB meets the circle (O), the points of intersection will be the limiting points of the system with centres in CB. Hence, this proposition is only a different mode of stating (201) and some of the deductions from it.

217. *If on the three diagonals of a complete quadrilateral, as diameters, circles be described, they shall have the same radical axis, and cut orthogonally the circle circumscribing the triangle formed by the three diagonals.*

Let $ABCD$ be the quadrilateral, EF its third diagonal, L, M,



N the middle points of its three diagonals. Therefore (22) LMN is a straight line. Let G, H, K be the points of intersection of the three diagonals taken two at a time.

Because (169) AH is cut harmonically in G and C, and AC is bisected in M, therefore (163) $MG \cdot MH = MC^2$, and therefore the circle on AC as diameter will cut orthogonally the circle described about the triangle GHK (110).

Similarly, the circles on BD and EF as diameters will cut the same circle orthogonally. Therefore (216) the radical axis of the three circles on the diameters AC, BD, EF will be the perpendicular on the straight line LN from the centre of the circle about GHK. Q. E. D.

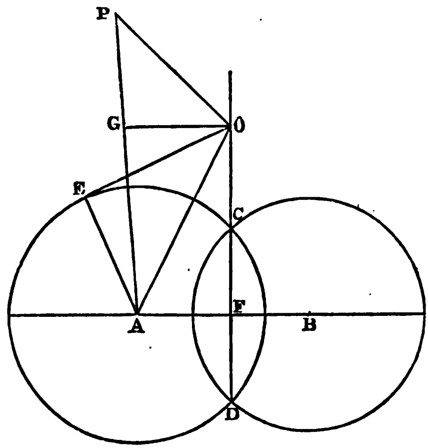
218. *If on the three diagonals of any quadrilateral, as diameters, circles be described, any transversal meeting them is cut in six points in involution.*

We have proved in (217) that the three circles form a co-axal system, and therefore (209, Cor. 3) the six points in which any transversal is met by the three circles form a system in involution.

219. *Describe a circle which shall pass through a given point, and cut orthogonally two given circles.*

First, let the radical axis of the two given circles be without them, then (201) every circle cutting them orthogonally passes through the two limiting points of the system with the same radical axis. Therefore the circle described through the given point and the two limiting points will be the required one.

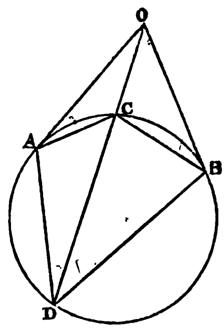
Next, let the circles touch, then (201, N.B.) every circle cutting the given circles orthogonally touches the line joining their centres at their point of contact. Therefore the circle described through the given point and touching this line at the point of contact of the two given circles is the required one. (See 132.)



Lastly, let the given circles intersect in C and D, and let A and B be their centres, and P the given point. Join AP, and cut it at G, so that the difference of the squares on AG and GP equals the square on radius of circle (A). Through G draw GO perpendicular to AP, and meeting the radical axis CD in O. The circle described from the centre O with the tangent OE as radius will be the required one. For it will cut the given circles orthogonally (200, Cor.); also the difference of the squares on AG and GP is equal to the difference of the squares on AO and OP.

Therefore the squares on AE and EO are together equal to the squares on OP and AE, and therefore OP and OE are equal. Therefore the circle with centre O and radius OE passes through P, and cuts the given circles orthogonally. This last method is applicable to the other two cases.

220. *If any secant be drawn through the intersection of two tangents to a circle, and if the points of intersection be joined to the points of contact of the tangents, the rectangles under the pairs of opposite sides of the quadrilateral formed by the joining lines are equal.*



Let OA, OB be the tangents, and OCD the secant. Then shall $DA \,.\, CB = AC \,.\, BD$.

Because the angle OAC is equal to the angle ADC in the alternate segment (III. 32), therefore the triangles ADO and OAC are equiangular, and therefore (VI. 4)

$$AD : DO :: AC : AO;$$

therefore alternately, $AD : AC :: DO : AO$.

Similarly, the triangles ODB and BCO are equiangular, and therefore $DB : BC :: DO : OB$ or OA.

Therefore $AD : AC :: DB : BC$.

Therefore (VI. 16) $AD \,.\, BC = AC \,.\, DB$. Q.E.D.

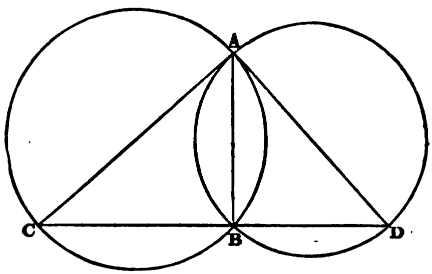
221. *If two tangents and a secant be drawn from any point outside a circle, the two points of contact and the points of section will subtend an harmonic pencil at any point on the circle.*

Fig. to (220). For (192) $[ACBD] = DA \cdot BC : AC \cdot BD$, which is a ratio of equality by (220). Q.E.D.

COR. Hence, *given two points on the circumference of a circle; draw a transversal passing through a given point and cutting the circle in two points, which shall be harmonic conjugates to the given points.* Let the tangents at the given points A and B meet in O. Join the given point to O. This is manifestly the required transversal.

222. *If two circles intersect, and if from either point of intersection two diameters be drawn, the straight line joining their extremities will pass through the other point of intersection, and be at right angles to the chord of intersection.*

For, let the circles intersect in A and B, and let AC, AD be



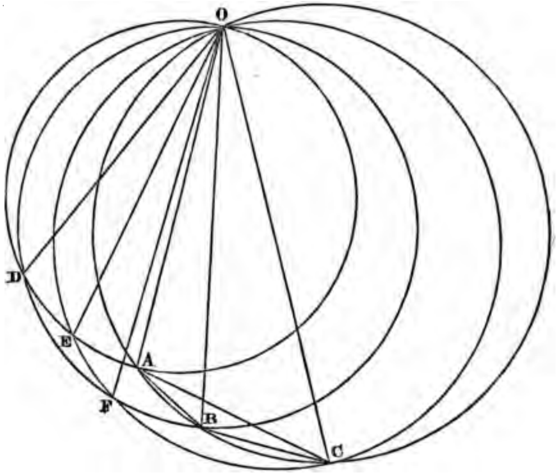
two diameters. Join CB, DB and AB. The angles ABC and ABD in semicircles are right, and therefore CB and BD must coincide, and form one straight line, and CD is perpendicular to AB. Q.E.D.

223. "*If through any point* O, *on the circumference of a circle, any three chords be drawn, and on each, as diameter, a circle be described, these three circles (which, of course, all pass through* O) *will intersect in three other points, which lie in one right line.*"

Let OA, OB, OC be the three chords, and let the circles on them, as diameters, intersect in D, E, F.

By (222) AB produced will pass through D and be at right angles to OD, BC will pass through F and be at right angles to OF, and CA will pass through E and be at right angles to OE.

Therefore D, E, F are the points in which perpendiculars from any point O on the circumference of the given circle meet



the sides of the inscribed triangle ABC, and therefore (118) D, E and F lie in one straight line. Q. E. D.

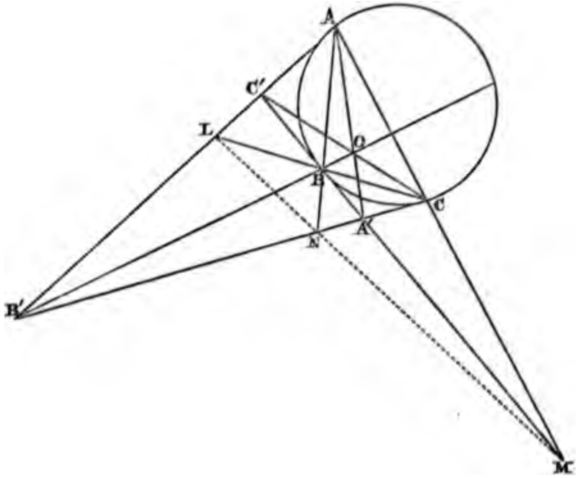
224. *The tangents at the angular points of any triangle inscribed in a circle intersect the opposite sides in three points which are situated in a straight line.*

Let ABC be a triangle inscribed in a circle, and $A'B'C'$ the triangle formed by drawing tangents at the angular points of ABC, and let these tangents meet the opposite sides of ABC in L, M, N. Then shall L, M, N lie in a straight line.

Because the triangle $A'B'C'$ is described about a circle, the straight lines AA', BB', CC' joining its angles to the points of contact of the opposite sides pass through the same point O (168, Cor.).

Now, in the triangle $A'B'C'$, the straight lines drawn from O to the angles A', B', C' intersect the opposite sides in A, B, C respectively, therefore (169) the intersections of BC and $B'C'$, CA and $C'A'$, AB and $A'B'$, viz. the points L, M, N lie in a straight line.

Or thus, since the triangles ABC, $A'B'C'$ are co-polar, they are also co-axial, that is, AB and $A'B'$, BC and $B'C'$, CA and $C'A'$ intersect on the same straight line (183). Q. E. D.



DEFS. (1) If the straight line joining the centres of two circles be cut internally and externally in the ratio of the corresponding radii, the points of section are called respectively *the internal* and *external centres of similitude* of the two circles.

It will be evident, by constructing for the two centres of similitude, that when the circles are external to one another, the centres of similitude are outside both circles, when the circles touch externally the internal centre of similitude is the point of contact, when one circle touches the other internally the point of contact is the external centre of similitude, when the circles intersect, the internal centre of similitude is within both circles, and when one circle is wholly within the other, the centres of similitude are within both circles.

(2) If any transversal be drawn through a centre of similitude, which is without both circles, intersecting the two circles, the *near* points of section to the centre on the two circles are called *corresponding points,* and the *remote*

points of section are also called *corresponding points*, but a *near* point on one circle and a *remote* point on the other are called *non-corresponding points*.

Some writers speak of these points as *points corresponding directly and inversely.*

When the circles intersect, the *corresponding points* with respect to the internal centre of similitude are the *two near* and the *two remote points*, and *each pair of non-corresponding points lies at the same side of the internal centre of similitude.*

When one circle lies wholly within the other, the *corresponding points* are on the *same* side of the *external* centre of similitude, and the *non-corresponding points* on the *same* side of the *internal* centre of similitude.

(3) When two circles touch one another externally, the contact may be called *external contact*, and when one circle touches another internally, the contact may be called *internal contact.*

(4) If one circle touch two others, the *contacts* are said to be of *the same kind*, when they are *both internal* or *both external*, but when one of the contacts is *internal* and the other *external*, the *contacts* are said to be of *different kinds.*

N. B. We have proved in (124, Cor.) that the *direct tangents* to two circles intersect at the *external* centre of similitude, and the *transverse tangents* at the *internal* centre of similitude.

225. *Every transversal drawn through a centre of similitude, and intersecting the two circles, is cut similarly by the circles, and the radii drawn to two corresponding points are parallel.*

Let A and B be the centres of the two given circles, K and O their internal and external centres of similitude respectively.

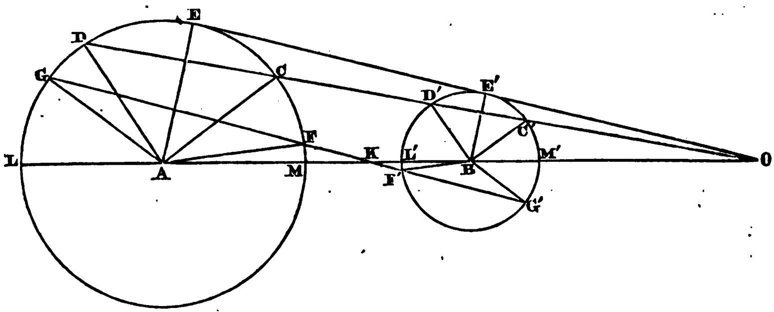
Through K and O draw the transversals GG' and OD and radii to their points of intersection with the circles.

By the definition of the centre of similitude,

$$AO : OB :: AC : BC',$$

or, alternately, $$AO : AC :: BO : BC',$$

but the angle AOC is common to the two triangles AOC, BOC', and the angles ACO and $BC'O$ are both obtuse, therefore (VI. 7) the triangles AOC and BOC' are similar, and therefore AC and BC' are parallel, and $OC : OC' :: AC : BC'$.



Similarly, AD and BD' are parallel, and

$$OD : OD' :: AD : BD'.$$

In like manner it can be proved that AF and BF' are parallel, and AG and BG', and that $KF : KF' :: AF : BF'$ and $KG : KG' :: AG : BG'$.

Therefore, *the distances of any two corresponding points from a centre of similitude are proportional to the corresponding radii.*

N.B. The proof is similar, whatever be the relative positions of the two circles with respect to one another. Since the common tangents of two circles pass through the centres of similitude, it is evident that when one of the circles is within the other the centres of similitude must be within both circles, since the circles have no (real) common tangents. This will also appear from the construction for determining the two centres of similitude. When the circles touch *externally*, the point of contact is the *internal* centre of similitude, and when one circle touches the other *internally*, the point of contact is the *external* centre of similitude. The learner should make the figures for each of these cases.

Further, if the tangent OE be drawn touching the two circles at E and E', the radii AE and BE' are parallel, since the angles at E and E' are right.

Again L and L', M and M' are corresponding points with respect to O, and

$$OL : OL' :: AL : BL'.$$

For $AO : AL :: BO : BL'$,

therefore $AO + AL : AL :: BO + BL' : BL'$,

or, alternately, $OL : OL' :: AL : BL'$.

Similarly $OM : OM' :: AM : BM'$.

Therefore $OL : OL' :: OM : OM'$,

and therefore $OL . OM' = OL' . OM$.

Also, M and L', M' and L are corresponding points with respect to K, and $AK : AL :: BK : BL'$.

Therefore $AK + AL : AL :: BK + BL' : BL$,

or, alternately, $KL : KM' :: AL : BL'$.

Similarly, $KM : KL' :: AL : BL'$.

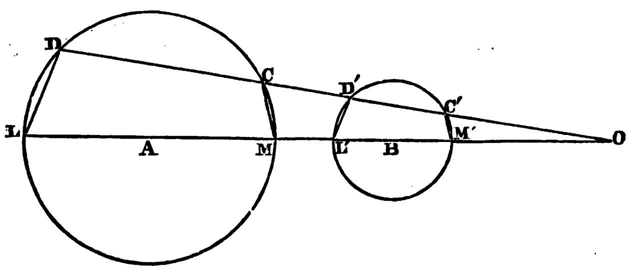
Therefore $KL : KM' :: KM : KL'$,

and therefore $KL . KL' = KM . KM'$.

But these results are only particular cases of the next proposition.

226. *If a transversal be drawn through a centre of similitude intersecting the two circles, the rectangle under the distances of either pair of non-corresponding points from the centre of similitude is constant.*

Let A and B be the centres of the two circles, and through O



either of their centres of similitude draw the transversal OD, so that C and C', D and D' are corresponding points.

We have proved that

$$OC : OC' :: AM : BM' :: OM : OM'.$$

Therefore CM and $C'M'$ are parallel. Similarly DL and $D'L'$ are parallel. Therefore the angle $OC'M'$ is equal to OCM, which (III. 22) is equal to DLM, since $DLMC$ is a quadrilateral in a circle.

Therefore the quadrilateral $DLM'C'$ is circumscribable by a circle.

Similarly, $D'L'MC$ is also circumscribable by a circle.

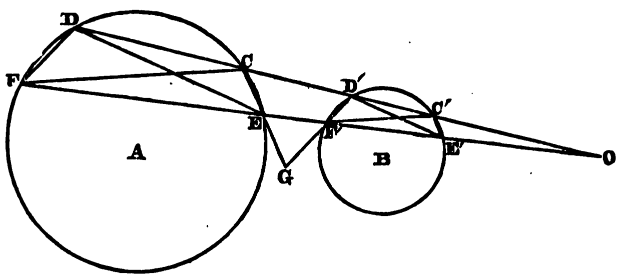
Therefore

$$OD \,.\, OC' = OL \,.\, OM' \text{ and } OD' \,.\, OC = OL' \,.\, OM,$$

but (225, N.B.) $$OL \,.\, OM' = OL' \,.\, OM.$$

Therefore $OD \,.\, OC'$ and $OD' \,.\, OC$ are each equal to the same constant rectangle. Q. E. D.

227. *If through a centre of similitude of two circles two transversals be drawn, meeting the circles in four pairs of points, the straight line joining any pair of points on one circle (not lying on the same transversal), will be parallel to the straight line joining the corresponding pair on the other circle, and it will meet the straight line joining the non-corresponding pair on the other circle, on the radical axis of the two circles.*



Let O be a centre of similitude of two circles whose centres are A and B, and OD, OF any transversals through O, meeting the circles. Join the points by straight lines as in the Fig., and let CE and $D'F'$ meet in G.

To C, E on the one circle belong the corresponding points C', E', and the non-corresponding D', F' on the other circle.

Because (225) $OC : OC' :: OE : OE'$, therefore CE and $C'E'$ are parallel.

Because (226) $OC . OD' = OE . OF'$, therefore the quadrilateral $CEF'D'$ is circumscribable by a circle, and therefore (III. 36) $CG . GE = D'G . GF'$, therefore (199, Def.) G is a point on the radical axis of the two given circles.

In the same manner it can be proved that DE and $D'E'$, FC and $F'C'$, DF and $D'F'$ are parallel, and that DE and $C'F'$, FC and $E'D'$, FD and $E'C'$ meet on the radical axis of the two circles.

N.B. The learner should bear in mind that the quadrilaterals $FDC'E'$, $CEF'D'$, $DEF'C'$, $CFE'D'$ are circumscribable by circles.

It is also worth noticing, that the quadrilaterals $DFEC$ and $D'F'E'C'$ are similar, and have their corresponding (or homologous) sides proportional to the radii of their respective circles.

For DF and $D'F'$ are parallel, and

$$DF : D'F' :: OD : OD'$$

:: radius of (A) : radius of (B). So $DC : D'C'$ as the radii, and so on.

Therefore $FD : DC :: F'D' : D'C'$, and similarly for the other sides about the equal angles. Also

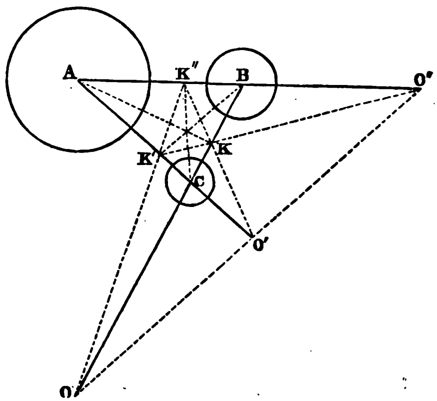
$$DE : D'E' :: OD : OD'.$$

COR. Hence the tangents at corresponding points will be parallel, and the tangents at non-corresponding points will meet on the radical axis of the two circles. For the radii drawn to corresponding points are parallel (225), and if a tangent be drawn at C it will be inclined to CD' at an angle equal to DEC in the alternate segment (III. 32), and the tangent at D' will be inclined to $D'C$ at an angle equal to the angle $D'E'C'$, but DEC and $D'E'C'$ are equal, therefore the tangents at C and D' meet on the radical axis of the two circles.

228. *Given three circles; taken two at a time they form three pairs of circles. The lines joining the centre of each circle to the internal centre of similitude of the other two meet in a point. The external centre of similitude of any pair, and the two internal centres of similitude of the other two pairs lie in the same straight line, and the external centres of similitude of the three pairs lie on a straight line.*

Let A, B, C be the centres of the three given circles, K and O the internal and external centres of similitude of (B) and (C), K

and O' of (C) and (A), K'' and O'' of (A) and (B); then shall AK, BK' and CK'' meet in a point, and O, K', K''; O', K, K''; O'', K, K'; O, O', O'' shall lie on four straight lines, respectively.



Since the centres of similitude of any two circles divide the line joining their centres internally and externally in the ratio of the radii, the two centres of similitude are harmonic conjugates to the centres of the circles.

Again, $AK' : K'C$ :: radius of (A) : radius of (C).

Also,

$$\begin{Bmatrix} AK'' : K''B \\ BK : KC \end{Bmatrix} :: \begin{Bmatrix} \text{radius of } (A) : \text{radius of } (B) \\ \text{radius of } (B) : \text{radius of } (C) \end{Bmatrix} :: \begin{matrix} \text{radius of } (A) \\ : \text{radius of } (C). \end{matrix}$$

Therefore $AK' : K'C :: \begin{Bmatrix} AK'' : K''B \\ BK : KC \end{Bmatrix}.$

Therefore (168, N.B.) AK, BK' and CK'' pass through the same point.

Therefore (169), considering the triangle ABC, $K''K$ will meet AC in the harmonic conjugate to K', that is, in O'; similarly, $K''K'$ will pass through O, and $K'K$ through O''. Also (169) O, O', O'' will lie on a straight line. Q. E. D.

DEF. The line $OO'O''$ on which the three external centres of similitude lie is called the *external axis of similitude*, and the three lines $OK'K''$, $O'KK''$ and $O''KK'$ are called the three *internal axes of similitude*.

We can now concisely enunciate the latter part of (228). *The six centres of similitude of three circles taken two at a time lie in groups of three on the four axes of similitude.*

229. *If a variable circle touch two fixed circles, the chord of contact passes through their external centre of similitude when the contacts are of the same kind, and through the internal centre when the contacts are of different kinds.*

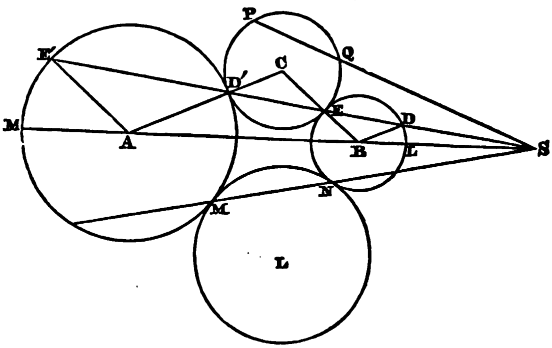


Fig. 1.

Let A and B be the centres of the two fixed circles, and C the

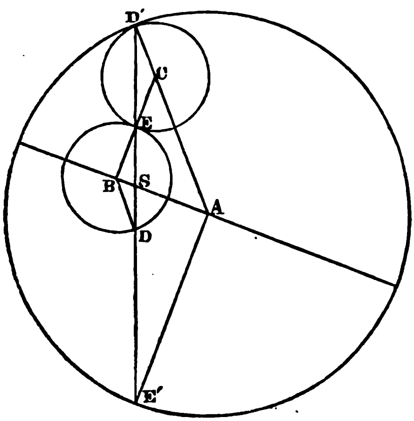


Fig. 2.

centre of the variable circle touching the two former in the points D', E. Join AC and BC, passing respectively through D' and E (III. 11, 12), and let $D'E$ and AB meet in S. Then shall S be a centre of similitude of the two circles (A) and (B). Since the tri-

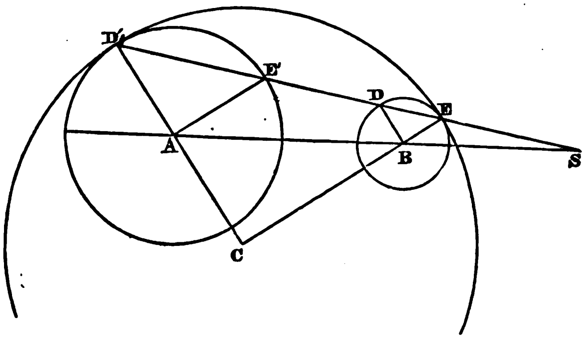


Fig. 3.

angles $AD'E'$, $CD'E$, and BDE are isosceles, it is clear that AD' and BD are parallel. Therefore, from the similar triangles ASD' and BSD, we have

$$AS : SB :: AD' : BD,$$

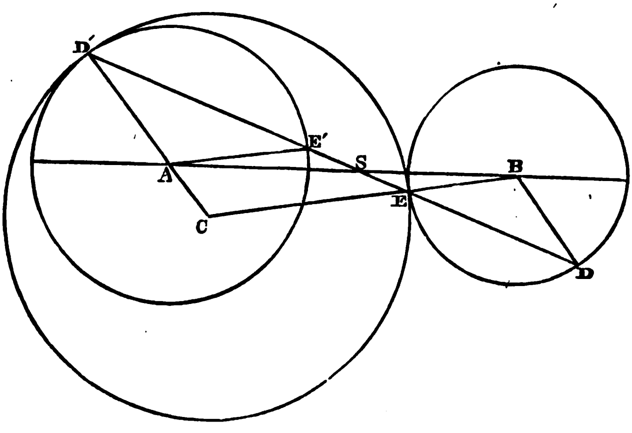


Fig. 4.

and therefore S is a centre of similitude of the two fixed circles (A) and (B).

In Figs. 1 and 3, the contacts are of the *same kind*, and S is the *external* centre of similitude of (A) and (B).

In Figs. 2 and 4, the contacts are of *different kinds*, and S is the *internal* centre of similitude of (A) and (B). Q. E. D.

230. *To describe a circle passing through a given point and touching two given circles.*

Fig. 1 to (229). Let P be the given point, A and B the centres of the two given circles, and S their external centre of similitude.

Join SP, and on SP take the point Q, so that

$$SP \,.\, SQ = SL \,.\, SM.$$

Through P, Q describe a circle touching (B) in E (131). This circle will also touch (A). For, produce SE to meet the circle PQE in D'. D' is a point on (A). Because

$$SP \,.\, SQ = SD' \,.\, SE \text{ (III. 36)},$$

but $$SP \,.\, SQ = SL \,.\, SM \text{ (constr.)},$$

therefore $$SD' \,.\, SE = SL \,.\, SM,$$

and therefore (226) D' must also lie on the circle (A).

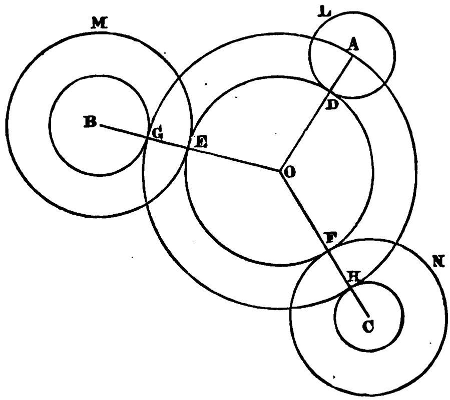
Again, let C be the centre of the circle PQE. Then CEB is a straight line. Join AD', CD'. We have now to prove that $AD'C$ is a straight line. Because D' and D are corresponding points, AD' and BD are parallel (225). Therefore the angles $AD'E$ and $CD'E$ are together equal to BDS and BDE together, that is, to two right angles, and therefore (I. 14) $AD'C$ is a straight line. Therefore the circle (C) touches (A) and (B) at D', E, and passes through the given point P.

Since (131) two circles can be described passing through two given points, and touching a given circle, two circles can be described as required by the aid of either centre of similitude. Therefore, *in general*, four circles can be described through a given point touching two given circles.

231. *To describe a circle touching three given circles.*

Let A, B, C be the centres of the three given circles L, M, N, and suppose that they are not all equal, and that L is not greater than M or N.

With centre B, and radius equal to the difference of the radii of M and L, describe a circle, and with centre C, and radius equal to the difference of the radii of L and N, describe a circle. Describe (230) a circle through A, and touching the two latter circles externally in G and H. Let O be the centre of this circle.



Join OB (which will pass through G) meeting M in E, and OC (which will pass through H) meeting N in F, and join OA meeting L in D.

GE and HF are each equal to DA (constr.), therefore OD, OE, and OF are equal, and therefore the circle described with O as centre, and OD as radius, will touch the given circles at the points D, E, F.

If we describe circles with centres B and C, and radii *exceeding* the radii of M and N by the radius of L, and then describe a circle through A, touching those circles externally, the centre of this circle will be the centre of a circle which will be touched internally by L and externally by M and N.

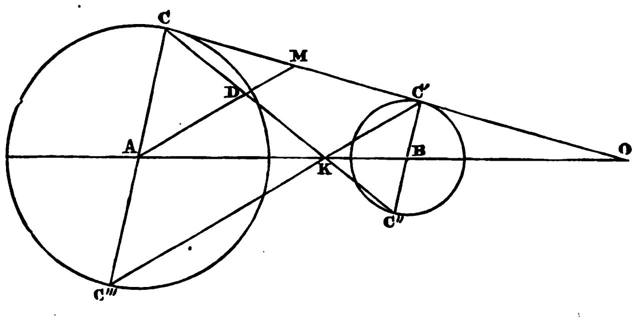
In general, eight circles can be described touching three given circles, but I will not here discuss the remaining cases, since another solution of the problem will be given farther on. See (254).

232. *If two variable circles touch two given circles, their radical axis will always pass through the external centre of similitude of the given circles when the contacts are both of* *the same kind, but through the internal centre of similitude when the contacts are of different kinds.*

Fig. 1 to (229). Let A and B be the centres of the given circles, and let the variable circles PQE and LMN touch them in D', E and M, N respectively. By (229) the straight lines $D'E$ and MN pass through S, the external centre of similitude. Therefore (226) $SD' . SE = SM . SN$, and therefore (199, Def.) S is a point on the radical axis of the variable circles; the proof is exactly similar when the contacts are of different kinds. Q. E. D.

233. *A centre of similitude of two circles is joined with the point of contact of one of the circles, with either common tangent through the other centre of similitude. Prove that the line joining the middle point of the line so drawn, and the centre of the circle, bisects that common tangent.*

Let A and B be the centres of the two given circles, K and O their centres of similitude, and CC' a common tangent through O. Join CK, and let C'' be the corresponding point to C, where CK meets the circle (B). Bisect CK in D, and join AD, meeting CC' in M; then shall M be the middle point of CC'. For, join KC', $C'B$, and BC''. Because the angles ACO and $BC'O$ are



right, AC and BC' are parallel, and because C and C'' are corresponding points, AC and BC'' are parallel (225). Therefore $C'BC''$ is one straight line.

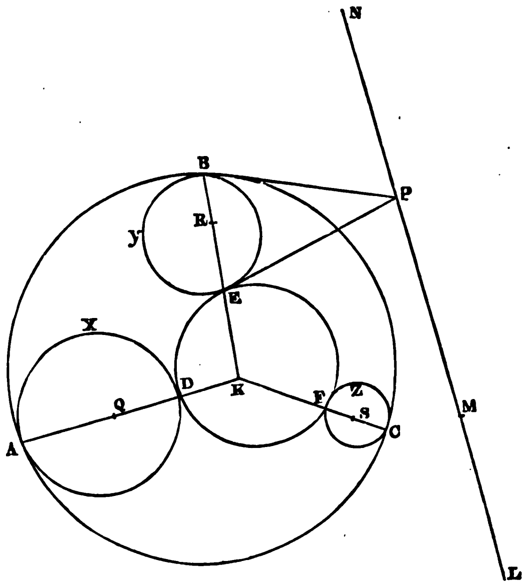
Similarly, CAC''' is a straight line, where C''' corresponds to C'.

Now, AD joins the middle points of the sides of the triangle

$CC'''K$, therefore AM is parallel to $C'''C'$, and therefore CC' is bisected in M. Q. E. D.

234. *If two circles touch three others, the contacts being of the same kind, the radical axis of the two is the external axis of similitude of the three, but if the contact of the two circles with one pair of the three be of the same kind, and with the other two pairs of different kinds, the radical axis of the two circles will be that internal axis of similitude which passes through the external centre of similitude of the first pair of the three circles.*

Let the two circles ABC, DEF touch the other three at the points A, B, C and D, E, F respectively, the contacts being of the same kind, and let L be the external centre of similitude of the circles BYE and CZF, M of CZF and AXD, and N of AXD and BYE, so that LMN is the external axis of similitude of the three circles.



Now, since the circles ABC and DEF touch BYE and CZF, their radical axis passes through L (232).

Similarly, it passes through M and N.

Therefore LMN is the radical axis of the two circles ABC and DEF.

The other cases of the theorem can be proved in the same manner. Q. E. D.

235. *If two circles touch three given circles, as in* (234), *the three chords of contact meet in a point, which is the radical centre of the three, and a centre of similitude of the two.*

Fig. to (234). Let the circles ABC and DEF touch the other three.

Since each of the three circles touches the two ABC and DEF and the contacts are of different kinds, therefore (229) the three chords of contact AD, BE and CF meet in K, the internal centre of similitude of ABC and DEF.

Again, the circles AXD and BYE touch ABC and DEF, and the contacts are of different kinds, therefore (232) the radical axis of AXD and BYE passes through K. Similarly, the radical axes of BYE and CZF and of CZF and AXD pass through K. Therefore K is the radical centre of the three. Q. E. D.

236. *If two circles touch three others, as in* (234), *the tangents at the extremities of the chords of contact of each of the three circles meet on that axis of similitude, which is also the radical axis of the two circles.*

Fig. to (234). Let the two circles ABC and DEF touch the other three, and let LN be the radical axis of the two, and therefore (234) an axis of similitude of the three.

The tangents to the circle BYE at B and E are equal, but these are also tangents to ABC and DEF, and therefore these equal tangents to ABC and DEF must meet on their radical axis, as at P.

In the same manner it can be proved, that the tangents at A and D, C and F, meet on LN. Q. E. D.

DEF. If through a fixed point any straight line be drawn intersecting a circle, and a point be taken on it, such that it and the fixed point are harmonic conjugates, with respect to the two points of intersection, the locus of the assumed point is called the *polar* of the fixed point, which is called the *pole.*

237. *Prove that the polar of a given point, with respect to a circle, is a straight line on the same side of the centre as the pole; that the straight line joining the centre and pole is perpendicular to the polar, and that the rectangle under the distances of the pole and polar from the centre is equal to the square on the radius of the circle.*

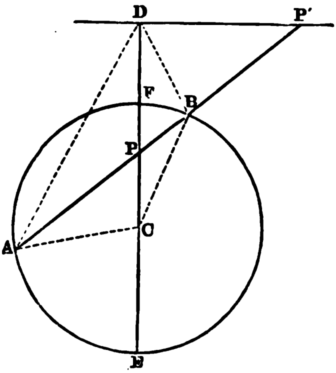


Fig. 1.

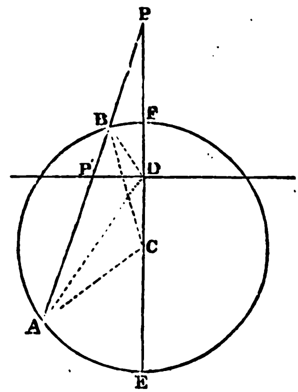


Fig. 2.

Let C be the centre of the given circle, and P the given point.

Through P draw the diameter EF, and take D so that P and D may be harmonic conjugates to E and F, and draw DP' perpendicular to DE. Then DP' is the polar of P. For, draw through P any transversal cutting the circle in A, B and DP' in P'.

Because P and D are harmonic conjugates to E and F and the mean EF is bisected in C, therefore (163) $DC \,.\, CP = CF^2$ or AC^2.

Therefore the difference between CP^2 and $DC \,.\, CP$ is equal to the difference between CP^2 and AC^2. But the former difference is equal to $CP \,.\, PD$ (II. 2, 3), and the latter is equal to $AP \,.\, PB$ (50), since ABC is an isosceles triangle. Therefore $CP \,.\, PD = AP \,.\, PB$, and therefore the four points A, B, C, D lie in the circumference of a circle. Therefore the angles BDP

and BAC are equal, but BAC equals ABC, which is equal to ADC. Also the angle PDP' is right; therefore PD and $P'D$ are the bisectors of the internal and external vertical angles of the triangle ADB. Therefore (171) $D . APBP'$ is an harmonic pencil. Therefore the conjugate to P, viz. P' with respect to A and B, always lies upon the fixed line DP'.

DP' is therefore the polar of P. Also the pole P and its polar DP' lie at the same side of the centre, and we have proved that the line CP joining the centre and pole is perpendicular to the polar DP', and that $CP . CD = AC^2$. Q. E. D.

COR. Hence, given a straight line we can find its pole, or given a point we can find its polar.

For, if DP' be the given line, draw CD perpendicular to it and take P so that $PC . CD = AC^2$, then P is the required pole; and if the pole P be given, on CP take D so that

$$PC . CD = AC^2,$$

and draw DP' perpendicular to CD. DP' is the required polar; or we may find the polar of P thus; through the pole draw any two secants, and take a point on each, such that it and the given pole shall be harmonic conjugates to the points in which the secant intersects the circle; the straight line passing through the two assumed points is the required polar.

N.B. Some writers take for their definition of *pole* and *polar* the following.

Join any point with the centre of a circle, and take a point on the joining line such that the rectangle under the distances of it and of the given point from the centre shall be equal to the square on the radius. The perpendicular through the assumed point to the joining line is called the *polar* of the given point which is called the *pole*.

The definition first given has been preferred, since it is applicable to all curves.

Some writers also lay down as their definition of *pole* and *polar* the property proved in the next proposition.

It is evident that when the polar is without the circle, the pole is within it, and when the polar cuts the circle the pole is without the circle; also that the polar of a point on the circle is the tangent at the point, and the pole of a tangent is its point of contact.

DEF. The foot of the perpendicular from the centre or from the pole on the polar is sometimes called the *middle point of the polar.*

238. *A chord is drawn through a fixed point either inside or outside a circle, and tangents at its extremities; the locus of their intersection is the polar of the fixed point.*

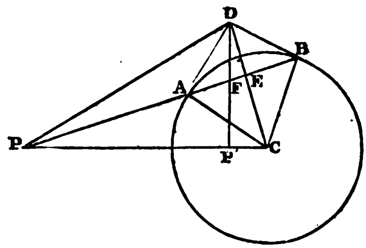


Fig. 1.

Let C be the centre of the circle and P the fixed point, and let the tangents AD, BD at the extremities of the chord AB, passing through P, meet in D.

Through D draw DP' perpendicular to CP, meeting AB in F.

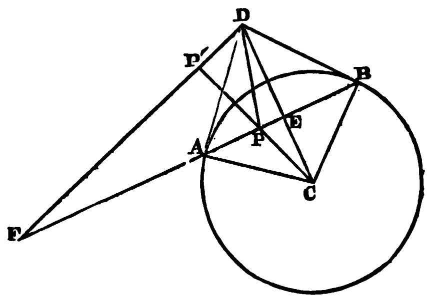


Fig. 2.

Since CD bisects AB at right angles in E, and the angles DAC, DBC are right; therefore $DC \cdot CE = AC^2$ (39).

But the angles $DP'P$ and DEP are also right, therefore the four points D, E, P, P' lie on the same circumference, and therefore $DC \cdot CE = PC \cdot CP'$. Therefore $PC \cdot CP' = AC^2$, and therefore

(237) DP' is the polar of P, that is, the point D always lies on the polar of P. Q. E. D.

N.B. Since $DC . CE = AC^2$, therefore D is the pole of AB, or (α) *The tangents at the extremities of any chord intersect in the pole of the chord.* Conversely, (β) *If tangents be drawn from any point, the chord of contact is the polar of the point.* Again, since P is the pole of DP' and any line drawn through the pole, the polar and circle is cut harmonically, therefore $D . PAFB$ is an harmonic pencil. Now DF is any line through the intersection of two tangents DA, DB, and D is joined to P the pole of DF, and we have seen that $D . PAFB$ is an harmonic pencil. Hence we have the following theorem. (γ) *If any straight line be drawn through a point, and the pole of that line be joined to the point, the first line and the joining line form an harmonic pencil with the tangents from the point.*

239. *The polar of the point of intersection of any two straight lines is the line which joins their poles, and, conversely, the line joining two points is the polar of the intersection of the polars of these points.*

Let P be the intersection of PA and PB, and C, D the respective poles of these lines, then shall CD be the polar of P. First, let

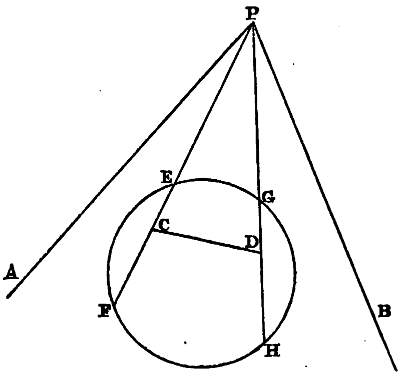


Fig. 1.

either P or both C and D lie within the circle. Since PF is drawn through C and its polar AP, it is cut harmonically in E and C. In like manner, PH is cut harmonically in G and D. Therefore CD is the polar of P (237, Cor.).

Next (Fig. 2), let C be without the circle, and D within it, and let PA cut the circle in K and M. By (238, *a*) the tangents at K

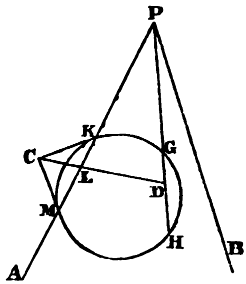


Fig. 2.

and M meet in C the pole of PA. Take L the harmonic conjugate to P with respect to M and K; then LD is the polar of P (237, Cor.), but (238) C lies on the polar of P, therefore LD produced passes through C.

Lastly (Fig. 3), let both C and D lie without the circle, and let PA and PB cut the circle in K, M and N, R respectively. The

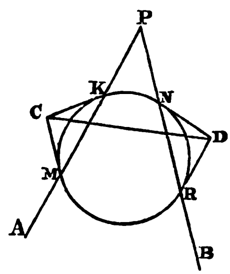


Fig. 3.

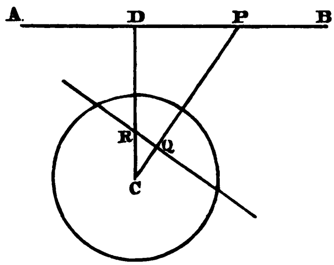
tangents at these points meet in C and D (238, *a*). By (238) C and D both lie on the polar of P, therefore CD is the polar of P. In all cases the converse is obvious. Q. E. D.

Cor. Hence, (*a*) *if any number of straight lines pass through a point, their poles all lie on the polar of that point.* For the line joining the poles of every two of the intersecting lines is the polar of the point of intersection; in other words, all the poles of the intersecting lines lie on the polar of the point of intersection.

(β) *If any number of points lie on a straight line, their polars all pass through the pole of the line.* For the polars of any two of the points intersect in the pole of the line joining them, that is, the polar of every point on a given straight line passes through the pole of that line. For the benefit of the learner I will give another proof of these principles in the next proposition.

240. *If a point move along a fixed straight line, its polar always passes through a fixed point, viz. the pole of the fixed line; and if a straight line always pass through a fixed point, its pole always lies on a fixed straight line, viz. the polar of the fixed point.*

Let AB be the fixed straight line, and P any point on it. Then



shall the polar of P pass through the pole of AB. From the centre C, draw CD perpendicular to AB, and let QR the polar of P meet CP (or CD produced) in R.

Because the angles PDR and PQR are right, the four points P, Q, R, D lie on the same circumference, therefore

$$PC \cdot CQ = DC \cdot CR,$$

but $PC \cdot CQ$ equals the square on the radius, therefore $DC \cdot CR$ equals the square on the radius, and therefore R is the pole of AB. Therefore the polar QR of any point P on AB always passes through the pole of AB. The proof is similar when AB cuts the circle.

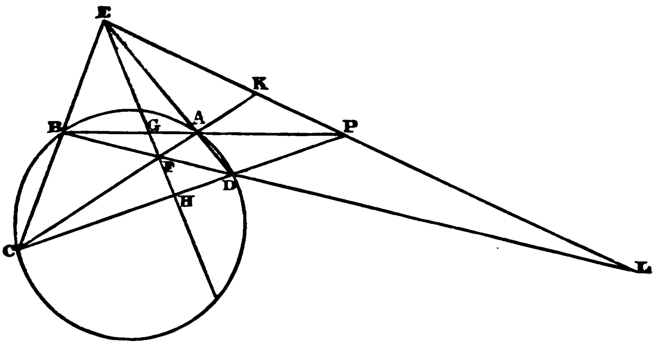
Now, suppose P a fixed point, and AB any straight line through it; then shall the pole of AB always lie on the polar of P. From the centre C draw CD perpendicular to AB, and meeting QR the polar of P in R. Therefore the points P, Q, R, D lie on the same circumference, and therefore

$$PC \cdot CQ = DC \cdot CR;$$

but $PC \cdot CQ$ equals the square on the radius, therefore $DC \cdot CR$ also equals the square on the radius, and therefore R is the pole of AB. Therefore the pole of any straight line through P always lies on the polar of the fixed point P. The proof is similar when P is within the circle. Q. E. D.

241. *If through any point inside or outside a circle secants be drawn, the straight lines joining the extremities of the chords intersect on the polar of that point.*

Let P be the fixed point, and through it draw any two secants



cutting the circle in A, B, and C, D respectively. Join AC and BD intersecting in F, and DA and CB intersecting in E. Join EF, and let it meet PB and PC in G and H, respectively. Then EF is the polar of P.

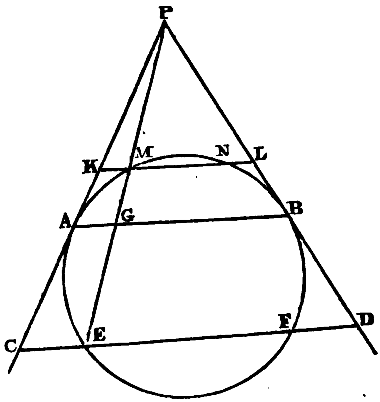
Because, through F a point within the triangle CED, straight lines are drawn to its angles, therefore (169) PB is cut harmonically in A and G, and PC in D and H. Therefore GH is the polar of P, and therefore AC and BD, CB and DA intersect on the polar of P.

Now, suppose F a fixed point within the circle, and through F draw any two chords AC and BD; then shall CB and DA, BA and CD intersect at the points E and P respectively, on the polar of F.

For join EP, and let CA and BD meet EP in K and L, respectively. Because K is the harmonic conjugate to F, with respect to A and C, and L to F with respect to D and B, therefore KL is the polar of F. Q. E. D.

242. *Prove (by a method applicable to any conic section) that a secant from the intersection of two tangents to a circle is cut harmonically by the circumference and the chord of contact.*

Let the tangents PC and PD intersect in P, and touch the



circle at A and B respectively. Through P draw any secant PE, meeting the circle at M and E, and the chord of contact AB at G, and through M and E draw KL and CD parallel to AB. If P be joined with the centre, it is clear that the joining line will bisect KL and MN, CD and EF, and AB at right angles; therefore KM and LN are equal, as also CE and FD.

On account of the parallels, we have the following proportions:

$$EP : PM :: CE : KM,$$

$$EP : PM :: DE : LM.$$

Therefore, compounding these ratios, we have,

$$EP^2 : PM^2 :: CE.DE : KM.LM :: CE.CF : KM.KN$$
$$:: CA^2 : AK^2.$$

Therefore $\quad CA : AK :: EP : PM.$

But $\quad CA : AK :: EG : GM,$

and therefore $\quad EG : GM :: EP : PM.$

Therefore EM is cut harmonically, for it is cut internally and externally in the same ratio. Q. E. D.

N.B. This proof is here given chiefly for the sake of the more advanced student, as the theorem has been already proved more simply.

243. *Prove* (149) *by* pencils, *and also by* polars.

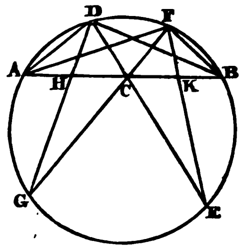


Fig. 1.

Let C be the middle point of the chord AB, and through C draw any other two chords DE and FG; then shall AH and BK be equal.

First, *by pencils*, Fig. 1.

$$\{D . AGEB\} = \{F . AGEB\}, \text{ (189).}$$

Therefore $[AHCB] = [ACKB]$, and therefore

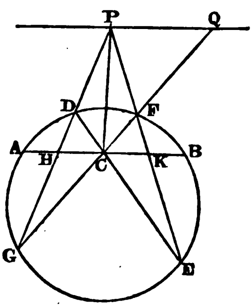


Fig. 2.

$$AB . HC : AH . CB :: AB . CK : AC . KB,$$

or, alternately,

$$AB . HC : AB . CK :: AH . CB : AC . KB.$$

Therefore $HC : CK :: AH : KB$,

or $CH : HA :: CK : KB$,

but AC and CB are equal; therefore HA and KB are also equal. Q. E. D.

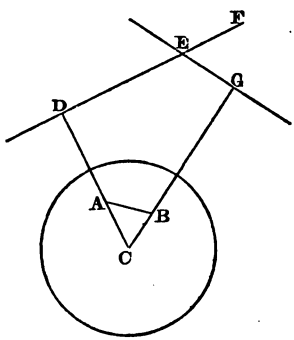
Next, *by polars*, Fig. 2.

Produce GD and EF to meet in P, which is a point on the polar of C (241). Let PQ be the polar of C. PQ is obviously parallel to AB.

Produce GF to meet PQ in Q, and join PC.

Because GQ is drawn through C and its polar PQ, $P . GCFQ$ is an harmonic pencil, but AB is parallel to PQ, therefore HC and CK are equal. Q. E. D.

244. *Any two points subtend at the centre of a circle an angle equal to that between their polars.*



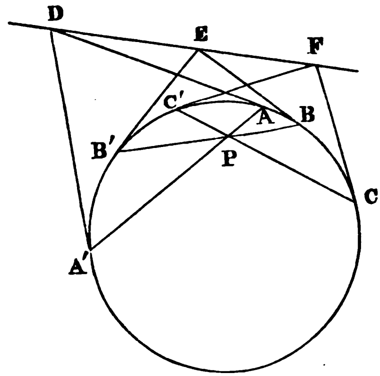
Let C be the centre of the circle, and A, B the two points. Find DE the polar of A, and EG the polar of B. Because the angles CDE and CGE are right (237), therefore the angle FEG between the polars is equal to the angle ACB subtended by AB at C. Q. E. D.

245. *The anharmonic ratio of four points in a straight line is equal to that of the pencil formed by their four polars.*

Since the points are in a straight line their four polars pass through the pole of this straight line (240), and therefore form a pencil. Also, if the four poles be joined to the centre of the circle, the pencil thus formed has its angles respectively equal to the angles of the pencil formed by the four polars (244). Therefore the anharmonic ratio of four points in a straight line is equal to that of their polars. Q. E. D.

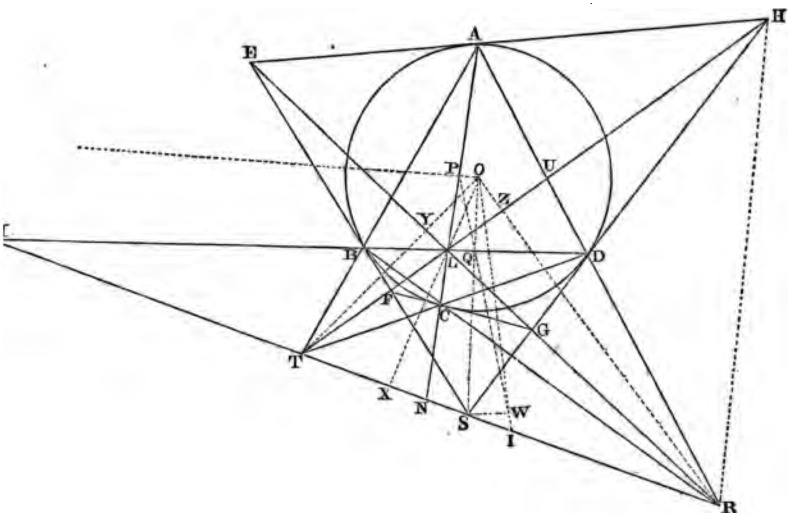
246. *If three pairs of tangents be drawn to a circle from three points in a straight line, they will cut any seventh tangent in involution.*

Let D, E, F be three points in a straight line, and A, A'; B, B'; C, C' the points of contact of the pairs of tangents from D, E, F respectively.



Since the chords of contact AA', BB', and CC' are the polars of three points D, E, F in a straight line, therefore (240) they pass through P the pole of DF. Now, if a seventh tangent cut the six tangents, the anharmonic ratio of any four points of intersection is equal to that of the four points of contact of the corresponding tangents (191, N.B.). But since the three chords AA', BB', CC' pass through the same point, the anharmonic ratio of any four of the six points of contact is equal to that of their four conjugates (215, 207). Therefore also, the anharmonic ratio of any four of the six points in which the six tangents meet any seventh tangent is equal to that of their conjugates, and therefore (208) the six points are in involution. Q. E. D.

247. *If a quadrilateral be inscribed in a circle, and another circumscribed touching at the angular points, prove that, (α) their diagonals intersect in the same point, and form an harmonic pencil; (β) their* third *diagonals are in the* *same straight line, and their extremities form an harmonic range;* (γ) *the intersection of each* pair *of the* three *diagonals of the circumscribed quadrilateral is the pole of the remaining diagonal.*



Let $ABCD$ be the inscribed quadrilateral, and $EFGH$ the circumscribed touching at the angular points of the former. Let O be the centre of the circle, and let AB, CD meet in T; BC, AD in R; EF, HG in S, and suppose that V is the point in which FG, HE, if produced, would meet, and let AC and BD intersect in L.

Because, through the point R two secants RA and RB are drawn, therefore (238, 241) the points T, F, L, H lie on the polar of R, and because through T two secants TA and TD are drawn, the points R, G, L, E lie on the polar of T; therefore $TFLH$ is the polar of R, and $RGLE$ is the polar of T. Therefore the *four* diagonals of the two quadrilaterals intersect in the same point L.

Now R is the pole of FH, S of BD, T of EG, and V of AC; but the four straight lines FH, BD, EG, and AC pass through the same point; therefore (240) their four poles R, S, T, V are in the same straight line. Therefore the *third* diagonals of the two quadrilaterals lie in the same straight line.

Again, because the secant RA is drawn through the point R and its polar HT, therefore $H.RDUA$ is an harmonic pencil, but its rays meet the transversal RV in R, S, T, V; therefore R, S, T, V, the extremities of the *third* diagonals, form an harmonic range, and therefore (245) also the *four* diagonals which are the polars of these points form an harmonic pencil.

Also, since FH and EG, the polars of R and T, intersect in L, therefore (239) L is the pole of RT. Therefore any diagonal of the circumscribed quadrilateral is the polar of the intersection of the other two.

Hence the proposition has been completely proved. Q. E. D.

N.B. It has been proved, that L is the pole of RT, R of TL, and T of LR. Hence, *if a quadrilateral be inscribed in a circle and the intersection of its two diagonals joined to the extremities of the third, the triangle formed by the two joining lines and the third diagonal is such that each vertex is the pole of the opposite side.* The proposition (247) furnishes a simple proof of (111).

DEF. A triangle which is such that each vertex is the pole of the opposite side, with respect to a circle, is called a *self-conjugate triangle.* The circle is also sometimes called *self-conjugate,* with respect to the triangle, and the three angular points of the triangle are said to form a *conjugate triad.*

248. *Given a circle and the lengths of the three diagonals of a quadrilateral inscribed in it; construct the quadrilateral.*

In Fig. to (247) suppose $ABCD$ the required quadrilateral. Then since V (viz. the point on RT where FG and HE intersect) is the pole of AC, therefore (237) OV bisects AC at right angles in P and $VO \cdot OP$ is equal to the square on the radius. Similarly, $SO \cdot OQ$ is equal to the square on the radius. But AC and BD are given in magnitude; therefore OP and OQ their distances from the centre are known, and therefore OV and OS are known. Also, if I be the middle point of the third diagonal RT, the points P, Q, I are in a straight line (22). Therefore (167) since the transversal PI cuts the sides of the triangle VOS,

$$VI : IS :: \begin{cases} VP : PO \\ OQ : QS \end{cases}.$$

Therefore $VI : IS$ is a known ratio.

Again, the tangent from I is equal to half the third diagonal (111), therefore OI is known. Hence the problem is reduced to the following. (*a*) *Given the two sides* OV, OS *of a triangle and the length of a line* OI *dividing the base* VS *in a given ratio in* I; *construct the triangle.*

Draw SW parallel to OV and meeting OI in W. Then

$$VI : IS :: VO : SW,$$

but $VI : IS$ is a given ratio and VO is given, therefore SW is known.

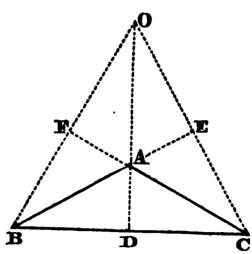
Similarly, OW is known. Therefore the sides of the triangle OSW are known, and it can be constructed with an angular point at O, and hence the triangle OVS can then be constructed. But OP and OQ are known. Therefore AC and BD perpendicular to OV and OS respectively through P and Q determine the required quadrilateral $ABCD$.

N.B. We have seen that $VO \cdot OP$ and $SO \cdot OQ$ are each equal to the square on the radius. Therefore P, V, S, Q lie on the same circumference, and therefore $PI \cdot IQ = VI \cdot IS = IT^2$, since R, S, T, V form an harmonic range, and I is the middle point of the mean RT (163).

Hence we have the following theorem. (β) *The rectangle under the whole line joining the middle points of the diagonals of a complete quadrilateral and its segment adjacent to the third diagonal is equal to the square on half the third diagonal.*

Since the line joining the centre to any point is perpendicular to the polar of the point, it is clear that the perpendiculars of the triangle LRT intersect in O the centre of the circle.

249. *Given a triangle, to describe the circle with respect to which the triangle is self-conjugate.*



Since the pole and polar are both on the same side of the centre, and the perpendicular from the pole on the polar passes through the centre, it is evident that the perpendiculars of the triangle meet in the centre of the required circle, and that this point must be *without* the given triangle, and therefore the triangle must be obtuse-angled when a solution of the problem is geometrically possible.

Let then ABC be the given triangle, and let its perpendiculars meet in O. Because the figures $BFAD$ and $DAEC$ are circumscribable by circles, therefore

$$BO \, . \, OF = DO \, . \, OA = CO \, . \, OE,$$

but each of these rectangles is equal to the square on the radius of the required circle (237). Therefore the centre O and the radius of the required circle are known. The circle may be thus constructed.

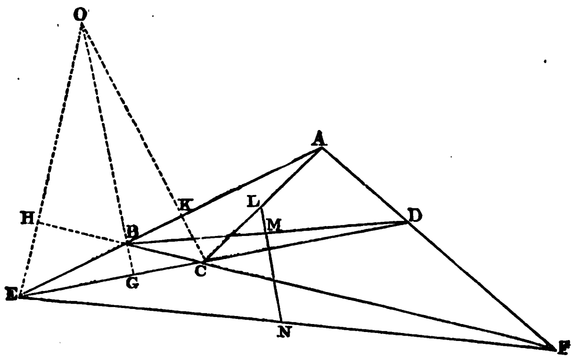
On BC, as diameter, describe a circle, and from O draw a tangent to this circle. The circle with O as centre and this tangent as radius will be self-conjugate with respect to the given triangle. The two circles evidently cut one another orthogonally.

COR. The circle self-conjugate to a given triangle cuts orthogonally the circle described on the side of the triangle, opposite the obtuse angle, as diameter.

250. *The four circles each self-conjugate to one of the four triangles formed by the sides of a quadrilateral, and the circle circumscribing the triangle formed by the three diagonals of the complete quadrilateral, form a system of five circles, which cut orthogonally the three circles described on the three diagonals, as diameters, and the straight line joining the middle points of the diagonals is the radical axis of the five circles.*

Let $ABCD$ be the quadrilateral, EF its third diagonal, and L, M, N the middle points of its diagonals, so that LMN is a straight line (22). The circle circumscribing the triangle formed by AC, BD and EF, and the four circles self-conjugate to the triangles BEC, AED, CFD and AFB shall have LMN for their radical axis. Let the perpendiculars of the triangle BEC meet in O, then (249) O is the centre of the circle, self-conjugate to the triangle BEC, and the square on its radius is equal to $EO \, . \, OH = GO \, . \, OB = CO \, . \, OK$. Therefore the circle ($O$) will cut orthogonally the circle on EF, as diameter, for this circle passes through H since EHF is a right angle. Similarly, $CO \, . \, OK =$ square on tangent from O to circle on diameter AC, since K is a right angle;

and since BGD is a right angle, circle on diameter BD passes through G, and therefore $GO \cdot OB$ equals square on tangent from O to circle on diameter BD.



Therefore the circle self-conjugate to the triangle BEC cuts orthogonally the circles on the three diagonals, as diameters.

In like manner it can be proved that the circles self-conjugate to the other three triangles cut the same three circles orthogonally. And it has been proved in (217) that the circle circumscribing the triangle formed by the three diagonals, also cuts orthogonally the circles on the three diagonals, as diameters.

Therefore (201, β) LMN is the radical axis of the system of five circles, and the straight line passing through the centres of the five circles is the radical axis of the system of three circles on the diagonals, as diameters. Q. E. D.

N.B. A particular case of this theorem has been already given by the Rev. N. M. Ferrers, F.R.S., Fellow and Tutor of Gonville and Caius College (Mathematical Tripos, Jan. 16, 1862).

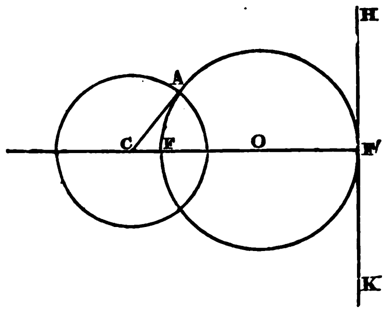
Another demonstration of Mr. Ferrers's theorem is given in the *Quarterly Journal of Mathematics* for June, 1862, p. 272.

251. *If a system of circles have a pole and polar in common, they shall have the same radical axis.*

Let F be the pole, and HK the polar. From F draw FF' perpendicular to HK, and on FF' as diameter, describe the circle with centre O. In FF' take any point C outside the circle, and so that F and F' are on the same side of C.

Draw the tangent CA. The circle described with centre C and radius CA is a circle of the system. For

$$CF \,.\, CF' = CA^2.$$



Therefore F is the pole, and HK its polar, with respect to the circle (C), and the perpendicular through O to FF' is clearly the radical axis of the system of circles, of which (C) is one (216). Q. E. D.

252. *If any circle cut two given circles orthogonally, the straight line joining any point in which it intersects one of the given circles to any point in which it intersects the other, always passes through a centre of similitude of the two given circles, and the limiting points of the two circles are harmonic conjugates with respect to their two centres of similitude. Also the limiting points have the same polars with respect to the two given circles.*

Let A and B be the centres of the two given circles, and let any circle cut them orthogonally in the points E, D, D', E'. Then shall EE' and DD' meet in the external centre of similitude, and ED', $E'D$ in the internal centre of similitude. Join AD, BD', and produce these lines to meet in C, and let DD' meet AB in O.

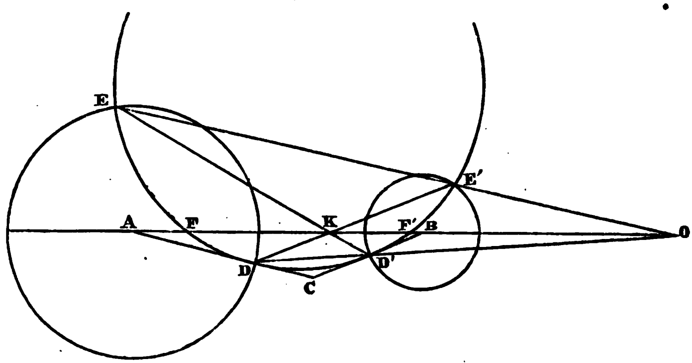
Because the transversal DO meets the sides of the triangle ACB, therefore (167)

$$AO : OB :: \begin{Bmatrix} AD : DC \\ CD' : D'B \end{Bmatrix},$$

but CD and CD' are tangents to the circle EDE' since it cuts the other two circles orthogonally, therefore CD and CD' are equal, and therefore

$$AO : OB :: AD : D'B.$$

Therefore O is the external centre of similitude of the two given circles.



In the same manner it can be proved that EE' passes through O, and that ED' and $E'D$ intersect in K the internal centre of similitude of the given circles.

Again, because from the point O two secants OD and OE are drawn meeting the circle EDE', therefore (241) K is a point on the polar of O with respect to this circle, and therefore O, F', K, F form an harmonic range, and F, F' are the limiting points of the given circles (201, *a*).

Also $AF \cdot AF' = AD^2$, and $BF \cdot BF' = BD'^2$, since AD and BD' are tangents to the circle EDE'.

Therefore the polar of F with respect to each of the given circles is the perpendicular through F' to AB, and the polar of F' is the perpendicular through F to AB. Q. E. D.

N.B. It is apparent that E, E' and D, D' are pairs of non-corresponding points.

253. *If two circles touch three given circles, as in* (234), *the pole of that axis of similitude of the three circles, which is also the radical axis of the two, with respect to any of the three circles, lies in the chord of contact of that circle.*

Fig. to (234). For by (236) the tangents to the circle BYE at B and E, meet at P on the axis of similitude LN of the three circles, and LN is the radical axis of the two circles ABC and DEF.

Therefore BE is the polar of P with respect to the circle BYE, and therefore (240) the pole of LN with respect to BYE lies on the chord of contact BE, as at R. Similarly, the poles of LN with respect to AXD and CFZ lie on AD and CF, as at Q and S. Q. E. D.

254. *To describe eight circles touching three given circles.* (See 231.)

Fig. to (234).

Let AXD, BYE, and CZF be the three given circles, and suppose ABC, DEF two circles touching the given circles, as in (234). It has been proved (234) that LN, the radical axis of the two circles ABC, DEF, is an axis of similitude of the three given circles, and (235) that K is the radical centre of the three given circles, and (253) that the poles of LN with respect to the three given circles lie on the three chords of contact, as at Q, R, S.

Hence we have the following construction.

Let K be the radical centre of the three given circles, and LN one of the four axes of similitude.

Find Q, R, S the poles of LN with respect to each of the three circles, and join KQ, KR, KS, meeting the circles in A and D, B and E, C and F respectively. The circles through the points D, E, F and A, B, C will touch the three given circles. In the same manner, by means of the other three axes of similitude, three other pairs of circles can be described touching the given circles. Q. E. F.

255. *If through any point, within or without a circle, four straight lines be drawn cutting the circle, the anharmonic ratio of four of the points of intersection is the same as that of the remaining four points.*

Let four straight lines drawn through the point O cut the circle in the four pairs of points A, A'; B, B'; C, C'; and D, D'. Then shall the anharmonic ratio of any four of the points, as A, B, C, D, be equal to that of the other four, A', B', C', D'.

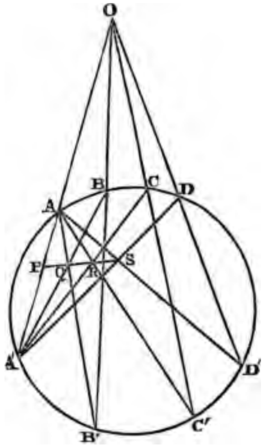
Join A' to the points B, C, D, and A to B', C', D'.

Since, through O two secants OA', OB' are drawn, therefore (241) the point Q, in which AB' and $A'B$ intersect, lies on the polar of O. Similarly the points R and S, in which AC', $A'C$ and AD', $A'D$ respectively intersect, lie on the polar of O.

Therefore SRQ is a straight line. Let this line meet OA' in P.

It is now obvious that

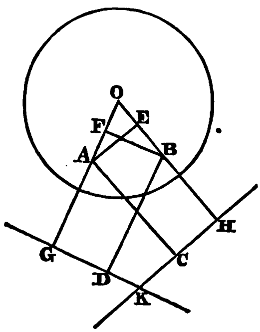
$$\{A \,.\, A'B'C'D'\} = [PQRS] = \{A' \,.\, ABCD\}.$$



The proof is similar when the point O is within the circle.
Q. E. D.

256. *The distances of any two points from the centre of a given circle are to one another as the distances of each point from the polar of the other.*

Let A, B be the two given points, O the centre of the given circle, KG the polar of A, and KH the polar of B.



Draw AE perpendicular to HO, BF to GO, AC to KH, and BD to KG. Then shall $AO : BO :: AC : BD$.

Because A and B are the poles of GK and KH, therefore $OA \cdot OG = OB \cdot OH$, since each of these rectangles is equal to the square on the radius, and because the angles AFB and AEB are right, the points A, F, E, B are in the same circumference, therefore $OA \cdot OF = OB \cdot OE$.

Therefore the difference of the rectangles $OA \cdot OG$ and $OA \cdot OF$ is equal to the difference of the rectangles $OB \cdot OH$ and $OB \cdot OE$, that is, $OA \cdot FG = OB \cdot EH$, but FG equals BD and EH equals AC, therefore $OA \cdot BD = OB \cdot AC$,

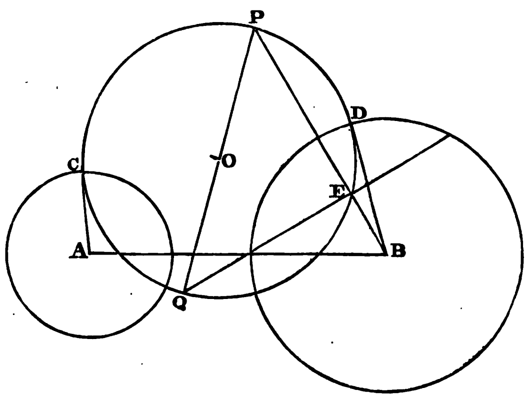
and therefore $OA : OB :: AC : BD$. Q. E. D.

N.B. This important theorem was discovered by the Rev. Dr Salmon, F.R.S. Fellow of Trinity College, Dublin, and Regius Professor of Divinity in the University.

257. *The polar of a given point, with respect to any circle of a co-axal system, will always pass through a fixed point.*

Let A and B be the centres of any two circles of the co-axal system, and let P be the given point.

Describe (219) the circle CPD, passing through P and cutting



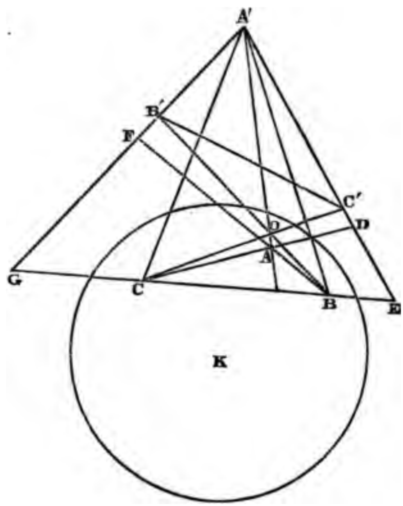
the two circles (A) and (B) orthogonally. This circle will cut the entire system orthogonally, since its centre O is on the radical axis of the system.

Join BP, meeting the circle CPD in E, and join E to Q, the extremity of the diameter PQ, opposite to P. Then Q is the fixed point through which all the polars pass.

Because BD is a tangent to the circle CPD, therefore $PB . BE = BD^2$; therefore QE is the polar of P with respect to the circle (B); and in the same manner it can be proved that the polar of P with respect to any other circle of the co-axal system passes through Q. Q. E. D.

258. *If* ABC, A′B′C′ *be two triangles, such that* A *is the pole of* B′C′, B *of* C′A′, *and* C *of* A′B′, *then the straight lines* AA′, BB′, CC′ *shall meet in a point, and the corresponding sides* BC, B′C′, CA, C′A′, *and* AB, A′B′ *shall intersect in three points situated on the same straight line.*

Let K be the centre of the circle with respect to which A, B, C are the poles of $B'C'$, $C'A'$, $A'B'$, respectively.



Let CC' and BB' intersect in O, and produce CB to meet $A'B'$ in G and $A'C'$ in E. Also let CA and BA meet $A'C'$ and $A'B'$ in D and F respectively, and join $A'B$, $A'C$.

Since the intersection of two lines is the pole of the line joining their poles, therefore D is the pole of BB' and E of $A'B$. Therefore the four points A', C', D, E are the poles of the four straight lines BC, BA, BB', BA'.

But the anharmonic ratio of four points in a straight line is the same as that of their polars (245), therefore

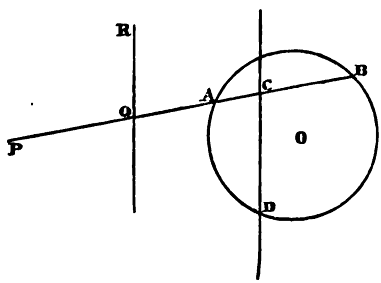
$$\{C\,.\,A'C'DE\} = \{B\,.\,GFB'A'\} = \{B\,.\,A'B'FG\}.$$

Therefore, since the ray BC is common, the intersections of the three pairs of corresponding rays lie in the same straight line (165), that is, the points A', O, A are in the same straight line, or AA', BB', and CC' pass through the same point O.

Again, A is the pole of $B'C'$ and A' of BC, therefore AA' is the polar of the intersection of $B'C'$ and BC. Similarly, BB' is the polar of the intersection of $C'A'$ and CA, and CC' of the intersection of $A'B'$ and AB, but the three polars AA', BB', CC' pass through the same point O, therefore (240) the three intersections lie in the same straight line. Q. E. D.

N.B. That the intersections of the corresponding sides lie in the same straight line follows immediately from (183), since the triangles ABC, $A'B'C'$ have been proved to be co-polar.

259. *Through a given point without a given circle, any transversal is drawn cutting the circle, and a point taken on it such that the reciprocal of its distance from the given point is equal to the sum of the reciprocals of the intercepts between the given point and the circle; find the locus of the point of section.*



Let O be the centre of the given circle, and P the given point.

Draw any secant PB, and take the point Q such that

$$\frac{1}{PQ} = \frac{1}{PA} + \frac{1}{PB}.$$

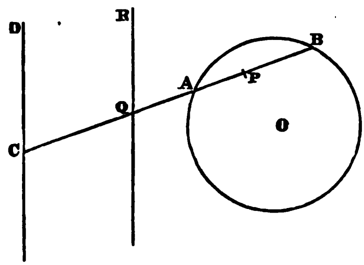
The locus of Q is required.

Let CD the polar of P meet AB in C; then P, A, C, B form an harmonic range, and therefore (176, N.B.)

$$\frac{2}{PC} = \frac{1}{PA} + \frac{1}{PB}; \text{ therefore } \frac{1}{PQ} = \frac{2}{PC},$$

and therefore PQ is equal to half PC. Hence the required locus is parallel to CD and at a distance from P equal to half the distance of CD from P.

260. *Through a given point within a given circle, any transversal is drawn, and a point taken on it, such that the reciprocal of its distance from the given point is equal to the difference of the reciprocals of the intercepts between the given point and the circle; find the locus of the point of section.*



Let O be the centre of the given circle, P the given point, and CD its polar. Draw any secant BC through P, and take Q such that

$$\frac{1}{PQ} = \frac{1}{PA} - \frac{1}{PB}.$$

The locus of Q is required.

Because BC is divided harmonically in P and A, therefore (176, N.B.)

$$\frac{2}{PC} = \frac{1}{PA} - \frac{1}{PB}; \text{ therefore } \frac{1}{PQ} = \frac{2}{PC},$$

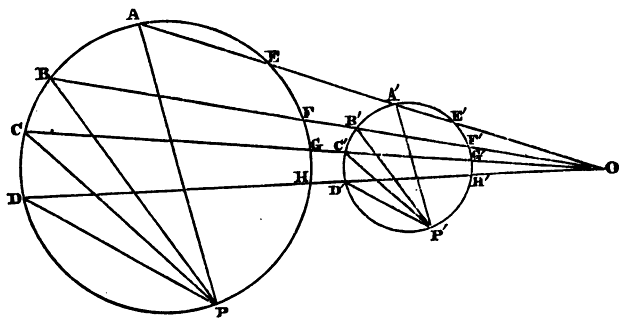
and therefore PQ is half PC. Therefore the required locus is a parallel to CD midway between P and CD.

261. *If through a fixed point without or within any number of given straight lines and circles, any transversal be* *drawn intersecting them, and a point taken on it, such that the reciprocal of its distance from the fixed point is equal to the excess of the sum of the reciprocals of the intercepts between the given point, and the lines and circles on one side of it over the sum of the reciprocals of the intercepts on the other side of it; find the locus of the point of section.*

By the last two problems, the circles can each be replaced by fixed straight lines. Hence the problem is reduced to (180) or (181), and the required locus is therefore a determinate straight line.

262. *If four secants be drawn through either centre of similitude of two circles, the anharmonic ratio of any four of the points where the secants cut one of the circles is the same as that of the four corresponding or non-corresponding points on the other circle.*

Let four secants be drawn through O the external centre of similitude, as in the figure, and let P, P' be corresponding points. Since the lines joining pairs of corresponding points are parallel (227), the rays of the two pencils $P.ABCD$ and $P'.A'B'C'D'$ are respectively parallel.



Therefore $\{P.ABCD\} = \{P'.A'B'C'D'\}$.

But since four secants are drawn through O meeting the circle ABC, therefore (255)

$$\{P.ABCD\} = \{P.EFGH\}.$$

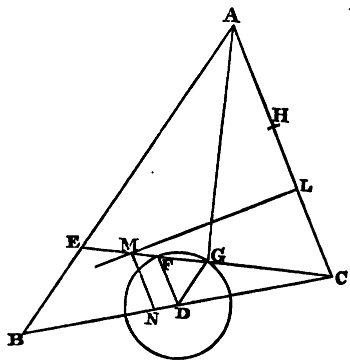
Therefore the anharmonic ratio of A', B', C', D' is the same as that of the four corresponding points A, B, C, D, and also of the four non-corresponding points E, F, G, H.

The proof is similar when the four secants are drawn through the internal centre of similitude. Q. E. D.

DEF. The pole of a straight line or the polar of a point, with respect to a given circle, is said to *correspond* to the line or point, and the centre of the circle is often called the *origin*, and the circle itself the *auxiliary circle*.

263. *Given the base and the difference of the sides of a triangle; the polar of the vertex with respect to one extremity of the base as origin always touches a fixed circle.*

Let BC be the given base, and ABC any triangle with the given difference of sides BE. Take C as origin, and let CH be the radius of the auxiliary circle. Let D be the middle point of



BC and AG the bisector of the vertical angle BAC. Then DG is half BE, and the locus of G is the fixed circle, with centre D and radius DG (23). Let this circle meet CE again in F, and let LM the polar of A with respect to the origin C meet EC in M. Draw MN parallel to AC. Then N is a fixed point, and NM is constant.

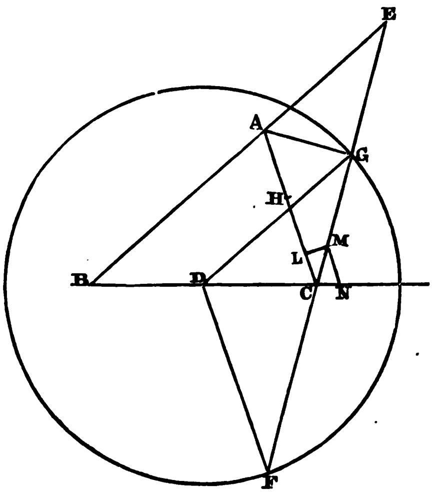
Because DF and DG are equal, the angles DFG and DGF are equal, but DG is parallel to AB, therefore the angles DGF and AEC or ACE are equal, and therefore DF and AC are parallel. Again, because the angles ALM and AGM are right, therefore $AC \cdot CL = MC \cdot CG$, but $AC \cdot CL = CH^2$, since LM is the polar of A with respect to the origin C; therefore $MC \cdot CG$ is given, but $FC \cdot CG$ is also given, since (D) is a fixed circle.

Therefore $MC \cdot CG : FC \cdot CG$, that is, $MC : CF$ is a given ratio, but MN and FD are parallel.

Therefore $NC : CD$ and $MN : FD$ are given ratios, and since CD and FD are given, therefore N is a fixed point and MN is of constant length. Also, since MN and AC are parallel and CLM is a right angle, therefore LMN is a right angle, and therefore LM always touches the fixed circle with centre N and radius NM. Q. E. D.

264. *Given the base and the sum of the sides of a triangle; the polar of the vertex with respect to one extremity of the base as origin always touches a fixed circle.*

Let BC be the given base and ABC any triangle, with the given sum of sides BE, C the origin, and CH the radius of the auxiliary



circle. Let D be the middle point of BC and AG the bisector of the external vertical angle CAE.

Therefore DG is half the sum of the sides, and the locus of G is a fixed circle with centre D and radius DG (24). Let this circle meet CG again in F, and let LM the polar of A with respect to the origin C meet CG in M, and draw MN parallel to AC. Because DG and DF are equal and DG is parallel to AB, therefore DF is parallel to AC.

Also, since the angles ALM and AGM are right, and LM is the polar of A, $GC \cdot CM = AC \cdot CL = CH^2$. Therefore $GC \cdot CM$ is constant, but $GC \cdot CF$ is also constant, since C is a fixed point and (D) is a given circle. Therefore $GC \cdot CM : GC \cdot CF$, that is, $CM : CF$ is a given ratio, but MN and DF are parallel, therefore

$$NC : CD :: MC : CF, \text{ and } MN : DF :: MC : CF.$$

Therefore N is a fixed point and MN is of constant length, since CD and DF are both given. Therefore LM always touches the fixed circle with centre N and radius NM. Q. E. D.

265. *If a circle touch two given circles (the nature of the contacts being assigned), the polar of its centre, with respect to one of the given circles, always touches a given circle.*

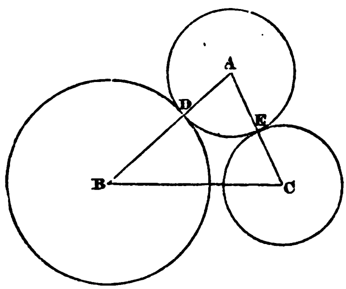


Fig. 1.

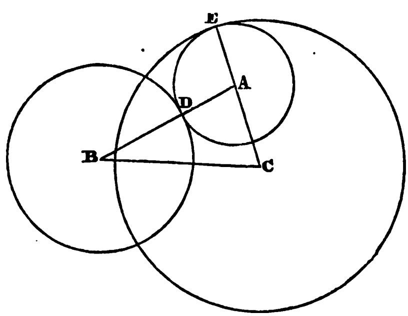


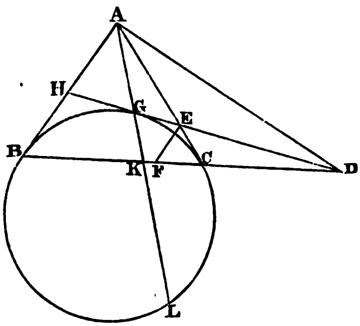
Fig. 2.

Let B and C be the centres of the given circles, and A the centre of the variable circle touching the others at the points D and E respectively.

In Fig. 1, it is clear that the difference of AB and AC is equal to the difference of the radii BD and CE of the given circles, and in Fig. 2, that BA and AC are together equal to the sum of the radii BD and CE of the given circles.

Therefore the theorem is reduced to this. *Given base* BC *and difference or sum of the two sides* AB, AC *of a triangle, the polar of the vertex* A *with respect to the circle* (C) *always touches a given circle;* which has been proved in (263) and (264). Q. E. D.

266. *If two tangents be drawn to a circle, any third tangent will be cut harmonically by its point of contact, the two former tangents and their chord of contact.*



Let AB, AC be the two fixed tangents, and let any tangent HD touch the circle at G and meet the chord of contact BC in D, then shall H, G, E, D form an harmonic range. Join AG, meeting BD in K.

Because G is the pole of HD, and A of BC, therefore (239) D is the pole of AG, and therefore DB is cut harmonically in C and K; therefore $A\,.\,BKCD$ is an harmonic pencil, and therefore HD is cut harmonically in G and E, which proves the proposition.

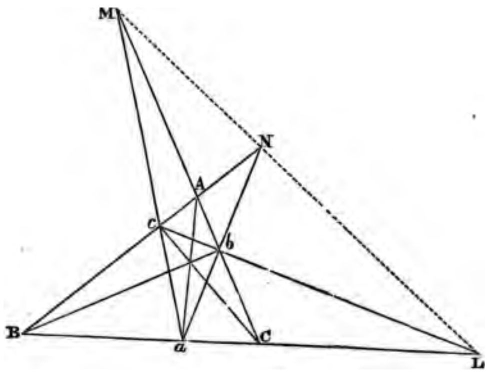
Or thus; draw EF parallel to AB, then, since ABC is an isosceles triangle, EF and EC are equal; also EC and EG are equal, and HG and HB.

Therefore $\quad HD : DE :: HB : EF,$

or $\quad HD : DE :: HG : GE.$

Therefore HE is cut internally in G and externally in D in the same ratio, and is therefore cut harmonically. Q. E. D.

267. *If the perpendiculars* Aa, Bb, Cc *be let fall from the angular points* A, B, C *of a triangle upon the opposite sides, prove that the intersections of* BC *and* bc, *of* CA *and* ca, *and of* AB *and* ab, *will lie on the radical axis of the circles circumscribing the triangles* ABC *and* abc.



Because the angles BcC and BbC are right, the four points B, C, b, c lie on the same circumference; therefore

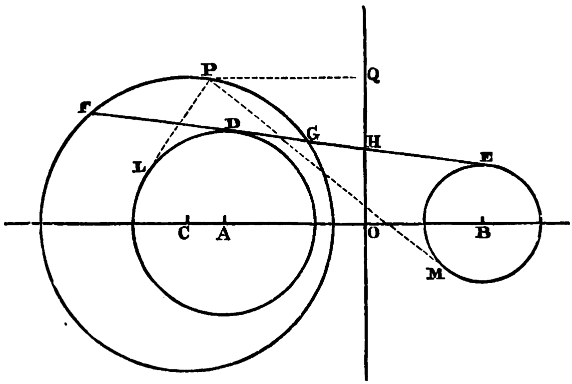
$$cL \,.\, Lb = BL \,.\, LC.$$

But $cL \,.\, Lb$ equals the square on the tangent from L to the circle described about the triangle abc, and $BL \,.\, LC$ equals the square on the tangent from L to the circle described about the triangle ABC, therefore L is a point on the radical axis of the circles about the triangles ABC and abc. Similarly, M and N are points on the radical axis of the same two circles. Q. E. D.

268. *A common tangent to any two circles is divided harmonically by any other circle having the same radical axis with the two given circles.*

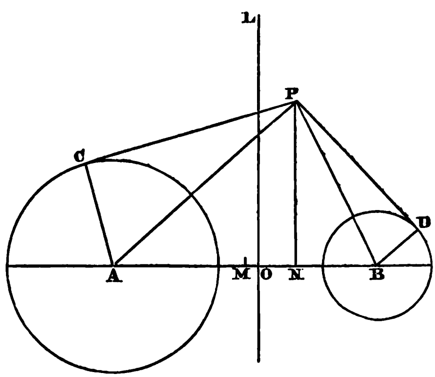
Let DE the common tangent to the two circles (A) and (B), meet the circle (C) in F and G, and the common radical axis OH in H, then shall F, D, G, E form an harmonic range.

Because OH is the radical axis, HD equals HE, and



$HG . HF = HD^2$, therefore (163) F, D, G, E form an harmonic range. Q. E. D.

269. *The difference of the squares on the tangents from any point to two circles is equal to double the rectangle under the perpendicular let fall from the point on their radical axis, and the line joining their centres.*



Let A and B be the centres of the two circles, LO their radical axis, and PC, PD tangents from any point P to the two circles.

Bisect AB in M (which will evidently lie between the centre of the greater circle and the radical axis), and draw PN perpendicular to AB. Then NO is equal to the perpendicular from P on LO.

Now $$PA^2 - PB^2 = 2AB \cdot MN.$$

But, since the angles PCA and PDB are right, the difference of the squares on PA and PB exceeds the difference of the squares on PC and PD by the difference of the squares on the radii, or by the difference of the squares on AO and OB.

But $$AO^2 - OB^2 = 2AB \cdot MO.$$

Therefore $PC^2 - PD^2 = 2AB \cdot MN - 2AB \cdot MO = 2AB \cdot NO.$

In the same manner the theorem can be proved for any other position of the point P. Q. E. D.

COR. Hence, if the point P be situated on either of the two given circles we have the following theorem.

If from any point on either of two given circles a tangent be drawn to the other, the square on this tangent is equal to double the rectangle under the perpendicular let fall from the point on their radical axis, and the line joining their centres.

This may also be proved independently.

270. *Given a system of three co-axal circles; if from any point on one, tangents be drawn to the other two, these tangents will be in a constant ratio.*

Fig. to (268). Let A, B, and C be the centres of the three given circles, and HO their radical axis. From any point P on the circle (C) draw the two tangents PL, PM to the other two circles, and PQ perpendicular to HO.

By (269, Cor.)

$$PL^2 = 2PQ \cdot CA, \text{ and } PM^2 = 2PQ \cdot CB.$$

Therefore $PL^2 : PM^2 :: CA : CB$, which is a constant ratio. Q. E. D.

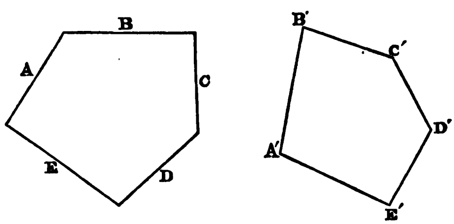
N.B. After what has been done, the learner should experience very little difficulty in solving the following problem, and constructing the locus.

Find the locus of intersection of tangents to two given circles, the tangents having a given ratio of inequality to each other.

The Method of Reciprocal Polars.

It will sometimes be found convenient to use the following notation. Let a straight line be denoted by a single letter A, and its pole by the same letter accented, viz. by A'. Also let the intersection of two lines A and B be denoted thus (AB), while the line joining the points A' and B' is denoted in the usual way by $A'B'$. In problems or theorems of position, as for example, where it is required to prove that any number of points lie in the same straight line, or that any number of straight lines pass through the same point, it is seldom necessary to actually exhibit the auxiliary circle with respect to which poles and polars are taken. Also, in forming a new figure by taking the poles of the sides of a given rectilinear figure, we may frequently disregard both the *relative* positions of the two figures on the paper and their *relative* magnitudes. Thus,

271. *Given a polygon* ABCDE; *construct another polygon* A′B′C′D′E′ *such that the vertices of each polygon shall be the poles of the corresponding sides of the other with respect to a given circle.*



Let A', B', C', D', E' be the poles of A, B, C, D, E, respectively.

Then, since the line joining the poles of two lines is the polar of their intersection (239), therefore $A'B'$ is the polar of (AB). Similarly, $B'C'$ is the polar of (BC), and so on. Therefore the polygon $ABCDE$ may be formed from $A'B'C'D'E'$ in the same manner as $A'B'C'D'E'$ was formed from it, viz. by joining the poles of the sides of $A'B'C'D'E'$ in order.

Hence the two polygons are called *polar reciprocals* of one another with respect to the auxiliary circle.

The process of generating one figure $A'B'C'D'E'$ from the other $ABCDE$, in this way, is called *reciprocating* ABCDE.

N.B. It is worthy of remark, that the theory of reciprocal polars *doubles* the number of theorems of geometry. Thus, when any theorem of *position* is once admitted as true, the consideration of reciprocal polars gives immediately another corresponding to the first.

This will appear more clearly from the examples of *reciprocation* which follow.

272. *The three perpendiculars of a triangle meet in the same point. By* reciprocating *this theorem deduce the following. If any point whatever be joined to the vertices of a triangle, and perpendiculars drawn to those joining lines, they will meet the sides opposite to the corresponding vertices in three points in the same straight line.*

Let the perpendiculars of the triangle ABC meet in I, and

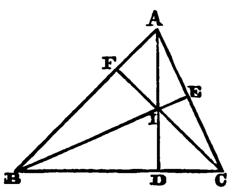


Fig. 1.

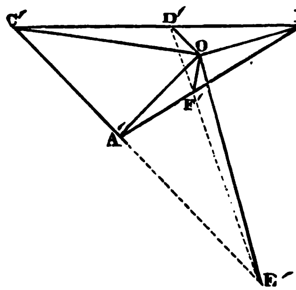
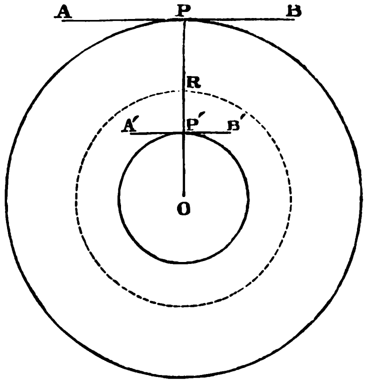


Fig. 2.

take A' the pole of BC with respect to the origin O, B' the pole of CA, and C' the pole of AB. Therefore (239) $B'C'$ is the polar of A, $C'A'$ of B, and $A'B'$ of C. Let $D'F'E'$ be the polar of I, therefore (239) D' is the pole of AI or AD. Similarly, E' is the pole of BE, and F' of CF. Again, because A' is the pole of BC, and D' that of AD, and that the angle ADB is right, and the poles of two lines subtend an angle at the origin equal to the angle between their polars (244), therefore the angle $A'OD'$ is a right angle, that is, OD' is perpendicular to OA', and it meets the side opposite to A' in D'. In like manner it can be proved that the angles $B'OE'$ and $C'OF'$ are right angles; also D', F', E' are in the same straight line, viz. the polar of I. Hence the reciprocal theorem enunciated above is true.

273. *Prove that the locus of the pole of a variable tangent to a given circle, with respect to its centre as origin, is a concentric circle.*



Let O be the centre of the given circle, AB a tangent to it at any point P; join OP meeting the auxiliary circle in R, and make $OP \cdot OP' = OR^2$, then P' is the pole of P.

Therefore OP' is constant. Hence the required locus is the concentric circle with radius OP'. Q. E. D.

N. B. If we draw the tangent $A'B'$ at P', it is obvious that P is the pole of $A'B'$ with respect to the auxiliary circle with radius OR. Hence the circle with radius OP can be generated from the circle with radius OP' in the same way as the latter was generated from the former. Thus each circle is the *polar reciprocal* of the other. It is evident that, when the auxiliary circle coincides with the given circle, the given circle and its polar reciprocal also coincide.

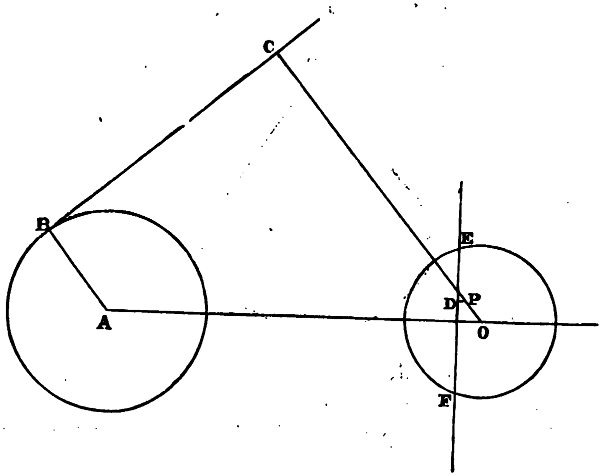
274. *If any tangent be drawn to a given circle, and its pole taken with respect to any origin; the distance of the pole from the origin is to its distance from the polar of the centre as the distance of the centre from the origin is to the radius of the given circle.*

Let A be the centre of the given circle, and BC any tangent to it.

Let O be the centre of the auxiliary circle, P the pole of

BC, and EF the polar of A. Draw PD perpendicular to EF, then shall

$$OP : PD :: OA : AB.$$



Because BC and EF are the polars of the two points P and A, therefore (256)

$$OP : OA :: PD : AB,$$

or, alternately, $OP : PD :: OA : AB$. Q. E. D.

N.B. We have seen in (273), that when the origin O coincides with A the centre of the given circle, the locus of P is a circle concentric with (A). When O does not coincide with A, the locus of P is an ellipse, hyperbola or parabola, according as the point O is within, without, or on the circle (A); but as the learner is supposed to be still unacquainted with these curves, and as their discussion is beyond the scope of the present work, I will not here further pursue this subject.

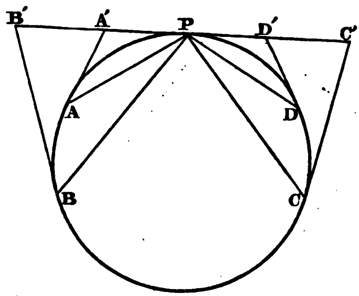
The learner will now see that, in reciprocating any figure partly composed of a circle, the centre of the circle must be taken as origin, in order to confine the reciprocal figure within the limits of the Elements of Euclid. It will generally be found most convenient to take the given circle as the auxiliary circle.

275. *Prove that* (189) *and* (191) *are polar reciprocals.*

I shall assume the truth of (189), and by reciprocating it deduce (191).

Let A, B, C, D be four fixed points on a circle, and P any variable point, then by (189),

$$\{P \,.\, ABCD\} \text{ is constant.}$$



Let the tangents at A, B, C, D meet the tangent at P in the points A', B', C', D' respectively.

By (238, *a*) A' is the pole of PA, B' of PB, C' of PC, and D' of PD, and by (245) the anharmonic ratio of four points in a straight line is the same as that of the pencil formed by their polars.

Therefore $$[A'B'C'D'] = \{P \,.\, ABCD\}.$$

That is, *if a variable tangent cut four fixed tangents, the anharmonic ratio of the four points of section is the same as that of the four points of contact.* Q.E.D.

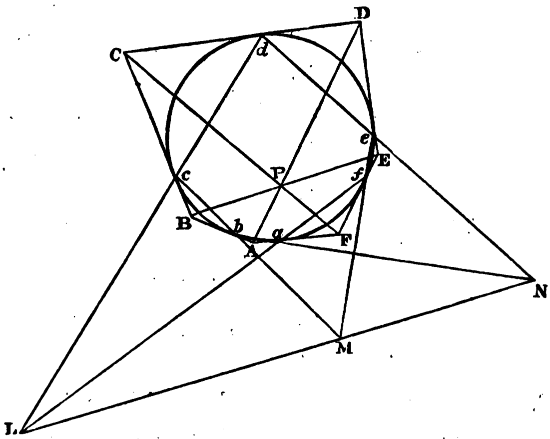
276. *By reciprocating Pascal's Theorem* (193), *deduce Brianchon's Theorem* (194).

Let *abcdef* be any inscribed hexagon, and let its opposite sides meet in L, M, N, then by Pascal's Theorem, LMN is a straight line.

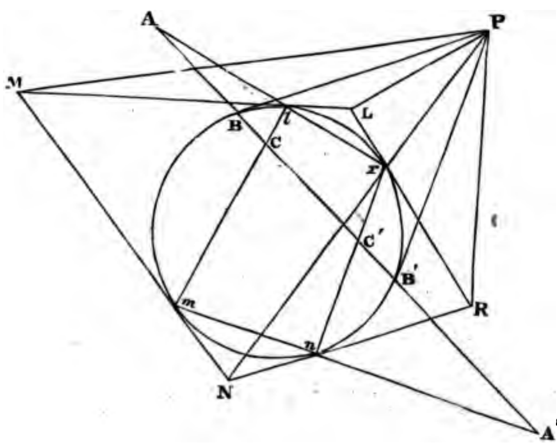
Form the circumscribed hexagon $ABCDEF$ by drawing tangents at the vertices of the inscribed hexagon. It is now required to prove that the three diagonals AD, BE, CF pass through the same point.

Because from the point L two secants Ld and Lf are drawn, and tangents at the extremities of the chords, therefore (238) CF is the polar of L. Similarly, BE is the polar of M, and AD the polar of N.

But the three points L, M, N are in the same straight line, therefore (239, β) AD, BE and CF pass each through P the pole of LN.



277. *Prove that if any point outside a circle be joined to the vertices of any circumscribed quadrilateral, and two tangents drawn from the point, these six lines form a pencil in involution.*



I will prove this theorem, by shewing that it is the polar reciprocal of (213).

Let P be the point outside the circle, and $LMNR$ the quadrilateral touching the circle at the points l, m, n, r. Draw the two tangents PB, PB'. Then shall PM, PB, PL, PN, PB', PR form a pencil in involution. Form the inscribed quadrilateral $lmnr$ by joining the points of contact of the circumscribed one, and let BB', the polar of P, meet its sides in A, A', C, C'. Therefore (213) the six points A, B, C, C', B', A' are in involution, and therefore their polars will form a pencil in involution.

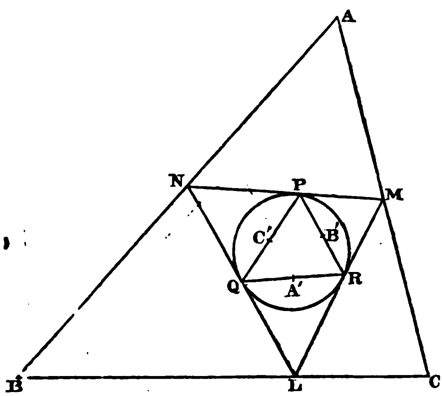
Now P is the pole of BB' and M of lm, therefore (239), PM is the polar of C. In like manner, it can be proved that PL is the polar of A, PN of A' and PR of C'.

Also PB is the polar of B and PB' of B'.

Therefore the six lines PM, PB, PL, PN, PB' and PR form a pencil in involution. Q. E. D.

278. *Describe a triangle which shall have its vertices on three given straight lines, and its sides tangents to a given circle.*

Let the three given straight lines form the triangle ABC, and take A' the pole of BC with respect to the given circle, B' the pole of CA, and C' the pole of AB. Inscribe in the given



circle the triangle PQR, having its sides passing through A', B', C' (145).

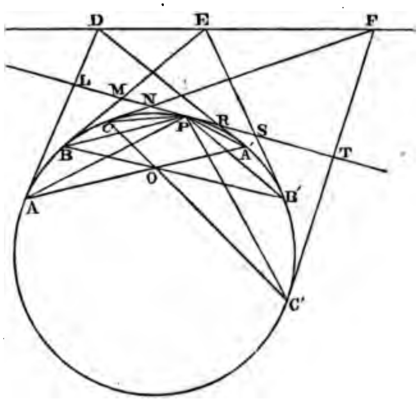
Draw tangents to the circle at the vertices of the inscribed triangle PQR. These tangents will form the required triangle.

Because AB is the polar of C', therefore (238) the tangents at P, Q will meet on AB, as at N. Similarly, the tangents at P and R will meet at M on AC, and the tangents at R and Q will meet at L on BC. Therefore LMN is the required triangle. Q. E. F.

N.B. Since (145), *in general*, two triangles can be inscribed in a circle having their sides passing through three fixed points, two triangles can generally be described as required in this proposition.

279. *Prove that* (246) *is the reciprocal of* (215).

Let the three chords AA', BB', CC' pass through the same point O, and let P be any point on the circumference, then by (215) $P.ABCA'B'C'$ is a pencil in involution, and therefore the poles of its six rays will be six points in involution. Let the

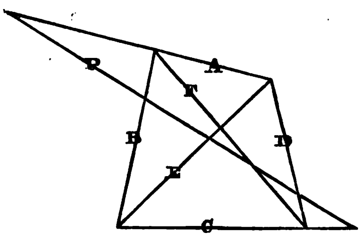


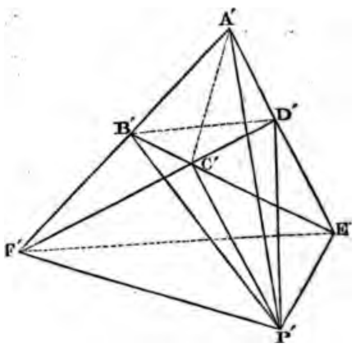
tangents at the extremities of AA', BB', CC' meet in D, E, F respectively; then (240) the points D, E, F lie on the polar of O. Therefore DEF is a straight line. Let the tangent at P meet the three pairs of tangents from D, E, F in the six points L, M, N, R, S, T; then these six points are evidently the poles of PA, PB, PC, PA', PB', PC' respectively. Therefore these points are in involution; that is, *Three pairs of tangents drawn to a circle from any three points* D, E, F *in a straight line, cut any seventh tangent* LPT *in six points in involution.* Q. E. D.

280. *Prove* (214) *by reciprocating* (212).

Let the sides and diagonals of the given quadrilateral be denoted by the single letters A, B, C, D, E and F respectively, and the transversal cutting them by P, then (PA) will denote the point in which P and A intersect. Let A' be the pole of A with respect to any circle, B' of B, C' of C and D' of D. Since the line joining the poles of two lines is the polar of their point of intersection, therefore $A'B'$ is the polar of (AB) and $C'D'$ the polar of (CD).

Therefore F' the point of intersection of $A'B'$ and $C'D'$ is the pole of F.





In like manner, it can be proved that E' is the pole of E, and that $E'F'$ is the polar of (EF).

Now, find P' the pole of P, and join P' to A', B', C', D', E', F'.

Then $P'A'$ is evidently the polar of (PA), $P'B'$ the polar of (PB), and so on.

Hence the pencil $P'.A'B'C'D'E'F'$ is formed by the polars of the six points in which the transversal P cuts the sides and diagonals of the quadrilateral $ABCD$, and since these six points of intersection are in involution by (212), therefore the pencil which is formed by their polars is also in involution, which proves (214), since the pencil is formed by straight lines drawn from any point P' to the extremities of the three diagonals of a complete quadrilateral. Q. E. D.

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