Everymind's Euclid

Euclid's Elements
Books I and II

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Comments, corrections, and criticisms are welcomed.

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Dedication

This book expresses my gratitude for the work and wisdom and, in some cases, wit of:

Euclid of Alexandria

Augustus De Morgan

Isaac Todhunter

Elias Loomis

Sydney Luxton Loney

and pretty much in that order, the top two running neck and neck.

And this book would not have been possible without the Internet Archive and the digitization of the world's libraries by giant corporations in their moment of uncustomary altruism.

Finally, I have taken the diagrams that accompany the propositions from Todhunter's Euclid, just as Lewis Carroll did for his Euclid I & II. The propositions' proofs are largely Todhunter's versions of Euclid's, condensed by my notation. The problems, excepting a handful, are from the first and larger part of the problems at the back of Todhunter's 1867 edition. The correct English in my text is due to my wife Ann having lovingly proofread the manuscript. Everything else is mine own. Mea culpa.

R. Earle Harris 2018

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Instructions

For Learners

Your mind, just as it is, can learn to comprehend Euclid. This book is designed for individuals who want to learn. So I will do all I can to help you learn Euclid on your own. I could tell you how all the great mathematicians love Euclid and why. But after you have experienced this book, you will see Euclid everywhere and when you find that great thinkers love Euclid as you do, you will rejoice in their company. Or I could tell you all the things Euclid will do for you. But I will say only this: you will realize that your consciousness is more powerful than you thought it was. After working through Todhunter's Euclid -- every page, every problem -- I found multi-page algebraic computations much easier and found reading and understanding difficult authors, Kant or Spinoza for example, much easier, too. Whatever you do with thought will benefit from attaining an understanding of Euclid.

I am myself self-taught in Euclid. If I told you all the things that I have studied on my own to the extent of being competent at them, you would think I was either bragging or lying. But I just like to learn. Euclid is one thing I've learned. So I am presenting Euclid in a manner informed by all my self-learning -- enhanced by what I have learned about teaching self-learners from Augustus De Morgan and Isaac Todhunter, both of whom loved self-driven thinkers. You should read every page of this book, write down every proposition and its proof, work every problem, study every solution. Just as Euclid builds from simplicity to complexity, I will do the same. Of axioms and such, I only present what you need as you need it, listing them all in the back for reference. I develop a clear shorthand of notation as we go. And problems are presented in the best way I know how, which I will describe as we reach the first ones.

For Teachers

This text could be used in a classroom. If you would teach five propositions a week, speeding though the auxiliary ones, it would fit a semester. If I were to teach it, I would spend ten or fifteen minutes covering the propositions for the day on the board and then help the students work on a problem until the last ten minutes, when I would share the solution with them and let them grade their own work by how much they grasped. If this were done in the right spirit, the students would learn mathematical maturity, which is simply honest judgment and integrity, and the teacher and students could go home with no math or grading to do.

I would have the students create a notebook from what I wrote on the board each day as their textbook. All problem-solving would be open-notebook except on Fridays. And on Thursday, I would tell them that Friday's "test" would be one of three problems. They could go home and solve or simply study the solutions of the test selection problems. They could even conspire to do it in groups. Traditionally, students had to memorize Euclid and cough it back up on exams. I think they would learn more by studying three problems a week so they could cough those back up. The end result would be that each student would copy two books of Euclid into a notebook, spend part of each day learning to solve problems, and study over thirty problems well enough to cough them up.

All of this would exercise their mind. And Euclid is about much more than geometry. It is about judging your own work, making your assertions support your claims, and learning how to stare into the blackness of a problem until you can reach up to the light. Teach these things and geometry will follow.

Euclid - Book I

Most Euclid texts simply state the propositions and their proofs, note a few mathematical things in passing, hit you with some problems, perhaps offer some solutions, and they're done. We're not doing things that way. For starters, most Euclids begin by listing all the axioms, postulates, and definitions when you hardly need any to start and some have never been used. I will supply these things as needed. But we should first spell out the

Guiding Principle of Euclidean Geometry

We consider only lines made with a straight-edge, curves made with a compass, and their relations. Neither the straight-edge nor the compass may be used for measurement.

And these are the practical rules for this principle:

Euclid's Postulates

1. A line may be drawn between any two points.
2. A line may be indefinitely extended.
3. Any point and any line from it may be used to construct a circle.

I should warn you that every other geometry text suffers from someone having decided that a "line" (except for a circular arc) could be any doodle between two points and that it was necessary to define a "straight line", which they did awkwardly, thereby cursing all of posterity into writing "straight line" instead of "line" every time a line came up. Do not blame Euclid -- he scratched all his lines in the dirt with a ruler. But you may have to write "straight line" in public.

All we need for Book I Proposition 1, besides the postulates, is a few definitions and an axiom. In our notation, the first proposition in Book I is 1.1, the first axiom is a.1, and the fifteenth definition of Book I is d.1.15. So:

a.1 Things which are equal to the same thing are also equal to one another.
d.1.15 A **circle** (⊙) is a plane figure bounded by its **circumference** which is equidistant (eqD) from its **center**.
d.1.20 A **triangle** (Δ) is bounded by three lines. Any of its angular points or **vertices** can be its **apex** which is opposite its **base**.
d.1.23 An **equilateral triangle** (eqSΔ) has three equal sides.

Now we need to backfill but only to clarify thought:

d.1.13 A **plane figure** is any shape enclosed by lines, which are its **perimeter** or **boundary**.
d.1.7 A **plane** is a surface such that, for any two points, their line lies entirely on the surface.
d.1.6 The **boundaries** of surfaces are lines.
d.1.5 A **surface** is length and breadth.
d.1.3 The **extremities** and **intersections** of lines are points
d.1.2 A **line** is length without breadth.
d.1.1 A **point** is position without magnitude.

In our notation, we use capital letters. Points are single letters. For points on or ending lines, we use A, B, C, ... and for points on their own P, Q, R, ... with O used for the center of figures. Lines are usually two letters, such as AB, where A and B are its endpoints. For any old triangle, we use Δ followed by its vertices, as in ΔABC. The first letter is the apex, the next two are the left and right endpoints of the base. In general, we label all points, whatever they belong to, from top to bottom and left to right. An equilateral triangle is eqSΔ, S for "sides". We have eqD for "equidistant" and eq∠ for "equiangular".

Circles are usually ⊙A,AB, where the point is its center and the line is its radius. Euclid never defines radius beyond that "any line from it" in postulate 3 (p.3). But we will use the term. We speak of an existing circle using only its center (⊙A).

I should also point out that Euclid's Elements are not Euclid's. He was not the ancient world's finest geometer. He was the ancient world's finest organizer and harmonizer and a decent number theorist. He took most of the geometry up to his time and organized it so that it built from a single first proposition into an edifice that continues to grow and forever will. He also standardized the form of geometrical proofs. Here is his Book I, Proposition 1, in the form he gave it, from Heath's translation of the Greek:

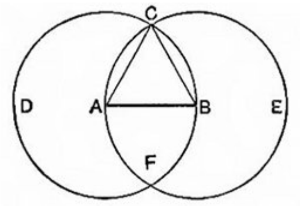
Proposition 1

On a given line to construct an equilateral triangle.

Let AB be the line. Thus it is required to construct an equilateral triangle on line AB. With center A and distance AB let the circle BCD be described (p.3); again, with center B and distance BA let the circle ACE be described (p.3) and from the point C, in which the circles cut one another, to the points A, B let lines CA, CB be drawn (p.1). Now since the point A is the centre of the circle CDB, AC is equal to AB (d.1.15). Again, since the point B is the centre of the circle CAE, BC is equal to BA (d.1.15). But CA was also proved to be equal to AB; therefore each of the straight lines CA, CB is equal to AB. And things which are equal to the same thing are also equal to one another (a.1); therefore CA is also equal to CB. Therefore the three lines CA, AB, BC are equal to one another. Therefore the triangle ABC is equilateral; and it has been constructed on the line AB. Being what it was required to do.

That's it. No diagram. Just a lump of text. You're on your own with ruler and compass in 350 B.C. It became better by 1867. Isaac Todhunter's Proposition 1 came with a diagram as you will see on the next page.

Proposition 1. Problem



To describe an equilateral triangle on a given straight line.

Let AB be the given straight line: it is required to describe an equilateral triangle on AB.

From the centre A, at the distance AB, describe the circle DCB. [Postulate 3.
From the center B, at the distance BA, describe the circle ACE [Postulate 3.
From the point C, at which the circles cut one another, draw straight lines CA and CB to points A and B [Postulate 1.
Because the point A is the centre of circle BCD, AC is equal to AB [Definition 15.
And because the point B is the centre of circle ACE, BC is equal to BA [Definition 15.
But it has been shown that CA is equal to AB;
Therefore CA and CB are each of them equal to AB.
But things which are equal to the same thing are equal to one another [Axiom 1.
Therefore CA, AB, BC are equal to one another,
Wherefore the triangle ABC is equilateral, [Definition 23.
and it is described on the given straight line AB [Q.E.F.

You can see the text is more organized after 2210 years. Loney's reworking of the last Todhunter Euclid in 1899 put every proposition either on one page or on facing pages. Todhunter put all the problems at the back of the book, ordered by groups of propositions from each book, graduated from easy to hard. Loney pulled the problems up to the propositions that made them possible.

But mainly, you can see that the diagram is the real improvement. Immanuel Kant described mathematics as "the science of diagrams." This is true and is the key to grasping Euclid. As you work through this book:

- write down each proposition
- copy each diagram
- write down the proof and absolutely follow the logic

Otherwise, unless your name is Ramanujan, you are missing half the book. We don't read mathematics. We study and comprehend mathematics. But back to the diagram. Everything we know about the objects under consideration is put into each diagram. Then we can consider what our knowledge implies. The power of the diagram is that it can be taken in as a whole. And its very existence suggests its implications.

In a sense, algebraic notation is also a diagram, in that it can be taken in very quickly -- certainly more quickly than the original syncopated algebra, which, like Euclid's 1.1, describes everything at length in the vernacular. Which makes it worse than Euclid: try describing x^2+2x+1 algebraically, with all its meaning, in words. Our notation will allow us to abbreviate Todhunter's version while making it so clear that we can take it in almost at a glance and read it off with ease.

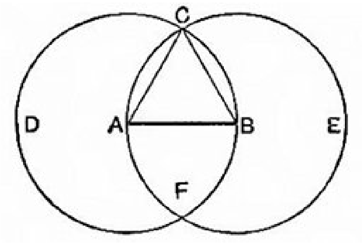
For our 1.1, we need a little more notation. We will use the multiplication sign (×) for intersection and "@" for "at". So "line AB intersects line CD at point E" becomes "AB × CD @ E". We will use "∴" for "therefore". "∀" means "any", "every", or "all", which are logically the same. When creating a line between A and B, we say "Join AB". When one argument is the same as the prior one, we can use "Sym." or "symetrically" to shorten the second one. And our conventions for circles allow us to never to write "circumference": if anything touches the center, it is "on center", if it touches the circumference, it is "∈⊙" or "on circle". And if it is in the circle's whitespace, it is "in ⊙". "Touch" in Book III means "tangent to". Here it just means "touches in any way." And there are two kinds of equality: equal in magnitude (quantity) "=" and equal in every way "≡".

I think we have everything we need for:

Proposition 1. Problem

Given: ∀line AB,

Required: eqS∆ on AB



Method

⊙A,AB × ⊙B,AB @ C,F (p.3, d.1.15)

Join AC,BC. (p.1)

∆ABC required

Proof

AB, AC radii ⊙A ∴ AB=AC (d.1.15)

Sym. for ⊙B, BC=AB

∴ AC=BC (a.1) ∴ AB=BC=AC

∴ ∆ABC ≡ eqS∆ on AB (d.1.23)

I am using Todhunter's 1867 diagrams. They may have more letters than we need and will not always follow our labelling conventions. Even without a diagram, our conventions tell us that if AB is horizontal, C is above F, and if vertical, to the left of F. We will rarely use Todhunter's "circle ACE" (⊙ACE) where the letters are points on circle. But in Book III, we learn that any three non-colinear points define a circle. A proposition is either a "theorem" proving something is true or a "problem" proving something can be constructed. Sym. "axioms" are true, "postulates" constructible; both are unproven and we are required to accept them without murmuring.

I have already told you how to miss out on half of this text. Let me tell you how to miss out on the other half.

- Do not work the problems. Actually, if you skip the problems, you will learn almost nothing. If you are going to do the problems, make sure you check the Problem Diagram appendix. This will correct your diagram if it is wrong before you use it and verify that your labels match those in the solution.

- Do not check the Problem Diagram appendix. Working hard on the wrong diagram is a good way to ruin your mood for the day. And having your labels different from those in the solution is a frustrating exercise in remapping your solution to the text's.

- Do not use the Problem Hints appendix. Every problem has a hint. If you can't come up with a bright idea in fifteen minutes, you may pass over into the dead zone. Todhunter notes that self-learners tend to look up the answer too late, rather than too early. Go ahead -- burn yourself out. Using the hint, you shouldn't spend more than ten minutes trying for a bright idea either. If you can't get out of the darkness, give it up and study the solution.

- Do not check your solutions in the Problem Solutions appendix. If you have a solution, check to see if it is correct. If you are sure yours is correct and the text's is different, think hard about your solution. Two or three times, mine has been just as right as Todhunter's. The other several dozen times, I was wrong. Copy the solution just as you copy the propositions and their proofs -- thoughtfully. If you simply wrote down each of the 162 problems and then carefully studied and wrote down each solution, you would learn a great deal. And if you went back and tried to solve them all, you'd find you had forgotten most of the solutions but had learned the tools to begin solving them with. Also, this appendix furthers our notation and notes what you should be getting out of all your hard work.

You will grasp the notation more quickly if you use it as instructions to create your diagram with a ruler and compass rather than copying the diagrams directly. A diagram must be relatively accurate in order to suggest its actual implications.

Finally, for the 1.1 problem, you will need:
d.1.33 A **rhombus** is an eqS 4-gon with no right angles (∟)
d.1.22 A **polygon** or **n-gon** is a plane figure with n lines for sides.
A figure with four sides is a 4-gon or "quadrilateral".
Euclid uses "polygon" for five or more sides and then gives them names with too many letters, just like "quadrilateral".

Problems

1. Problem

Construct a rhombus.

Okay. Now go do the problem, check the diagram, use the hint if necessary, and check your solution. The solution will often add a bit to our notation. I'll wait here.

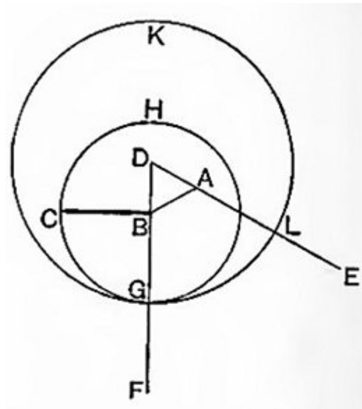
Welcome back. As I said in the hint, you have only one tool. Not only that, you have all of it. And you have everything there is to know about proposition 1.1. This is how mathematics is. It is not a big, dark room, where what you understand amounts to tiny dots of light. It is, for each of us, the sum of the simple things we know. And you know all there is to know about any line AB: it is the endpoints A, B and the straight line between them. You know everything there is to know about some ΔABC: it is made up of the lines joining those points and that's all. You will learn a great deal about the relations of these objects. But you can't know those until you are told about them, unless you are going to re-invent every wheel -- which most of us can't. Truly, at every point, you know all you need to know. Euclid's grace is sufficient for thee.

What you don't have is experience in deciding how to use what you know. The only way to gain the necessary experience is to try to solve the problems and then to study their solutions. And do not concern yourself with comparing your abilities with those of other people. "You are alone with your own being and the reality of things."

Proposition 2. Problem

Given: ∀point A, ∀line BC, A∉BC

Required: Line on A = BC



Method

Join AB (p.1)

On AB, eqSΔDBA (1.1)

DA(pr) to E, DB(pr) to F (p.2)

⊙B,BC × DF @ G (p.3)

⊙D,DG × DE @ L (p.3)

AL = BC required

Proof

BC,BG radii ⊙B ∴ BC=BG (d.1.15)

Sym. ⊙D, DL=DG (d.1.15)

DA=DB (d.1.23) ∴ DL - DA = DG - DB (a.3) ∴ AL=BG

BG=BC ∴ AL=BC (a.1)

P∉BC means "P is not on BC". P∈BC would mean "P is some point on BC". Postulate 2 says that lines can be extended indefinitely. When we do so, we "produce" them. In our notation, "Produce AB to C" is "AB(pr) to C". To go the other direction, "BA(pr) to C".

For the problems we need:

d.1.24 An **isosceles triangle** (isosΔ) has two equal sides.

Problems

2. Problem

Given: ∀AB,CD: CD > AB

Required: isosΔ on AB w/sides=CD

3. Problem

Given: data of proposition 1.2

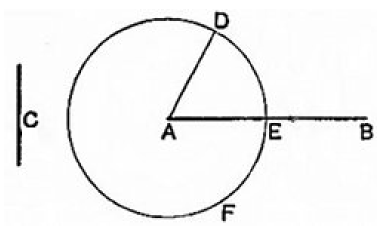
Required: Place A and alter the method of 1.2 such that both circles have the same radii.

Real problems are those for which no solutions are given. It is not mathematics to do a thing and expect someone to tell you it is right or wrong. Who, pray tell, would perform this service for you?

In any mathematical activity, you are aware of knowing a thing is true, of knowing you are unsure, or of knowing that you are certainly wrong. The middle one comes with a spectrum of uncertainty.

You must make each problem your own and take responsibility for it. You must work from what you certainly know and establish certainty where you are in doubt. No one can do this for you or give it to you. This is what mathematicians call "mathematical maturity" and some mathematicians never acquire it. You can acquire it right now. It is, in a real sense, a moral choice.

Proposition 3. Problem



Given: lines AB,C: AB > C
Required: AB - C

Method

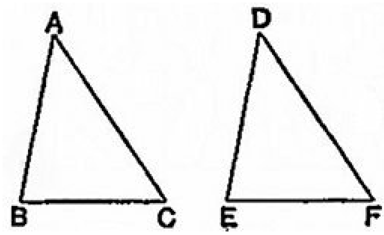
Copy C to A as AD (1.2)
⊙A,AD × AB @ E (p.3)
EB required

Proof

AD,AE radii ⊙A ∴ AE=AD (d.1.15)
AD=C (con) ∴ AE=C ∴ EB = AB - C

a.3 Things taken from equals leave equals.
These "problem" propositions show what can be constructed in Euclid's geometry. For the Greeks, the Guiding Principle was taken seriously. Now, each construction is a permission to perform some action in our diagrams. So after 1.2 we can measure a line with our compass and copy it to some point. After 1.3, we can use our compass to make one line equal to another. As we go along, I will point out which constructions are actually better than estimations for practical work with problems. I recommend doing the constructions for the proposition diagrams through 1.22. After you have moved three lines around to build a triangle and seen what a mess diagrams can be, you will have enough experience to

construct anything you think is necessary. And there **are** problems for which an inaccurate diagram cannot imply the truth. But they are rare. In about eight hundred problems, I have encountered two or three of these. And there are none in this text, if your logic is good.



Proposition 4. Theorem

If two triangles share any two sides and their included angle, the triangles are equivalent.

∀ΔABC,DEF: if AB,AC = DE,DF

∠A = ∠D, then ΔABC ≡ ΔDEF

Proof

Let ΔABC be applied to ΔDEF, with A on D, AB on DE.

AB=DE ∴ B on E and AB on DE and ∠BAC=∠EDF (hyp)

∴ AC on DF

AC=DF ∴ C on F

B on E ∴ BC on EF

Else two lines enclose a space ↴ (a.10)

∴ BC on EF and BC=EF (a.9)

∴ ΔABC coincides with ΔDEF (a.9) and ∠ABC,ACB =∠DEF,DFE

∴ ΔABC ≡ ΔDEF

a.9 Magnitudes which can be made to coincide are equal.

a.10 Two lines cannot enclose a space. They must have 0, 1, or all points in common.

Let us make something clear. In d.1.20, any vertex of a triangle can be on top and its opposite side is the base. So ΔDEF can be on any of its three "bases" and still be equivalent to ΔABC. We can rotate ΔDEF in any way and ΔDEF ≡ ΔABC. But what if we flip ΔDEF over or "reflect" it so that its labels read ΔDFE? We still have the same labelled sides and angles equal to each other and so ΔDFE ≡ ΔABC. These relations are true in all such propositions (1.8,26) and for all Euclidean figures.

We can show several equalities in our notation by stacking them. For example if AB = CD and EF = GH, we have AB,EF = CD,GH. The (hyp) means "by hypothesis" meaning a part of our assumptions which we began with. And this laying of one triangle on top of another is called **superposition**. If it seems sketchy as a proof, Euclid didn't like it either. And it is never a method of solution.

The "↴" means "contradiction" and goes with the "Else". This is proof by contradiction or "reductio ad absurdum". Basically, to prove one thing true, you assume its opposite and show that assumption leads to contradiction or impossibility. Euclid uses this approach reasonably often.

For the problems, in our notation, we modify intersect "×" for bisect and bisector "×/2". If AB is at right angles to CD, we write "AB⊥CD". Colons (":") can be read "such that".

A few more axioms:
a.2 Things added to equals make equals.
a.6 Things twice the same thing are equal to each other.
a.7 Things half the same thing are equal to each other.

Problems

4. Theorem

∀AB,CD: if AB ×/2 CD: AB⊥CD, then ∀P∈AB is eqD C,D
(Any lines AB and CD: if AB bisects CD such that AB is perpendicular to CD, then any P on AB is equidistant to both C and D)

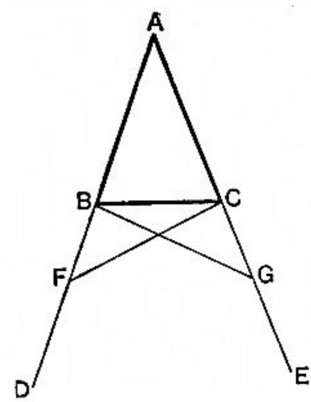
5. Theorem

∀ 4-gonABCD: AB=AD, AC ×/2 ∠BAD
Then 1) CB=CD 2) AC ×/2 ∠BCD
(Any quadrilateral ABCD such that AB equals AD and AC bisects angle BAD then 1) CB equals CD and 2) AC bisects angle BCD)

6. Theorem

∀eqSΔABC, if eqSΔABF,BCD,CAE added, then AD=BE=CF

Proposition 5. Theorem



∀isosΔABC: AB=AC AB(pr) to D, AC(pr) to E, then ∠B=∠C and ext∠B=ext∠C (or ∠CBD=∠BCE)

Proof

∀F∈BD, Copy AF to AG∈AE (1.3)
Join FC,GB
ΔAFC,AGB: AF=AG (con),
AB=AC (hyp), ∠FAG = ∠FAG (a.1)
∴ΔAFC≡ΔAGB (1.4)
∴FC=GB, ∠ACF=∠ABG, ∠AFC=∠AGB
AF=AG, AB=AC ∴AF - AB = AG - AC (a.3) ∴BF=CG
ΔBFC,CGB: BF=CG, FC=GB, ∠BFC = ∠CGB (proven)
∴ΔBFC≡ΔCGB (1.4)
∴ ∠BCF = ∠CBG, ∠FBC = ∠GCB (ext∠B=ext∠C)
∴ ∠ABG - ∠CBG = ∠ACF - ∠BCF (a.3)
∴ ∠ABC = ∠ACB (or ∠B=∠C)

Corollary 1

∀eqSΔ is an eq∠Δ

For clarity, the external angle of ∠ABC (ext∠ABC) here is ∠CBF and vice versa. The angle and its external angle sit on the same line so their sum is 2∟. I marked a line "(proven)" in order to point out that, for instance, when we prove ∠AFC=∠AGB, we've proven ∠BFC=∠CGB. You see this naturally when you write a proof. But when you read one, it's confusing unless you look back to see how a current equality refers back to a prior, perhaps different, one.

A "corollary" is a theorem that immediately, or with trivial additions, follows logically from the proposition itself. Corollaries, like the above, are denoted as 1.5.C1 in our notation.

Proposition 1.5 is known as the Bridge of Asses (pons asinorum). Let me explain. Euclid taught and wrote in Alexandria. As time went on, the Christians and then the Muslims burned the libraries there. In both cases, those cultures had reached that point where if a book wasn't scripture, the people in power destroyed it. The Muslims commandeered a bathhouse near the main libraries and kept it roaring for days, burning the books. Burning books, in whatever culture, goes with killing intellectuals. The Muslim scholars, the premiere intellectuals of their age, fled to Europe with their beloved books.

Let me digress a moment. People speak of the "Classics" as if they were the arbitrary choices of aged white intellectuals. Nothing could be further from the truth. When one becomes homeless, one keeps what is most valued. I can vouch for this. So every time barbarians roar across the border or consume their own culture, some of the world's libraries are reduced and refined to what one carries on one's back. Thus was Euclid (and every other great work) preserved as a "Classic." End digression.

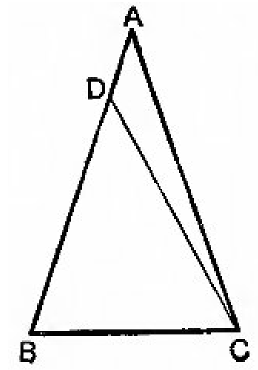
So Euclid's Elements comes to Europe and is assimilated into the Roman Church's universities. At that time, students were required to learn Euclid all the way through Book 1, Proposition 6. Poor babies, how ever did they make it? Proposition 5 is the hardest of the half-dozen and many "scholars" failed to cross the bridge. In 1899, we still have Loney writing, "This proposition is often found hard for beginners." Oh, please.

The whole proof comes down to this. It uses 1.4 twice to equate big angles ∠ABG,ACF and then small angles ∠BCG,CBF. In doing so, it picks up the second target of the proof, the equality of the ext∠B,C. Then it subtracts small from big to hit the first target, the equality of the internal base angles ∠B,C.

Another axiom: a.8 The whole is greater than its part.

Proposition 6. Theorem

∀ΔABC, if ∀2 angles (∠B,∠C) are equal, then their opposite sides (AC,AB) are equal.



Proof

Else in ΔABC, let ∠B=∠C and AB≠AC.
Then one side is greater. Let AB > AC.
On AC, make DB=AC (1.3) Join CD (p.1)
ΔABC,DBC: DB=AC (con) BC=BC (a.1)
∠DBC = ∠ACB (hyp)
∴ΔABC≡ΔDBC (1.4)
The less equal to the greater ⦣ (a.8)
∴AB=AC

Line 5 is a twist on using 1.4 to show triangles are equivalent. You would normally state here that ∠DBC=∠ABC. But back in the data of the hypothesis, we have ∠ABC=∠ACB. Euclid substitutes one for the other to make his point. This is another proof by contradiction and letting the lesser equal the greater is his most common ploy in this type of proof. Proposition 1.6 is the converse of 1.5. There we have: If sides are equal then base angles are equal. Here we have: If base angles are equal, then sides are equal. Converses are **usually** false. Consider: If you live in Uganda, then you live on Earth.

Problems

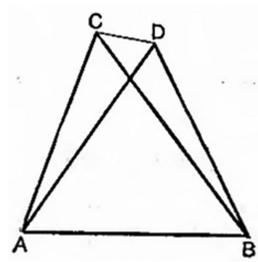
7. Theorem

∀isosΔABC, if ×/2∠B × ×/2∠C @ D
(if bisector of angle B intersects bisector of angle C at D)
Then ΔDBC≡isosΔ

I should also point out that Euclid sometimes talks about a triangle's three sides and sometimes distinguishes between two sides and a base. I have tried to make this clearer than in older Euclids. But you still have to determine this for yourselves. Missing this distinction can cause confusion.

Proposition 7. Theorem

∀ΔCAB,DAB sharing same side of base AB.
If CA,CB = DA,DB then ΔCAB ≡ ΔDAB.



Proof

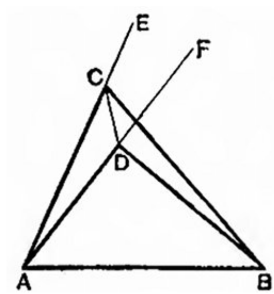
Else ΔCAB ! ≡ ΔDAB

Case 1: D outside ΔACB.

AC=AD (hyp) ∴ ∠ACD = ∠ADC (1.5)
∠ACD > ∠BCD (a.8) ∴ ∠ADC > ∠BCD
∴ ∠BDC > ∠ADC > ∠BCD
BC=BD (hyp) ∴ ∠BDC=∠BCD (1.5) ↴
∴ ΔCAB ≡ ΔDAB

Case 2: D in ΔACB.

AC(pr) to E, AD(pr) to F
AC=AD ∴ΔACD: ∠ECD = ∠FDC (1.5)
∠ECD > ∠BCD (a.8) ∴ ∠FDC > ∠BCD
∴ ∠BDC > ∠FDC > ∠BCD
BC=BD (hyp) ∴ ∠BDC=∠BCD (1.5) ↴
∴ ΔCAB ≡ ΔDAB



Case 3: D on ΔACB

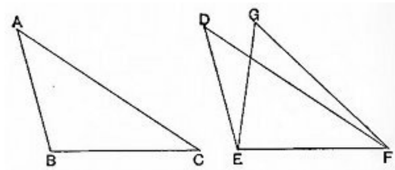
Contradictory on inspection.
∴ΔCAB ≡ ΔDAB

In these proofs, contradictions are not referenced to axioms or propositions because in each case we show two things unequal and then equal, which is contradictory. The "!" means "not" in "! ≡ ".

This proposition 1.7 is really only a lemma for 1.8. A **lemma** is smaller proof used within other larger ones. Mathematicians have shown 1.7 could have been easily proven within 1.8 by using a different approach to that proof. But I think that Euclid considered this proposition important, as he did 1.4. The earlier one states that if two triangles are equivalent, one would precisely cover the other on its base. This states that no other triangle can be present on that base. Greek mathematics dealt with physical bodies and these propositions nail down what bodies can exist on the same base.

Proposition 8. Theorem

∀ΔABC, DEF: if AB=DE,
AC=DF, BC=EF, then ∠A=∠D



Proof

Let ΔABC be applied to ΔDEF:
B on E, AC on DF, ∴BC=EF (hyp)
∴C on F
∴AB,BC,CA on DE,EF,FD.
Else they differ as in EG,GF.
∴ΔDEF,GEF: DE,DF = GE,GF share same base. ⤵ (1.7)
∴All sides coincide and ∠A = ∠D.

Corollary

ΔABC coincides with ΔDEF, ΔABC≡ΔDEF

It is weird that Euclid's theorem here isn't about equivalent triangles. One could take each side in turn and the proof is the same. This means that here the angles, not the triangles, were important to Euclid. Our mathematic views it the other way round and we simply quote 1.8 as a proof that if two triangles have equal sides, then they are equivalent. No one mentions the corollary by name.

Problems

8. Theorem

Opposite angles of a rhombus are equal

9. Theorem

Diagonals of a rhombus bisect opposite angles

10. Theorem

ΔABC,DBC on same side BC: if AB=DC, AC=DB, AC × BD @ E
Then ΔEBC≡isosΔ

11. Theorem

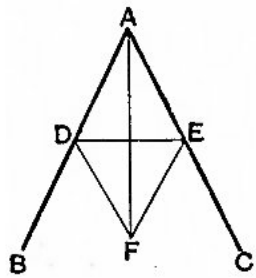
∀isosΔABC,DBC on same side BC, then AD(pr) ×/2 BC

Proposition 9. Problem

Given: ∀∠BAC

Required: Bisect ∠BAC

(divide into two equal angles)



Method

∀D∈AB, AE∈AC AE=AD (1.3)

Join DE. On DE, opposite side of A,

eqSΔDEF (1.1)

Join AF. AF required

Proof

ΔDAF,EAF: AD=AE (con) AF=AF and DF=EF (d.1.23)

∴ ∠DAF=∠EAF (1.8)

One of the problems below is a theorem for any triangle (∀Δ). So let's take a moment for a lesson in drawing "any" diagrams. In general, you want your diagrams three or four notebook lines tall, depending on how much you have to write inside them.

∀Δ: Make a steep AB. Using an inch or cm mark on your ruler, hold your ruler at a right angle on A and then decline it slightly for AC. If you decline it much, you get an isosΔ, which will mislead you. Join BC.

∀isosΔ: Mark apex A. With a compass, mark the ends of the base on a line far enough below to avoid an eqSΔ. Join the dots.

∀eqSΔ: Make a base BC long enough to produce a three or four line tall triangle. Set compass once and swipe apex A from both sides. Connect the dots.

∀4-gon: On a notebook line, mark sides of a two-line tall ∀Δ above and a three-line one below. Or a one-line above and two below. Make all sides unequal. Looking forward, put the short sides towards the bulk of the page and make the 4-gon so opposite sides intersect above and to one side within a third of a page. In post-Euclid "Modern Geometry", this is both a "quadrilateral" and a "quadrangle".

Problems

12. Theorem

∀ ∠ACB if BC(pr) to D, CE ×/2 ∠ACB, CF ×/2 ∠ACD then ∠ECF=∟
(CE bisects ∠ACB and CF bisects ∠ACD)

13. Theorem

On diagram for 1.5: if AD=AE, AB=AC, CF ×/2 ∠BCE, BG ×/2 ∠CBD, BG × CF @ H then FH=GH

14. Theorem

On diagram for 1.5: if AB=AC, CF ×/2 ∠BCE, BG ×/2 ∠CFD,
BG × CF @ H then AH ×/2 ∠A

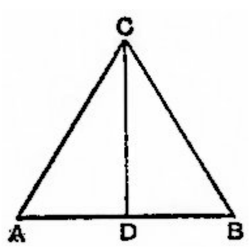
15. Theorem

∀ΔABC: if ∠A = 2∠B, AD ×/2 ∠A × BC @ D then AD=BD
(AD bisects ∠A and intersects BC at D)

When solving problems, learn to think in terms of "**or**". Let's say I have to show AB⊥CD. Gathering all the data from the diagram I then use all my tools to create "**or**" equivalents for the soln: AB⊥CD **or** ∠ABC = ∟ **or**, joining AB and CD with a line, the other two angles of the Δ add up to a ∟ **or** AB is parallel to an existing line EF⊥CD. You may not have all these tools yet but you get the idea.

Proposition 10. Problem

Given: ∀line AB
Required: bisect AB
(divide AB into two equal parts)



Method

On AB, eqSΔCAB (1.1)
CD ×/2 ∠C × AB @ D (1.9)
D required

Proof

ΔACD,BCD: AC=CB (d.1.23) CD=CD (a.1) ∠ACD=∠BCD (con, 1.9)
∴ AD=DB (1.4)

I suppose we should talk about angles:
d.1.8 A **plane angle** is the inclination of two lines to one another which meet on the plane.
d.1.9 A **plane rectilinear angle** is the plane angle of two straight lines which meet at their **vertex**.
d.1.10 When a line meets another so that the two angles created by the former on one side of the latter are equal, these are **right angles** (∟) and the lines are **perpendicular**.
d.1.11 An **obtuse angle** is greater than a right angle.
d.1.12 An **acute angle** is less than a right angle.

But you knew all that. No one says "plane" or "plane rectilinear" because all of them in Books I - VI are plane and rectilinear. What Euclid does not say is how we measure an angle and all his propositions dodge that question. In Book III, we discover that angles are based on sectors of circles. Long story short, make an angle, use vertex for center and the angle is the same for every circle on that center. But we would have to measure that angle as a fraction of π, which cannot be represented by any ratio of magnitudes (fraction). That's why Euclid dodges the question. Another long story short: if you bisect a plane angle, it bisects the chord on the circle's arc (AB in 1.10 ∈⊙C,CA) and bisects the angle itself. But you can trisect any line and, usually, you will not trisect the angle's arc. So 1.9 and 1.10 are actually misleading in this way when it comes to angles. I will unmislead you for now and all time: For any n, integer or rational fraction, you can n-sect a line. This does not, generally, n-sect the arc of a circle when the chord on that is the base of a triangle, the sides of which make the angle of arc.

Proposition 11. Problem

Given: ∀line AB, C∈AB

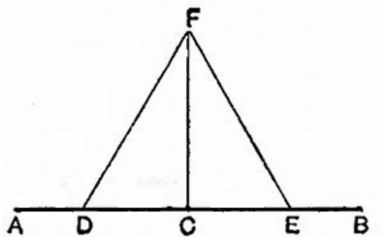
Required: line on C perpendicular to (⊥) AB

Method

∀D∈AC, make CE=CD (1.3)

On DE, eqSΔFDE (1.1) Join CF

CF required



Proof

ΔDCF,ECF: DC=CE (con)

CF=CF (a.1) DF=EF (d.1.23)

∴ ∠DCF=∠ECF=∟ (1.8, d.1.10)

∴ FC⊥AB and FC on C

Problems

16. Problem

Given: AB, S,T∉AB

Required: 1) P∈AB: PS=PT 2) conditions of ↴

(↴ can mean "contradiction" or "impossibility" so the question is "What choices of S,T make the solution impossible?")

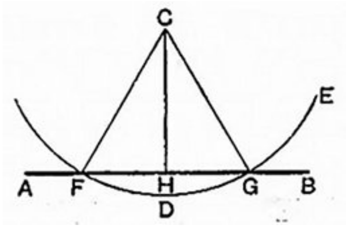
17. Problem

Given: AB between (·|·) points P, Q

Required: Q∈AB: AB ×/2 ∠PQR

In the diagram of 1.11, we encounter a common element of a triangle. FC is the **median** of ∠F or FC≡med∠F. ∀ΔABC, the median from ∠A runs from the vertex A to the midpoint (mdpt) of BC. And med∠B ×/ 2 AC and med∠C ×/2 AB. Another common element is the **altitude** on an angle. ∀ΔABC, AD≡alt∠A ⊥ BC or "the line AD is the altitude on ∠A and, at D on BC, AD is perpendicular to BC." All three angles can have altitudes. It can be shown that the three medians **concur** or mutually intersect at a point. Altitudes also concur. Old Euclids never use these terms out of some kind of weird respect for ancient ways. Or something. I have seen altitudes described multiple ways in the same text and some descriptions were ambiguous. We'll just call them by name.

Proposition 12. Problem



Given: line AB, C∉AB

Required: line on C ⊥ AB

Method

∀D∉AB on side of AB opposite C,

⊙C,CD × AB(pr) @ F,G (p.3)

CH ×/2 FG @ H (1.10)

CH required

Proof

∆FHC,GHC: FH=HG (con) HC=HC (a.1) CF=CG (d.1.15)

∴ ∠CHF=∠CHG (1.8) ∴ CH⊥AB

In line 2 of Method, it says AB(pr) or "AB produced" because your D might move your F and G off of the existing AB. If it does, just lengthen it.

What we are doing with Euclid is called **synthetic geometry**. This means we start from premises and build a ladder of logic up to the conclusion. This is the natural, more or less naive, way to approach the problems you've been solving. There is another, often better, approach. Most Euclids and most geometry texts don't mention this second way until the end, as if it were some kind of sweet dessert. Let's eat dessert first and talk about **analysis**.

Analysis is the approach of starting with the result and working backwards. Let's go back to Problem 16 and solve it analytically. You have line AB and points S,T. And you want P on AB such that PS=PT. In synthesis, you begin by staring dumbly at the paper. In analysis, you start by drawing in PS and PT. Use a ruler or a couple of straight things and fudge PS and PT so they look equal. So your P has to be about there and so do PS and PT. Now stare at the paper but skip the dumbly. Ask yourself, what, of the things I know, does this diagram suggest? If nothing comes to you, rotate the paper and keep thinking. Pretty soon, PS and PT will strike you as equal sides ... on base ST ... which is bisected by its median ...

which is perpendicular to ST. And there you go. Now turn it around: join ST, bisect ST, run a ⊥ from the bisection to AB, and that defines P. Analysis is always better than synthesis for construction problems. Sometimes it works for theorem problems but not so often. And then sometimes you find that you can go backwards with the analysis but that it doesn't work in reverse for a solution. This is rare and you'll just have to try a different analysis based on what you learn from the first one.

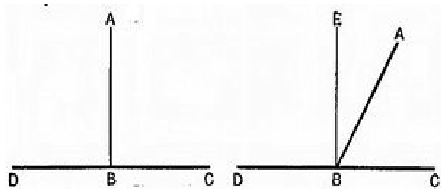
The problems you are solving were constructed by teachers of geometry. Let's think about what that means. It means the problems are solvable with only what you know about Euclid, which isn't much. They were often inspired by the proposition diagrams and the question, "What else is true here?" So it pays to look at those diagrams and see if your problem isn't obviously derived from them. Look at them from all sides. Sometimes the connection is obvious.

Problems make all of us feel stupid, unless the answer is within easy reach of what we understand. Graduate students in mathematics feel stupid facing their work. And at the front of the line, professors feel stupid when they face the leading edge of their research. The difference between you and those ahead of you is that they are comfortable in that stupid darkness. Get comfortable.

And if you are so talented that the problems are easy, get uncomfortable. If you get any kind of degree in mathematics you will observe the phenomenon of natural talent reaching its limit. For some it happens before a bachelor's degree; for some, in grad school; for some, after a doctorate. Whenever it happens, the problem is that the victim has never learned how to learn. The victim has coasted on talent. If you are talented, push harder until you run up against that wall. And then learn how to learn.

Proposition 13. Theorem

∀AB,CD: if AB×CD then the angles formed on one side of one line by the other are either 2∟ or sum to 2∟



Proof

Let AB × CD @ B

Case 1: If ∠ABC=∠ABD then both = ∟ (d.1.10)

Case 2: Let ∠ABC < ∠ABD. Add BE⊥DC.

∠DBE+∠EBC=2∟ (con, d.1.10) and ∠EBC=∠EBA+∠ABC

∴ ∠DBE+∠EBA+∠ABC=2∟ (a.2)

∠DBE+∠EBA=∠DBA ∴ ∠DBA+∠ABC=2∟

Two angles adding up to one right angle are **complementary** and they are **complements** of each other. Two angles adding up to two right angles are **supplementary** and each is the **supplement** of the other. In triangles, an angle and its external angle are supplementary.

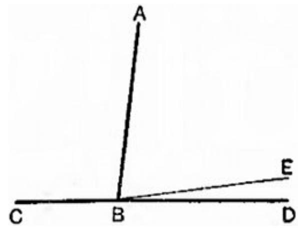
You may have noticed that problems 15, 16, and 17 were all solved using isosceles triangles. Propositions 1.5 and 1.6 are very powerful tools. Let me just mention some of the ways they are used. If you need a line equal to another line, then two equal angles on a base touching the one line forces the other line into existence. Maybe you will have to add the base to the diagram to do this. Symetrically, you can force an equal angle on a "base" by using two equal lines as sides. Let's pretend we know 1.32 and that the three angles in any triangle add up to two right angles. Say you need to force ∠X into a problem. Build an isosceles on its side and make the base angles equal to ½X. Then the apex angle is 2∟-∠X. But the apex's supplement is 2∟- (2∟-∠X) = 2∟- 2∟+∠X = ∠X. Draw a diagram to see this if you need to. Isosceles triangles are your most powerful tool so far.

Proposition 14. Theorem

∀BA,BC,BD. If BC,BD × AB:

∠ABC + ∠ABD = 2∟

then CBD is one line.



Proof

Else let BE, not BD, be one line w/BC.

∴ ∠ABC+∠ABE=2∟ (1.13)

∠ABC+∠ABD=2∟ (hyp)

∴ ∠ABC+∠ABE=∠ABC+∠ABD (a.1, a.11)

∴ ∠ABE=∠ABD (a.3) lesser equal greater ↴

Sym. No such BE in same line w/BC ∴ CBD one line

Mathematicians have claimed that a.11 (∀∟ are equal) should be a theorem and not an axiom. I think not. Just as we know what a straight line or a flat surface is without needing a proof, we know what a right angle is. Here is the context of axioms: "Let us grant that we know these few things. We know they are ideal things. And so we know that, first, we know everything about them and, second, that we shall never encounter them in perfect form in this world." And the context of the rest of Euclid is: "So let us see where these few ideas lead us."

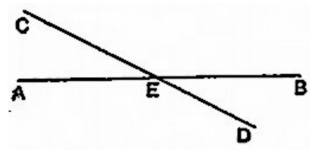
We have said that "=" means equal in magnitude. And you know what magnitude means without definition. Euclid does not even define it. But his idea of it is different from ours. For him, lines simply have length, plane figures simply have area. But our answer to "How big?" is a number that relies on agreeing upon a "number one" -- like an inch. Euclid has no "number one." So for him, this line is **this** long and that line is **that** long and they are either equal or not. Later he will note that one line fits twice into another at a ratio of, say, 1:2 and number slips in the back door.

When a problem says ∀Δ or ∀AB or ∀P it is perfectly legitimate to choose those elements to make the solution as easy as possible. What you cannot do is introduce relations, i.e., ∀Δ is not an isosΔ.

Proposition 15. Theorem

∀AB,CD: if AB × CD @ E

Then ∠AEC=∠BED, ∠BEC=∠AED



Proof

AE×CD ∴ ∠AEC+∠AED=2∟ (1.14)

DE×AB ∴ ∠AED+∠BED=2∟ (1.14)

∴ ∠AEC+∠AED=∠AED+∠BED (a.1, a.11)

∴ ∠AEC=∠BED (a.3) Sym. ∠BEC=∠AED

Corollary 1.

∀AB × CD @ E, then ∠AEC+∠BED+∠BEC+∠AED=4∟

Corollary 2.

∀[AB,CD,EF,...] x @ P, sum of angles around P = 4∟

Problems

18. Theorem

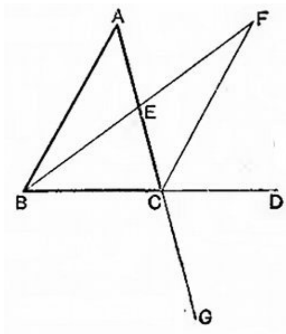
∀E[ABCD]: if opposite angles are equal

Then AED, BEC are single lines.

Remember that propositions tell us what is true about our diagrams. So do not over-focus on prominent propositions. See all the truth in a diagram: what is equal to what, what is related to what. If you don't get anything else from Euclid, get this: **see all the truth**. And not just in geometry. If the news boasts about full employment, see all the truth. In 2018, full employment means ten million fewer jobs than in 1950. And there are around 300 million people now compared to around 200 million then. So full employment is very bad news because if you plot the curve, soon almost no one will be employed. And note that seeing all the truth is apolitical. Just do the math.

When proving theorems, do not get caught up in accurate constructions. The proofs are logical sequences, the implications of which build a ladder to that which you are proving. You need only enough of a diagram to suggest these sequences.

Proposition 16. Theorem

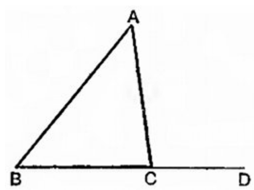


∀∆ABC, ext∠C > ∠A or ∠B.
Sym. for ext∠A,B

Proof

∀∆ABC, BC(pr) to D,
AC ×/2 @ E   Join BE
BE(pr) to F: EF=EB   Join FC
∆AEB,CEF: AE=EC and EB=EF (con)
∠BEA=∠CEF (1.15)
∴ ∠BAE=∠ECF (1.4)
∠ECD > ∠ECF (a.8) ∴ ∠ACD > ∠A
Sym. Bisect BC, produce AC then ext∠C > ∠B

Proposition 17. Theorem



∀∆ABC, ∀ two angles together < 2 ∟

Proof

BC(pr) to D ∴ ext∠C > ∠B (1.16)
∴ int∠C + ext∠C = 2∟ > ∠B + int∠C
Sym. for other two pairs.

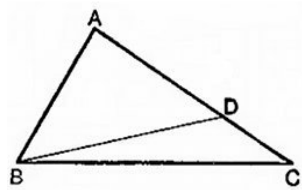
Problems

19. Problem

Given: ∀acute∆ABC, produce BC to D: BC=CD
Required: P∈BD: AP demonstrates ∠ABC + ∠ACB < 2∟

Everyone who truly works at Euclid gets what they need from Euclid. Not everyone is a mathematician. If your entire accomplishment is the ability to understand the propositions and the solutions to the problems and the use of the notation, then that is all you needed. You were able to work through a text that demanded concentrated attention and the application of reason. And that is no slight accomplishment. It is an intellectual achievement. And even if you achieve more, the ideas that are not consistent with your individuality will fall away. You will only keep what you need.

Proposition 18. Theorem



∀Δ, if one side is greater than a second side, the angle opposite the first is greater than the angle opposite the second.

Proof

Let AC > AB, then let AD=AB D∈AC. Join BD.
∠ADB ≡ ext∠BDC, ∠ADB > ∠DCB (1.16)
AB=AD (con) ∴ ∠ADB=∠ABD (1.5)
∴ ∠ABD > ∠ACB and even more is ∠ABC > ∠ACB

As the problems become more substantial, it becomes more important to muse upon the solutions. If you simply solve or study the solution and move on to the next thing, you will find yourself slipping behind. Solutions are a matter of thought. You need to consolidate your thinking about what the circumstances of the problem were and what enabled the solution. These thoughts constitute your permanant gain in the study of Euclid. You are not making this effort for someone else. You are strengthening your own ability to think with consistency and with rigour. Do not cheat yourself in this effort.

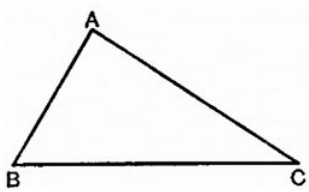
And don't worry about how many problems you can solve. Be grateful for any you do solve and study the solutions of the ones you can't solve. Some of them you almost certainly cannot solve. Some are "clever" test questions from the Cambridge Tripos and you are probably not First Wrangler material. So what? Truly engage with the problems and solve what you can.

Problems

20. Theorem

∀4-gon ABCD: If AD > (AB or CD) > BC
Then ∠B > ∠D and ∠C > ∠A

Proposition 19. Theorem



∀Δ, if one angle be greater than a second angle, the side opposite the first is greater than the side opposite the second.

Proof

If ∠B > ∠C, then AC > AB (hyp) Else AC ≤ AB
∠B ≠ ∠C (hyp) ∴ AC ≠ AB (1.5)
∠B >∠C, AC !< ∠B (1.18)
∴ AC > AB.

This symbol "∑" means "sum of". In ΔABC, "∑sides" means "AB + BC + CA". You get the idea.

Problems

21. Theorem

∀ΔABC, if AD ×/2 ∠A × BC @ D then BA > BD and CA > CD

22. Theorem

∀AB, ∀C∉AB:
1) ⊥ shortest line from C to AB
2) Of others, nearer to ⊥ shorter than further from ⊥
3) Given ∀line from C to AB, at most, only one other is its equal

23. Theorem

∀square ABCD: if AF × CD,BC(pr) @ E,F then AF > AC

24. Theorem

∀ΔABC,∀P Join P[ABC]
Then PA+PC+PC > ½ perimeter ΔABC (AB+BC+CA)
Prove for P in, on, and outside Δ.

25. Theorem

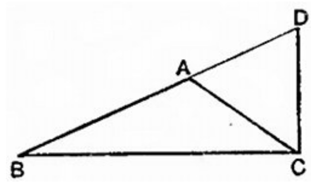
∀4-gon, ∑sides > ∑diagonals
(∑sides ≡ AB+BC+CD+DA)

26. Theorem

∀ΔABC, ∑A[BC] > 2 AD med∠A
(3 cases: ∠ADB=∟, ∠ADB=∠ABD, ∠ADB < ∠ABD)

Proposition 20. Theorem

∀ΔABC, ∑ ∀2 sides > 3d side



Proof

BA(pr) to D: AD=AC. Join DC.

AD=AC ∴ ∠ADC=∠ACD (1.5)

∠BCD > ∠ACD (a.8)

∴ ∠BCD > ∠BDC

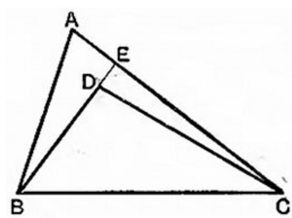
ΔBDC: ∠BCD > ∠BDC ∴ BD > BC (1.19)

But BD = BA + AC ∴ BA + AC > BC

Sym. for other two pairs of sides.

Proposition 21. Theorem

∀ΔABC, for ∀D in Δ, DB < AB, DC < AC, and ∠D > ∠A



Proof

BD(pr) × AC @ E

ΔABE: BA + AE > BE (1.20)

∴ BA + AE + EC = BA + AC > BE + EC

ΔDEC: DE + EC > DC (1.20)

∴ DB + DE + EC > DC + DB

BA + AC > BE + EC ∴ BA + AC > BE + EC > BD + DC

ΔCDE: ext∠BDC > ∠CEB (1.16)

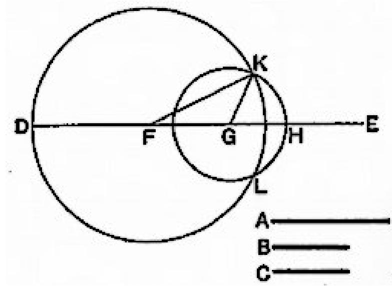
ΔABE: ext∠CEB > ∠BAE

∴ ∠BDC = ∠D > ∠CEB > ∠BAE = ∠A

In the past, students were required to memorize Euclid. They were tested on their ability to reproduce his propositions exactly and then to solve "clever" problems, called "riders," concerning those propositions. I don't see the point in memorizing Euclid's reasoning. It is more important to grasp his strategy for each proof. If you are conscious of these strategies, then his tools are your tools. What you gain from Euclid, in the end, are those things that remain available to you in your mind. Fill your mind with tools.

Proposition 22. Problem

Given: 3 lines A, B, C, any two greater than the third.
Required: Δ with sides equal A, B, C



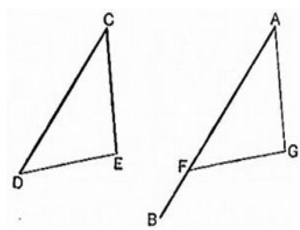
Method

∀DE > A+B+C: DF = A, FG = B, and GH = C (1.3)
⊙F,FD × ⊙G,GH @ K (p.3)
Join K[FG]
ΔKFG required

Proof

FD,FK radii ⊙F ∴ FK = FD = A (d.1.15, a.1)
GH,GK radii ⊙G ∴ GK = GH = C (d.1.15, a.1)
FG = B (con)

Proposition 23. Problem



Given: AB, ∠ECD
Required: Copy ∠ECD to A

Method

Join DE. F∈AB: AF = CD
ΔAFG: AF,FG,GA = CD,DE,EC (1.22)
∠GAF required

Proof

ΔDCE,GAF: FA=DC, AG=CE, FG=DE (con)
∴ ∠GAF = ∠ECD (1.8)

Problems

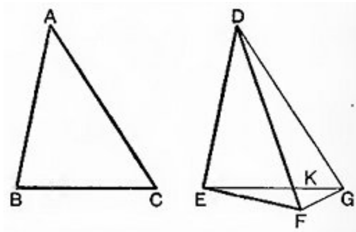
27. Theorem

∀ΔABC, if ∠A = ∠B + ∠C
Then ΔABC can be divided into two isosΔ

28. Theorem

∀ΔABC, if ∠A = ∠B + ∠C
Then BC = 2 AD (med∠A)

Proposition 24. Theorem



∀ΔABC,DEF, if ∀2 sides equal
(AB,AC=DE,DF) and ∠A > ∠D
Then BC > EF

Proof

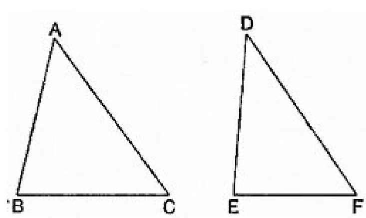
Let AB,DE < AC,DF
Copy ∠BAC to ∠EDG (1.23) ∴ DG = AC (1.3)
Join G[EF] EG × DF @ K
DG ≤ DF and DF = DG ∴ DGE ≤ DEG (1.5, 1.18)
∠DKG > ∠DEG ∴ DG,DF > DK
ΔABC,DEG: AB=DE (hyp) AC=DG and ∠BAC = ∠EDG (con)
∴ BC = EG (1.4) and DG=DF ∴ ∠DGF = ∠DFG
∠DGF > ∠EGF (a.8) ∴ ∠DFG >∠EGF
∴ ∠EFG > ∠DFG > ∠EGF (a.8)
ΔEFG: ∠EFG > ∠EGF ∴ EG > EF (1.19)
EG = BC ∴ BC > EF

Problem

29. Problem

Given: Base AB, base ∠B, sum of sides CD
Required: Implied triangle.

Proposition 25. Theorem



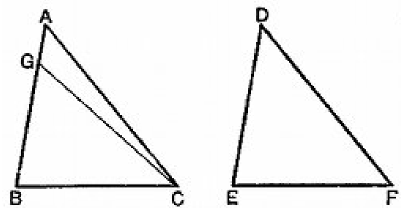
∀ΔABC,DEF, ∀2 sides equal
(AB,AC=DE,DF) and BC > EF
Then ∠A > ∠D

Proof

Else ∠A ≤ ∠D
BC not equal EF (hyp) ∴ ∠A ≠ ∠D (1.4)
BC not less than EF (hyp) ∴ ∠A !< ∠D (1.24)
∴ ∠A >∠D

The next proposition is more in the style of Euclid, with less symbolic condensation. Think of it as an inoculation.

Proposition 26. Theorem



∀ΔABC,DEF: If two angles and one side are equal, each to each, then ΔABC ≡ ΔDEF

Proof

Case 1: equal sides between equal angles

Let ∠B,C = ∠E,F and BC=EF, then ΔABC ≡ ΔDEF

Else let AB > DE. Add G: BG = DE (1.3) Join CG.

ΔGBC,DEF: GB=DE (con) BC=EF and ∠B = ∠E (hyp)

∴ΔGBC ≡ ΔDEF (1.4) and ∠GCB = ∠DFE. But ∠DFE = ∠ACB (hyp)

∴ ∠GCB = ∠ACB the lesser equals the greater ⮧ ∴ AB=DE

∴ΔABC,DEF: AB=DE (proved) BC=EF and ∠B = ∠E

∴ΔABC ≡ ΔDEF (1.4)

Case 2: equal sides not between equal angles

Let ∠B,C = ∠E,F and AB=DE Then ΔABC ≡ ΔDEF

Else let BC > EF. Add BH: BH=EF. Join AH.

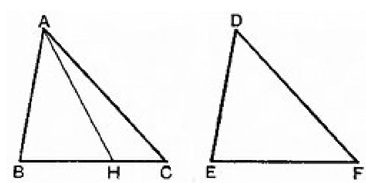
ΔABH,DEF: AB=DE, ∠B = ∠E (hyp)

BH=EF (con) ∴ΔABH ≡ ΔDEF (1.4)

∴ ∠BHA =∠EFD (1.4)

But ∠EFD =∠BCA (hyp)

∴ ∠BHA =∠BCA (a.1)



or ΔAHC: ext∠BHA = int∠BCA⮧ (1.16) ∴BC=EF

∴ΔABC,DEF: AB=DE (hyp) BC=EF (proved) ∠B = ∠D

∴ΔABC ≡ ΔDEF (1.4)

Problems

30. Theorem

∀ΔABC: if AD ×/2 ∠A, BD⊥AD, BD × AD,AC @ D,E then BD=DE

31. Theorem

∀ΔABC, if ∀P∈AD ×/2 ∠A, PQ,PR⊥AB,AC then PQ=PR

32. Problem

Given: AB,CD,EF, CD !∥ EF

Required: P∈AB: PQ⊥CQD, PR⊥ERF

(CQD,ERF are lines. For angles we always use ∠)

33. Problem

Given: AB,AC, ∀P outside ∠BAC

Required: line EFP (PEF): E∈AB, F∈AC, AE=AF

34. Problem

Given: ∀P,Q,R

Required line OP ·|·(Q,R): QS⊥ASB, RT⊥ATB

35. Theorem

∀ ⊿ABC,DEF: ∟B=∟E, AB=DE, AC=DF then ⊿ABC ≡ ⊿DEF
(⊿ is a right triangle)

On Parallel Lines

First, an axiom and a definition:

a.12 If a line cut two other lines such that, on one side of the first, the other two make angles summing to less than two right angles, the lines, extended on that side, must intersect.
d.1.29 **Parallel lines** are coplanar lines which cannot be produced to intersect.

If you pursue pure geometry much further you will find that this definition and axiom have come under a lot of discussion. Many have attempted improvements. In spite of all the big names involved, I will speak up for Euclid here and say why his choices are all we could ask for. First, parallel lines, like right angles and straight lines, need no definition. We understand two equidistant lines not meeting before we acquire the words "parallel" and "equidistant". The idea of such lines is another ideal we know everything about and will never encounter perfectly in the world. And his definition clearly brings this ideal to mind.

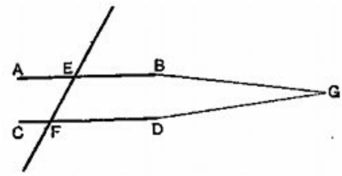
This definition could be stated in many ways. But it goes along perfectly with the axiom and with the practice of Euclid's pure geometry. Say we have two lines scratched in the dirt that do not meet. If we want to know if they will meet when produced, we need a test for intersection. And this is what the axiom gives. Cut the two lines with another line, measure the angles on one side of the cutting line. If they add up to less than two right angles, our lines intersect on that side; if to more than two right angles, on the other side; if equal to two right angles, they do not intersect. No proposed replacement for this axiom has offered such a test. And so, in my mind, all those suggestions are useless. We don't need heightened formal elegance here. We need a practical and useful test of parallelism between lines.

In Euclid, we are working with almost naive ideals in a small (no bigger than a sheet of paper or a sandbox) and practical way. When Euclid said lines could be produced indefinitely, he meant inside the sand box. He did not mean around the world, which he knew to be round, nor to infinity, which did not exist for him or for Archimedes, who computed how many grains of sand would fill the universe. (Quite a few, apparently.) By excluding ideas that Euclid could never have had, we realize how beautifully and elegantly his small world of ideals has achieved its completeness. That his small world is not a complete model of our only world is hardly shocking. And no amount of tweaking Euclid will fill that gap. It suffices that Euclid is complete within its own bounds of three postulates and twelve axioms. Euclid was the first to formally express a mathematic in such terms. And every formal mathematic that has followed him stands on his shoulders.

You will also discover that the later introduction of non-Euclidean geometry shocked the intellectual world to the core. I am completely mystified by their reaction. Even philosophy suffered under the blow. And all the minds affected were aware that we live on a somewhat lopsided sphere. And on this sphere, parallel lines, perpendicular to any given line, all meet at a point. The main case of this being lines of longitude perpendicular to the equator. The point here is that, in mathematics, we are dealing with ideals and their relations, not with reality. And we should keep this in mind.

Proposition 27. Theorem

If a line cut two others so as to make equal alternate angles (alt∠), then the two lines are parallel.



If EF×AB,CD: ∠AEF = ∠EFD

then AB∥CD

Proof

Else, AB(pr) × CB(pr) @ G towards B and D (a.12)

∴ figure GEF ≡ Δ and ext∠AEF > ∠EFG (1.16)

∴ ∠AEF = ∠EFD = ∠EFG (hyp) ↴

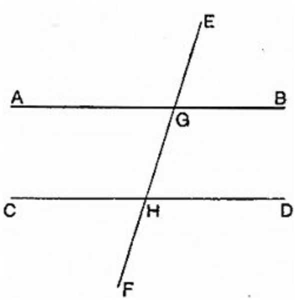
∴ AB !× CD @ G

Sym. AB !× CD towards A and C

∴ AB∥CD (d.1.29)

Proposition 28 Theorem

If a line cut two other lines to make an exterior angle equal to its opposite interior angle or to make interior angles on one side equal to two right angles, the other two lines are parallel.



If EF × AB,CD:

1) ext∠EGB = int opp∠GHD or

2) ∠BGH + ∠GHD = 2∟,

then AB∥CD

Proof

Case 1: ∠EGB = ∠GHD (hyp) and ∠EGB = ∠AGH (1.15)

∴ ∠AGH = ∠GHD (a.1) and they are alternate.

∴ AB∥CD (1.27)

Case 2: ∠BGH + ∠GHD = 2∟ (hyp)

and ∠AGH + ∠BGH = 2∟ (1.13)

∴ ∠BGH + ∠GHD = ∠AGH + ∠BGH (a.1, a.11)

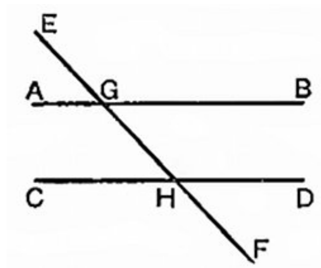
∴ ∠GHD = ∠AGH (a.3) and they are alternate.

∴ AB∥CD (1.27)

1.27 and 1.28 indicate what kind of angles show that a line has cut two parallel lines. 1.29 is the converse of both, assuming the parallel bit and showing those same angles are created.

Proposition 29. Theorem

If a line cut two parallel lines, it creates all the angular relations of propositions 1.27 and 1.28.



If EF × AB,CD, AB∥CD, then

1) alt∠AGH = alt∠GHD

2) ext∠EGB = int opp∠GHD

3) ∠BGH + ∠GHD = 2∟

Proof

Case 1: If ∠AGH ≠ ∠GHD, let

∠AGH > ∠GHD

∴ ∠BGH + ∠AGH > ∠GHD + ∠BGH (a.2)

∠AGH + ∠BGH = 2∟ (1.13)

∴ ∠GHD + ∠BGH < 2∟ ∴ AB × CD (a.12) ↴ (hyp)

∴ ∠AGH = ∠GHD

Case 2:

∠AGH = ∠EGB (1.15) ∴ ∠EGB = ∠GHD (case 1, a.1)

Case 3:

∠EGB = ∠GHD ∴ ∠BGH + ∠EGB = ∠GHD + ∠BGH (a.2)

∠EGB + ∠BGH = 2∟ (1.13)

∴ ∠GHD + ∠BGH = 2∟

Don't get too caught up in alternate, external, opposite, or internal angles. Look at the big picture. Let ∠EGA be ∠1, ∠EGB be ∠2, ∠BGH be ∠3, and the last one ∠4. Keep numbering the same angles around H the same way. Then ∠1 = ∠3 = ∠5 = ∠7 and ∠2 = ∠4 = ∠6 = ∠8. Take one from each group and they sum to 2∟. This is all from 1.13 and 1.15. One of the powers of parallel lines is to show us this equality of angles. And soon they will show us the equality of figures bounded by them. Note that alt∠ is "alternate" or "altitude". But the latter is always given as "XY alt∠Z".

Problems

36. Theorem

∀lines A,B,C,D: if A∥C, B∥D

Then the angle A makes with B equals the angle of C with D

37. Theorem

∀isosΔABC: if ∀DE∥BC, DE × AB(pr),AC(pr) @ D,E

Then ∠CED = ∠BDE

38. Theorem

∀ΔABC, if ext ×/2 ∠A∥BC then ΔABC ≡ isosΔ

39. Theorem

∀AB,CD: AB∥CD ∀E,F∈AB,CD, G mdpt EF

Then ∀line on G ·|·(AB,CD) has mdpt G

40. Theorem

∀lines A,B: A∥B, ∀P eqD A,B

Then ∀two lines, not∥A,B, on P intercept equal segments of A,B

41. Theorem

∀ΔABC: if AD ×/2 ∠A × BC @ D,

DE∥AC x AB @ E, DF∥AB × AC @ F then DE=DF

42. Theorem

∀ABC: if BC(pr) to D, CE ×/2 ∠C × AB @ E, CG ×/2 ext∠C,

EF∥BC × AC @ F, EF × CG @ G then EF=FG

(Or: Any triangle ABC: if BC is produced to some D; and CE, bisector of ∠C intersects AB at E; and CG bisects external ∠C; and EF, parallel to BC, intersects AC at F; and EF intersects CG at G, then EF=FG. You can see why we use symbols.)

43. Problem

Given: ⊿ABC, ∟C

Required: D∈AB: DB=DE DE⊥AC

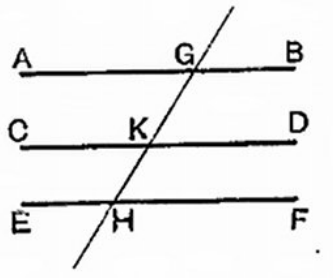
44. Theorem

∀isosΔABC: if ∀D∈BC, DEF⊥BC × AB(pr),AC(pr) @ E,F

Then ΔAEF ≡ isosΔ

(Note that, here, AB and AC may or may not **need** to be produced.)

Proposition 30. Theorem

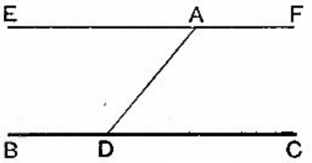


Lines which are parallel to the same line are parallel to each other.

Proof

AB∥EF, CD∥EF, GKH x AB,CD,EF
GKH × AB,EF
∴ ∠AGH = ∠GHF (1.29)
GKH × CD,EF
∴ ∠GKD = ∠GHF (1.29)
∴ ∠AGH = ∠GKD (a.1) and they are alt∠
∴ AB∥CD (1.27)

Proposition 31. Problem



Given: ∀point A, line BC, A∉BC
Required: line on A∥BC

Method

∀D∈BC, join AD.
Copy ∠ADC to A for ∠DAE (1.23)
Produce EA to F. EF required

Proof

AD x EF,BC ∴ ∠EAD = ∠ADC and they are alt∠ (con)
∴ EF∥BC and A∈EF

Problems

45. Problem

Given: point A, line CD, ∠E, A∉CD
Required: B∈CD: ∠ABC = ∠E

46. Problem

Given: ∀isosΔABC
Required: D,E∈AB,AC: BD=DE=EC

Problem-wise, prepare yourself. Proposition 1.32 is, in a sense, the culmination of 1.16-21, 24, and 25, the culmination of all angle relations of a triangle. It enables a boatload of problems. Even 1.47 (Pythagorean Theorem) has fewer problems following it.

Proposition 32. Theorem

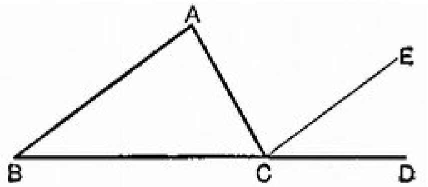
∀ΔABC, 1) if any side is produced, the external angle is equal to the sum of the two opposite internal angles.

2) The sum of the three interior angles is two right angles.

ΔABC, if BC produced to D, then

1) ext∠C (∠ACD) = ∠A + ∠B

2) ∠A + ∠B + ∠C = 2∟



Proof

CE∥AB.

1) AB∥CE and AC × AB,CE

∴ ∠BAC = ∠ACE (1.29)

BD × AB,CE ∴ ∠ECD = ∠ABC (1.29)

∴ ext∠ACD = ∠ACE + ∠ECD = ∠A + ∠B (a.2)

2) ∴ ∠ACB + ext∠ACD = ∠BAC + ∠ABC + ∠ACB (a.2)

∠ACB + ext∠ACD = 2∟ (1.32)

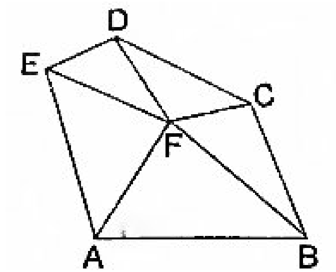
∴ ∠BAC + ∠ABC + ∠ACB = ∠A + ∠B + ∠C = 2∟

The next two corollaries to 1.32 were added by Robert Simson (18thC Scotland), who wrote an early Euclid text. In the notation, "∃" is read "there exist(s)" and some of the "∀" should be read "all". For example, in the proof of C1, line 2 reads: "Therefore there exist n triangles, such that for every triangle, the sum of their angles equals two right angles." And then for line 3: "But the sum of all of the triangles' angles equals". You will know you are reading it correctly when it is true. Reason it out.

Everything in Euclid is true. Euclid included what he did because it had been discovered to be true. Truth requires no human authorities to pass judgment upon it. This is because, when you understand the truth, it is demonstrable. You can demonstrate its truth. And, until you can do that, nothing in Euclid is true for you. Demonstration is the measure of your understanding. Truth is realized and you experience this when you suddenly see the particular truth that solves a problem.

Corollary 1

∀n-gon, ∑int∠ + 4∟ = n2∟



Proof

∀n-gon, ∀F in n-gon, join F[A-N]

∴ ∃nΔ: ∀Δ,∑∠ = 2∟ (1.32)

But ∑(∀Δ▦) =

∑(int∠n-gon) + ∑(∀∠ on F)

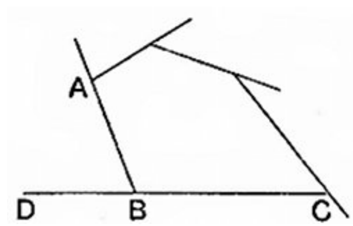
∀∑∠ on F = 4∟ (1.15.C2)

∴ ∑int∠ + 4∟ = n2∟

Corollary 2

∀convex n-gon, ∑ext∠ = 4∟

("convex" means no angles poke into the n-gon. No **re-entrant** ∠)



Proof

∀int∠ABC + ext∠ABD = 2∟ (1.13)

∴ ∑∀int∠ + ∑∀ext∠ = n2∟

But ∑∀int∠ + 4∟ = n2∟ (1.32.C1)

∴ ∑∀int∠ + ∑∀ext∠ = ∑∀int∠ + 4∟ = n2∟

∴ ∑∀ext∠ = 4∟ (a.3)

Problems

47. Theorem

∀Δ,∀∠X is obtuse, right, acute as ∠X >=< 2∟

48. Problem

Required: ×/3 ∟ (trisect a right angle)

49. Problem

Required: isosΔ: ∠A = 4∠B,C

50. Problem

Required: isosΔABC: ½∠A = 1/3∠B,C

51. Theorem

∀isosΔABC: produce BA to D: BA=AD. Join DC

Then ΔDBC ≡ ⊿

52. Theorem

∀isosΔABC: if BD,CE alt∠B,C then ∠DBC = ∠ECB = ½∠A

53. Theorem

∀isosΔABC: if BD,CE ×/2 ∠B,C, BD × CE @ F then ∠BFC = ext∠B,C

54. Problem

Given: line A, points P,Q∉A

Required: lines on P,Q forming eqSΔ on A

(Base is segment of A)

55. Problem

Given: AB, AC, DE, ∠F

Required: P,Q ∈ AB,AC: AP + PQ = DE, ∠APQ = ∠F

56. Theorem

∀ΔABC: if BD,CD ×/2 ext∠B,C then ∠BDC + ½∠A = ∟

57. Theorem

∀isosΔABC: if sides produced and

below BC: ∠BCD = ∠CBE = 1/3∠B,C

then three isosΔ created

58. Theorem

∀ ⊿ABC ∟A: AD med∠A = ½ BC

59. Theorem

∀ΔABC: if AD,BE alt∠A,B and F mdpt AB then DF=EF

60. Theorem

∀ΔABC: if AD,BE alt∠A,B, F mdpt AB, FG⊥AB then FG ×/2 DE

61. Theorem

∀isosΔABC: BD,CE ×/2 ∠B,C then DE∥BC

62. Theorem

∀AB,CD: if AB = CD, AB !∥ CD, ∠ABD = ∠CDB then BD∥AC

63. Problem

Given: hypotenuse and AD = ∑(other two sides)

Required: implied ⊿

64. Problem

Given: hypotenuse and AD = ~(other two sides)

Required: implied ⊿

("~" means "the difference of")

65. Problem

Given: hypotenuse AB and alt∟C

Required: implied ⊿

66. Problem

Given: ΔABC, perimeter DE

Required: Δ of perimeter DE with angles of ΔABC

67. Problem

Given: perimeter DE, ∠FGH

Required: implied ⊿

68. Problem

Given: AB, CD: AB‖CD ∀P·|·(AB,CD)

Required: S,T ∈ AB,CD: PS⊥PT

(Problems requiring a greater mastery marked with *)

69. Problem *

Given: AB,AC, ∀P∈AB

Required: PQ: Q∈AC, ∠APQ = 3∠AQP

70. Theorem *

∀ΔABC: AD,CF med ∠A,C, produce AD,CF to E,G: AD=DE, CF=FG

Then EBG is one line.

71. Problem *

Given: ∀AB

Required: ×/3 AB using eqSΔ or isosΔ on AB

72. Theorem *

∀AB,CD: AB × CD @ E, join AC,BD, BF ×/2∠B × CF ×/2∠C @ F

Then ∠CFB = ½(∠EAC + ∠EDB)

73. Problem

Given: regular 8-gon (regular ≡ eqS and eq∠)

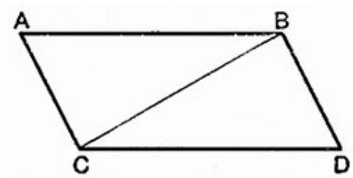
Required: Magnitude of its angles

74. Theorem

∀AB, ⊙A,AB × ⊙B,BA @ C,F, eqSΔCAB, Produce AB to E ∈ ⊙B

Then ΔCDE ≡ eqSΔ

Proposition 33. Theorem

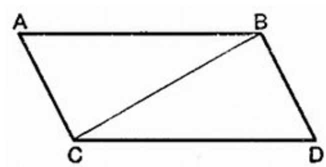


Lines joining the endpoints of equal and parallel lines are equal and parallel.
$\forall AB,CD$: if $AB=CD$, $AB \parallel CD$ then $AC=BD$ and $AC \parallel BD$.

Proof

Join BC
$AB \parallel CD$ and $BC \times AB,CD \therefore \angle ABC = \angle BCD$ (1.29)
$\Delta ABC,BCD$: $AB=CD$ (hyp) $BC=BC$ (a.1) $\angle ABC = \angle BCD$ (proven)
$\therefore \Delta ABC \equiv \Delta BCD$ and $AC=BD$, $\angle ACB = \angle CBD$ (1.4) $\therefore AC \parallel BD$ (1.27)

Proposition 34. Theorem



$\forall \parallel$gm ABCD: 1) $AB=CD$, $AC=BD$, $\angle A = \angle D$, $\angle B = \angle C$ and
2) AD,BC $\times/2$ $\parallel$gm

Proof

1) $AB \parallel CD$ and $BC \times AB,CD \therefore \angle ABC = \angle BCD$ (1.29)
$AC \parallel BD$ and $BC \times AC,BD \therefore \angle ACB = \angle CBD$ (1.29)
$\Delta ABC,BCD$: $\angle ABC = \angle BCD$, $\angle ACB = \angle CBD$ (proven) $BC=BC$ (a.1)
$\therefore \Delta ABC \equiv \Delta BCD$ and $AB=CD$, $AC=BD$ and $\angle BAC = \angle CDB$
$\angle ABC = \angle BCD$, $\angle ACB = \angle CBD$
$\therefore \angle ABC + \angle CBD = \angle BCD + \angle ACB \therefore \angle ABD = \angle ACD$ (a.2)
2) $\Delta ABC \equiv \Delta BCD$ (proven) $\therefore$ BC $\times/2$ $\parallel$gmABCD. Sym. for AD.

d.1.30 A **parallelogram** ($\parallel$gm) is a 4-gon of opposing parallel sides

Given any problem about a triangle, you can parallelogramize the triangle. Everything you learn about the one can often be applied to the other. In our notation, turning a Δ into a $\parallel$gm will be noted as **∥gmize** in the solutions. And here is how to draw **∀∥gm:** Use your six-inch ruler to strike the long off-set parallel horizontal sides. Then use the ruler to strike an angled end. Do not strike the last side without moving the ruler away from the third one or you will have a rhombus to skew your thinking.

Problems

75. Theorem

∀4-gon, if opp sides are equal, then 4-gon ≡ ‖gm.

76. Theorem

∀4-gon, if opp angles are equal, then 4-gon ≡ ‖gm

77. Theorem

∀‖gmABCD, AC,BD ×/2 e.o. (e.o. ≡ "each other")

78. Theorem

∀4-gonABCD, if AC,BD ×/2 e.o., then 4-gon ≡ ‖gm.

79. Theorem

∀‖gm, if a diagonal bisects opposite angles, all sides are equal.

80. Theorem

∀4-gonABCD: if two opp sides parallel, two equal but not parallel
Then ∑(opp∠) = 2∟

81. Theorem

∀∆ABC ∀CE,BF E∈AB F∈AC, CE,BF cannot ×/2 e.o.

82. Problem

Given: ∀AB,CD: AB‖CD, ∀P ∉AB,CD, line (magnitude) L
Required: Line on P intercepted by magnitude L ·|·(AB,CD)

83. Theorem

∀‖gm, bisectors of adjacent angles intersect at right angles.

84. Theorem

∀‖gm, bisectors of opposite angles either coincide or are parallel.

85. Theorem

∀‖gm, if diagonals are equal then ‖gm eq∠

86. Problem

Given: lines AB,CD, magnitudes L,M
Required: 1) P: perpendiculars from AB,CD to P equal L,M
2) Number of such points that exist

87. Problem

Given: ∀AB,CD, magnitudes E,F
Required: line equal to E, parallel to F, terminated by AB,CD

88. Theorem

∀‖gmABCD: eqS∆ AEB,CGD outside ‖gm, eqS∆BFC overlaying ‖gm
Then EF,GF=AC,BD

89. Theorem

∀line ABC: AB=BC, ∀line DF not passing between A and C,
AD,BE,CF ⊥ AC × DE @ D,E,F Then AD + CF = 2BE

90. Theorem

∀‖gm ABCD: ∀EF outside ‖gm, join A,B,C,D w/⊥ to EF
Then ∑(⊥on A,C) = ∑(⊥on B,D)
("w/⊥" ≡ "with perpendiculars")

91. Theorem

Given a ‖gm of constant sides, if the angle contained by two sides increases, then the diagonal on that angle decreases.

92. Theorem

∀6-gon, if opposite sides are pair-wise equal and parallel
Then the three diagonals **concur** (meet at a point)

93. Problem

Given: AB, AC, point D·|·(AB,AC)
Required: line w/endpoints on AB,AC, ×/2 @ D

94. Theorem

∀‖gm ABCD: E,F mdpt AD,BC then BE,DF ×/3 AC
(BE and DF trisect AC)

95. Theorem

∀4-gon: if AD‖BC then area ABCD equals that of ‖gm formed by line‖AB on E mdpt CD

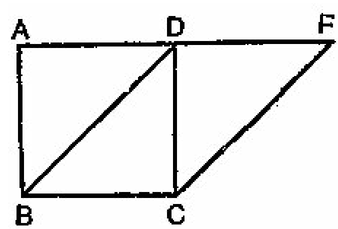
96. Theorem

∀∆ABC: D,E mdpt AB,AC,
Then ∆ADE = 1/4∆ABC

97. Problem *

Given: rhombus ABCD, P mdpt AB
Required: inscribed rhombus w/vertex P
("Inscribed" rhombus means all its vertices are on ABCD)

Proposition 35. Theorem



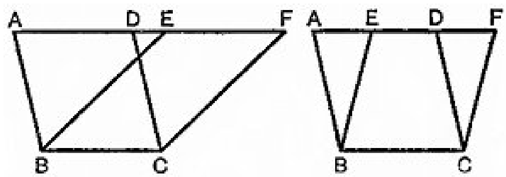
‖gms on same base, between same ‖s have equal area.

If ‖gm ABCD,DBCF ·|· (AF‖BC)
then ‖gmABCD = ‖gmDBCF

Proof

Case 1) sides AD, DF terminate on D.
Then by inspection both ‖gm equal 2ΔBDC (1.34)
∴ ABCD = DBCF (a.6)

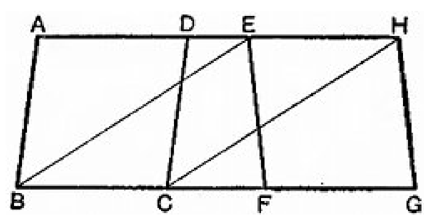
Case 2) Sides not terminated at same point.



ABCD ≡ ‖gm ∴ AD=BC
Sym. EF=BC ∴ AD=EF (a.1)
∴ AE=DF (a.2,3)
ΔEAB,FDC: AB=DC, AE=DF
∠FDC = ∠EAB (1.29)
∴ ΔEAB ≡ ΔFDC
∴ ABCF - ΔFDC = ABCF - ΔEAB (a.3) ∴ ABCD = EBCF

Case 2 is more clearly seen in RHS diagram. Once you see it there, you'll see it in the LHS one.

Proposition 36. Theorem



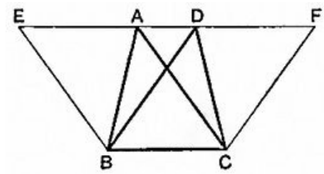
‖gms on equal bases between same ‖s have equal area.

‖gm ABCD,EFGH: if BC=FG, ABCD,EFGH ·|· (BG‖AH)
Then ABCD = EFGH

Proof

Join BE,CH.
BC=FG (hyp) FG=EH (1.34) ∴ BC=EH (a.1)
BC‖EH (hyp) and BC=EH ∴ BE=CH (1.33) ∴ EBCH ≡ ‖gm
‖gm EBCH,ABCD on BC ·|· (BC‖EH) ∴ EBCH = ABCD (1.35)
Sym. EBCH = EFGH ∴ ‖gm ABCD = ‖gm EFGH (a.1)

Proposition 37. Theorem



Triangles on same base between same ‖s have equal area.
If AD‖BC, ΔABC,DBC on BC
then ΔABC = ΔDBC

Proof

Produce EADF. BE‖AC, FC‖BD (1.31)
‖gm EBCA = ‖gm DBCF (1.35)
AB,DC ×/2 EBCA,DBCF ∴ ΔABC,DBC = ½(EBCA,DBCF) (1.34)
∴ΔABC = ΔDBC (a.7)

Get it very clear in your head that these last propositions are only about equal area. We use "=" for this, showing equal magnitudes. In Euclid, magnitude can be length, area, or volume and they never have any numbers to go with them. They are simply equal. And equivalence ("≡") means "equal in every way": sides, angles, area: all equal.

Problems

98. Theorem

∀‖gm ABCD, ∀line on D × BC,AB(pr) @ F,G. Join AF,CG.
Then ΔABF = ΔCFG

99. Problem

Given: ΔABC on line BCD
Required: Triangle w/base on BD of equal area ΔABC

100. Problem

Given: ∀ΔABC, ∀D∈BC
Required: Δ* ΔABC, apex D, base on AB(pr)
(AB produced either way)

101. Problem

∀4-gonABCD, ∀P∈CD
Required: 4-gonABEF, P∈EF, EF‖AB, area = ABCD

102. Problem

Given: ∀4-gonABCD, P∈CD
Required: Δ = ABCD, vertex P, base∈AB(pr)

103. Problem *

Given: ∀∥gmABCD

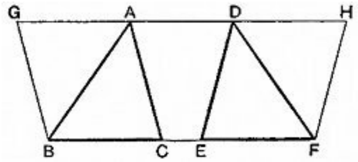
Required: rhombus = ABCD

104. Problem *

Given: ∀ΔABC, ∀MN∥AB

Required: Δ = ΔABC, base∈AB(pr), apex∈MN

Proposition 38. Theorem



Triangles on equal bases between same ∥s have equal area.

ΔABC,DEF: if BC=EF, AD∥BCEF,

Then ΔABC = ΔDEF

Proof

Produce GADH, BG∥AC, FH∥DE, join CE

∴GBCA, DEFH ≡ ∥gm (d.1.30)

BC=EF and AD∥BCEF (hyp) ∴GBCA = DEFH (1.36)

ΔABC,DEF = ½∥gmGBCA,DEFH (1.34)

∴ΔABC = ΔDEF (a.7)

Problems

105. Theorem

∀ΔABC: if D,E mdpt AB,AC, BE × CD @ F

Then ΔFBC = 4-gon ADFE

106. Theorem

∀ΔABC,DEF: if AB=DE, AC=DF, ∠A + ∠D = 2∟

Then ΔABC = ΔDEF

107. Theorem

∀∥gmABCD: AC,BD create 4 triangles of equal area.

108. Theorem

∀∥gmABCD, ∀P∈BD, join P[AC] then ΔPAD = ΔPCD

109. Theorem *

∀4-gonABCD: if a Δ has sides equal to 4-gon's diagonals and the included angle of those sides equals either of the opposite angles (1.15) of the diagonals, then the Δ = ABCD.

110. Problem *

Given: ∀ΔABC ∀P∈AC (nearer A than C)

Required: Bisect Δ with line on P

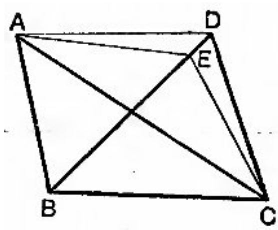
111. Problem *

Given: ∀4-gon ∀vertex

Required: Bisect 4-gon with line on vertex

(For minimal agony, see problem diagram instructions.)

Proposition 39. Theorem



Equal triangles, on same side of same base, are between same parallels.

If ΔABC = ΔDBC and A,D same side BC then AD∥BC

Proof

Join AD. Then AD∥BC.

Else let AE∥BC × BD @ E. Join EC.

ΔABC,EBC on BC, ·|·(AE∥BC) ∴ΔABC = ΔEBC (1.37)

ΔABC = ΔDBC (hyp) ∴ ΔDBC = ΔEBC (a.1) or greater = lesser ↴

∴AE !∥ BC Sym. no other line but AD∥BC ∴ AD∥BC

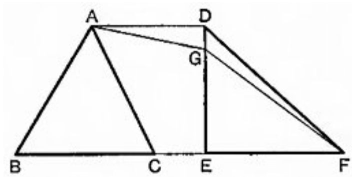
Problems

112. Theorem

∀AB,CD: AB × CD @ E, if ΔAEC = ΔBED then AD∥BC.

It can happen at some point that you are no longer able to solve the problems. This is not uncommon. It is perfectly valid to continue on, studying the solutions. But my approach to Euclid has been to restart the problems. Go back to Problem 1 and start over, bringing to bear all you have learned to solve the problems again. You will solve more this time than on your first pass. Annoyingly, you will be unable to solve some you solved before. In my case, I restarted twice and worked on all 625 problems in Todhunter's Euclid, solving a fair number and studying his solutions of all of them. This approach is easier if you remove all sense of limiting deadlines from your thought. Make it a free and joyful effort.

Proposition 40. Theorem



Equal triangles on equal bases on same side of same line are between the same parallels.
If ΔABC = ΔDEF on same side BF, BC=EF, then BF∥AD.

Proof

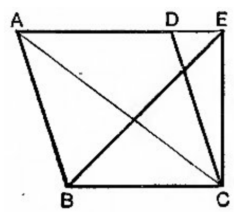
Join AD. Then AD∥BF. Else let AG∥BF × DE @ G. Join GF.
ΔABC,GEF: BC=EF, AG∥BF ∴ΔABC = ΔGEF (1.38)
ΔABC = ΔDEF (hyp) ∴ΔDEF = ΔGEF or greater = lesser ⤵
∴AG !∥ BF Sym. no other line but AD∥BF ∴AD∥BF

Problems

113. Theorem

∀ΔABC,DBC: A,D opp sides of BC: if ΔABC = ΔDBC then BC ×/2 AD

Proposition 41. Theorem



∀∥gmABCD ∀ΔEBC: if ADE∥BC
then ∥gmABCD = 2ΔEBC

Proof

Join AC. ΔABC,EBC on BC, ·|·(BC∥AE)
∴ΔABC = ΔEBC (1.37)
AC ×/2 ∥gmABCD (1.34) ∴∥gmABCD = 2ΔABC
∴∥gmABCD = 2ΔEBC

Problems

114. Theorem

∀∥gmABCD: if EF ×/2 ∥gmABCD, EF × AD,BC @ E,F,
Then ΔEBF = ΔCED

115. Theorem

∀4-gonABCD: if BC ∥AD, E mdpt CD then ΔAEB = ½ 4-gon

116. Theorem

∀∥gmABCD: if O mdpt BD then ∀line on O ·|·(AD,BC) ×/2 ∥gm

117. Problem

Given: ∀∥gmABCD, ∀P∈∥gm

Required: Bisect ∥gm with line on P

118. Theorem

∀ΔABC: Line joining midpoints of sides is parallel to the base.

119. Theorem

∀ΔABC: Line joining midpoints of sides = ½ base.

120. Theorem

∀ΔABC, ∀D∈BC, if E,F,G,H mdpt BD,DC,AB,AC then EG=FH

121. Theorem

∀4-gon: lines joining mdpts adj sides form ∥gm

122. Problem *

Given: mdpts of three sides of Δ

Required: implied Δ

123. Theorem

∀ΔABC: if E,F mdpt AB,AC, alt∠A × BC @ D

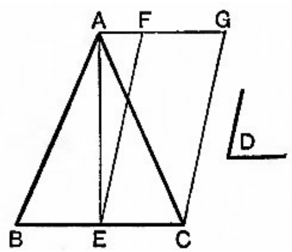
Then 1) ∠FDE = ∠BAC 2) AFDE = ½ΔABC

124. Theorem

∀∥gms ABCD = BEFC = EGHF: if DE,CG × BC,EF @ K,L

Then ∥gmKELC = ½ of each equal ∥gm

Proposition 42. Problem



Given: ΔABC, ∠D

Required: ∥gm with ∠D = ΔABC

Method

×/2 BC @ E (1.10) Join AE.

Copy ∠D to ∠CEF (1.23)

AFG∥BC and CG∥EF (1.31)

∥gmFGCE required

Proof

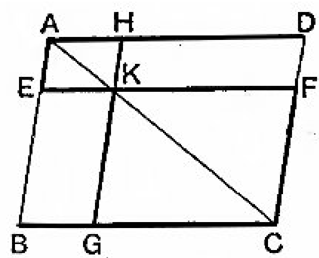
BE=EC and BC∥AG (con) ∴ΔABE = ΔAEC (1.38) ∴ ΔABC = 2ΔAEC

∥gmFGCE,ΔAEC: base AC ·|·(EC∥AG) ∴FGCE = 2ΔAEC (1.41)

∴∥gmFGCE = ΔABC and ∠CEF = ∠D

In the following figure is ∥gmABCD with diagAC. ∀K∈AC (or BD), add EKF∥AD and HKG∥AB. This creates four ∥gms. We can denote ∥gms by opposite corners, i.e., ∥gmAHKE ≡ ∥gmAK or simply AK. So in ∥gmABCD, we have AK and KC on the diagonal, BK and KD off the diagonal. BK and KD are called **complements**.

Proposition 43. Theorem



∀∥gm, complements are equal.
∀∥gm ABCD ∀K∈AC: BK = KD

Proof

AHKE ≡ ∥gm with diagAK
∴ΔAEK = ΔAHK (1.34)
Sym. ΔKGC = ΔKFC
∴ ΔAEK + ΔKGC = ΔAHK + ΔKFC
AC ×/2 ∥gmABCD ∴ ΔABC = ΔADC (1.34)
∴ΔABC - (ΔAEK + ΔKGC) = ΔADC - (ΔAHK + ΔKFC)
∴ BK = KD

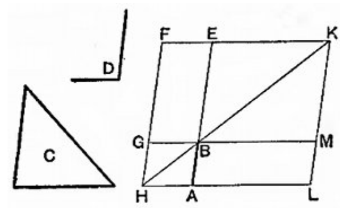
Problems

125. Theorem *

∀∥gmABCD, ∀O∈∥gm: if ∃two lines on O parallel to sides and ∥gmOB = ∥gmOD then O∈AC

I have a theory about the next three propositions. Greek geometry was greatly concerned with the perfect figure of a square. Proposition 46 allows us to create a square on any line. But what if we want to compare some other figure with that square? Proposition 44, with 42 as lemma, lets us put a parallelogram equal to the simplest figure, a triangle, on any line. And a square is a parallelogram. Then proposition 45, extending 44, lets us cut up any n-gon, starting with a 4-gon as example, and turn it into a parallelogram. So we can take any rectilineal figure (n-gon) and turn it into a square on a given line. The Greeks studied, geometrically, the form of number (Euclid, Books VII to X). And the side of a square gives us the square root of any n-gon's area.

Proposition 44. Problem



Given: ∀AB, ∀ΔC, ∀∠D
Required: ∥gm on AB with ∠D = ΔC

Method

∥gmFEBG = ΔC with ∠EBG = ∠D on ABE (1.42)
AH∥BG (1.31) × FG @ H. Join HB.
HF × ∥s AH,EF ∴ ∠AHF + ∠HFE = 2∟ (1.29)
∴ ∠BHF + ∠HFE < 2L ∴ HB(pr) × FE(pr) @ K (a.12) towards B
KL∥EA × HA,GB @ L,M
∥gmBL required

Proof

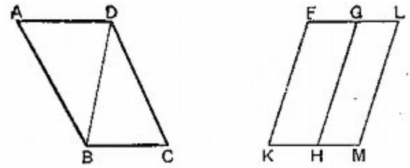
∥gmHLFK: ∥gmBL,FB complements ∴ ∥gmBL = ∥gmFB (1.43)
∥gmFB = ΔC (con) ∴ ∥gmBL = ΔC (a.1)
∠GBE = ∠D ∴ ∠ABM = ∠D (1.15)
∴ ∥gmBL on AB with ∠D = ΔC

We are all bozos on this bus. It is a very long bus. People change seats all the time. That idiot from way back behind you is now sitting up near the front. And when he talks mathematics now, all you can make sense of are the pronouns and some of the verbs.

With diligence, you will also move into seats now in front of you. Be nice to the people you sit with, even if they are snobs or bullies. Many riders choose to wear Smartie Pants. And if the wearers are smart enough, they get away with dressing so ridiculously while they are on our bus. Away from the bus, they pay a heavy price. Some of them get the smackdown they deserve and change their ways. Some of the really smart people in Smartie Pants never learn any better and pay a heavy price. The life of George Hardy is a cautionary tale along these lines.

Part of real mathematics is real humility -- not just now, while you are slow and thick, but always. And this quality will help to get you a life worth living.

Proposition 45. Problem



Given: ∀n-gon ∀∠

Required: ‖gm = n-gon w/∠

Method

∀n-gonABCD, ∀∠E. Join DB.

‖gmFGHK = ΔADB with ∠FKH = ∠E (1.42)

‖gmGLMH = ΔDBC with ∠GHM = ∠E (1.44)

‖gmKMLF required

Proof

∠E = ∠FKH,GHM (con) ∴ ∠FKH = ∠GHM (a.1)

∴ ∠KHG + ∠FKH = ∠GHM + ∠KHG (a.2)

∠FKH + ∠KHG = 2∟ (1.29) ∴ ∠GHM + ∠KHG = 2∟

∴ KHM one line (1.14)

HG × ‖s KM,FG ∴ ∠MHG = ∠HGF (1.29)

∴ ∠HGL + ∠MHG = ∠HGF + ∠HGL

∠HGL + ∠MHG = 2∟ (1.29) ∴ ∠HGF + ∠HGL = 2∟

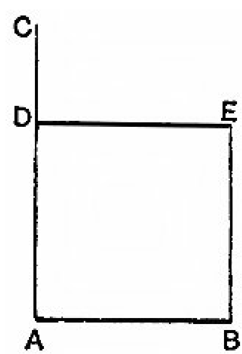
∴ FGL one line (1.14)

KF‖HG and HG‖ML ∴ KF‖ML (1.30)

KM‖FL (con) ∴ KMLF ≡ ‖gm

ΔABD,DBC = ‖gmHF,GM (con) ∴ABCD = ‖gmKMLF

Proposition 46. Problem



Given: ∀AB

Required: AB^2 (square on AB)

Method

AC⊥AB (1.11) AD=AB (1.3)

DE,BE‖AB,AD (1.31) ABED required

Proof

ABED ≡ ‖gm (con) ∴AB,AD=DE,BE (1.34)

AB=AD (con) ∴ AB=AD=DE=BE (a.1)

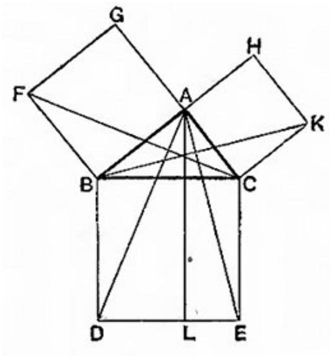
∠BAD = ∟(con) ∴ ABED ≡ square (d.1.31)

d.1.31 A **square** is an eqS 4-gon with one right angle.

That's right: one right angle. You can prove it for yourself. And 1.47, coming up, is the Pythagorean Theorem. But you knew that.

Proposition 47. Theorem

∀ ⊿ABC ∟A: $BC^2 = AB^2 + AC^2$



Proof

BC^2, AB^2, AC^2 ≡ BCED,ABFG,ACKH (1.46)

AL∥BD (1.31) Join AD,FC

∠BAC = ∟ (hyp) ∠BAG = ∟ (d.1.31)

∴ CG colinear (1.14) Sym. BH colinear

∠DBC = ∠FBA = ∟ (a.11)

∴ ∠ABC + ∠DBC = ∠FBA + ∠ABC

∴ ∠DBA = ∠FBC (a.2)

ΔABD,FBC: AB=FB and BD=BC (con) ∠DBA = ∠FBC

∴ ΔABD ≡ ΔFBC (1.4)

∥gmBL,ΔABD: on BD ·|·(BD∥AL) ∴ ∥gmBL = 2ΔABD (1.41)

Sym. FB^2 (ABFG) = 2ΔFBC (1.41)

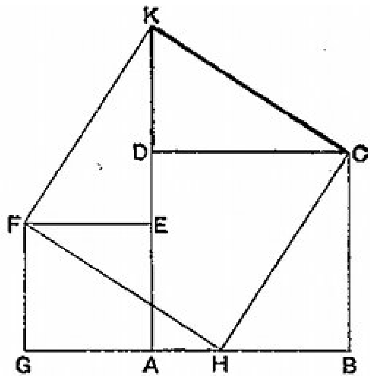
∴ ∥gmBL = $FB^2 = AB^2$ (a.6)

Sym. Join AE,BK and ∥gmCL = AC^2

∴ ∥gmBL + ∥gmCL = $AB^2 + AC^2$ (a.2)

∴ $BC^2 = AB^2 + AC^2$ (a.1)

Aliter



∀GB, ∀A∈GB: GA^2, AB^2 (1.46)

AB=GH=EK (1.3)

Join HC,CK,KF,FH

GH=AB (con) ∴ HB=GA=FE=FG

EK=AD (d.1.31) ∴ DK=AE=FG=HB

∴ ΔFGH,FEK,HBC,KDC equivalent

∴ AEFG + ADCB = FHCK

and CH=FH=FK=KC

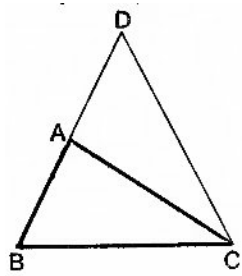
and ∠KCD = ∠HCB

∴ ∠HCK = ∠BCD = ∟ ∴ FKCH ≡ CH^2

CH ≡ hypotenuse of ⊿BCH and BH=AG and CB=AB

"Aliter" is Latin for "alternatively" and this alternative proof is one of the few things in mathematics so far that actually strikes me as "beautiful" in the sense of "elegantly reasoned."

Proposition 48. Theorem



∀ΔABC: If $BC^2 = AB^2 + AC^2$ then ∠A = ∟

Proof

AD⊥AC AD=AB Join DC (1.11,3, p.1)

DA=AB ∴ $DA^2=AB^2$

∴ $AC^2 + DA^2 = AB^2 + AC^2$ (a.2)

∠DAC = ∟ (con) ∴ $DC^2 = DA^2 + AC^2$ (1.47)

$BC^2 = AB^2 + AC^2$ (hyp) ∴ $DC^2 = BC^2$ ∴ DC=BC

ΔBAC,DAC: AC=AC (a.1) BA=AD (con) DC=BC ∴ ∠DAC = ∠BAC (1.4)

∠DAC = ∟ ∴ ∠BAC = ∟

Problems

126. Theorem

∀ΔABC, if AC^2 ≡ ACDE BC^2 ≡ BCFH then AF=BD

127. Theorem

∀ΔABC, if ∠A < ∟ then $BC^2 < AB^2 + AC^2$

128. Theorem

∀ΔABC, if ∠A > ∟ then $BC^2 > AB^2 + AC^2$

129. Theorem

Prove converse of #127 and #128

(if $BC^2 < AB^2 + AC^2$ then ∠A acute, etc.)

130. Theorem

∀ ⊿ABC ∟A: if ∀DE∥BC × AB(pr),AC(pr) @ D,E

Then $BE^2 + CD^2 = BC^2 + DE^2$

131. Theorem

∀rect∟ ABCD, ∀P: $PA^2 + PC^2 = PB^2 + PD^2$

132. Theorem

∀ ⊿ABC ∟A: if BE,CF med ∠B,C then $4(BE^2 + CF^2) = 5BC^2$

133. Theorem

∀ ⊿ABC ∟A: if $AC^2 = 3AB^2$, AD med ∠A, AE alt∠A

Then ∠BAE = ∠EAD = ∠DAC

134. Theorem

∀ ⊿ABC ∟A, if squares BDEC,AFGB,AHJC then $DG^2 + EJ^2 = 5BC^2$

Euclid - Book II

Euclid's second book is closely associated with algebra. Euclid, of course, could not have had this in mind. But the results are, for us, largely algebraic. However, in Euclidean space, there are no negative quantities. If AB > CD, there is no possible representation, hence no meaning, for CD - AB. And the same thing goes for CD^2 - AB^2. But keeping things positive, you can use algebra in your methods and proofs.

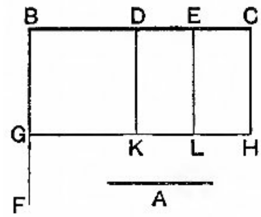
Let's say you have two lines L = AB + CD and M = AB - CD and you are required to find AB and CD. Then

$$L + M = (AB + CD) + (AB - CD)$$
$$= AB + CD + AB - CD$$
$$= 2AB$$
$$\text{Sym. } L - M = 2CD$$
$$\therefore AB = \tfrac{1}{2}(L + M) \text{ and } CD = \tfrac{1}{2}(L - M)$$

And there you are, no ruler, compass, or other theorems required. There are three definitions in Book II:

d.2.1 ∀ rectangle ABCD is **contained** by any two adjacent sides. In our notation, this is "rect∟ABCD ≡ AB•AD".
d.2.2 In ∀ ∥gm, there are two internal ∥gms on a diagonal and two complements. The complements combined with either internal ∥gms is a **gnomon**.
d.2.3 ∀ AB produced in both directions: if we choose a point (cut) between A and B, we divide AB **internally**. If we choose a point to either side, outside of AB, we divide AB **externally**.

Proposition 1. Theorem



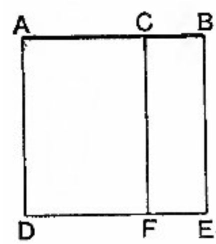
∀2 lines: if one is divided into parts, then rect∟ on lines equals sum of rect∟s on the parts.
∀lines A,BC: BC cut D,E: then
A•BC = A•BD + A•DE + A•EC

Proof

BF⊥BC: BG=A (1.11,3) and GH∥BC (1.31)
DK,EL,CH∥BG × GH @ K,L,H (1.31)
rect∟BH = rect∟BK + DL + EH
BG=A (con) ∴ rect∟BH ≡ A•BC
Sym. BK,DL,EH ≡ A•BD,A•DE,A•EC
∴ A•BC = A•BD + A•DE + A•EC

Algebra: a(b + c + d) = ab + ac + ad

Proposition 2. Theorem



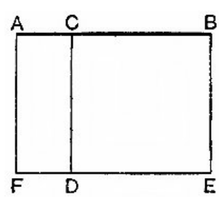
∀AB cut C: then AB•AC + AB•CB = AB^2

Proof

AB^2 ≡ ADEB (1.46)
CF∥AD × DE @ F (1.31)
rect∟AE = rect∟AF + rect∟CE and rect∟AE = AB^2 (con)
AB=AD (con) ∴ rect∟AF = AB•AC and rect∟CE = AB•CB
∴ AB•AC + AB•CB = AB^2

Algebra: $a(a + b) + b(a + b) = (a + b)^2$

Proposition 3. Theorem



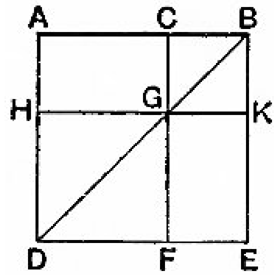
∀AB cut C: then AB•BC = BC^2 + AC•CB

Proof

BC^2 = CDEB (1.46) AF∥CD × ED(pr) @ F (1.31)
rect∟AE = rect∟AD + rect∟CE (con)
BE=BC (con) ∴ rect∟AE = AB•BC
CD=CB (con) ∴ rect∟AD = AC•CB
rect∟CE = BC^2 (con)
∴ AB•BC = BC^2 + AC•CB

Algebra: $(a + b)b = ab + b^2$

Proposition 4. Theorem



∀AB cut C:
$AB^2 = AC^2 + CB^2 + 2(AC•CB)$

Proof

AB^2 ≡ ADEB (1.46) Join BD.
CF∥AD × HK∥AB @ G
BD × ∥s CF,AD ∴ ∠CGB = ∠ADB (1.29)
∠ADB = ∠ABD and BA=AD (con)
∴ ∠CGB = ∠CBG (a.1) ∴ CG=CB (1.6)
CB,CG = GK,BK (1.34) ∴ ∥gmCK ≡ eqS
and ∠CBK = ∟ (1.46) ∴ ∥gmCK ≡ CB^2 (d.1.31)
Sym. ∥gmHF ≡ AC^2 (1.34, d.1.31)
∥gmAG = ∥gmGE (1.43)
CG=CB ∴ rect∟AG = AC•CB ∴ rect∟GE = AC•CB (a.1)
∴ rect∟AG + rect∟GE = 2(AC•CB)
AB^2 = ∑HF,CK,AG,GE ∴ $AB^2 = AC^2 + CB^2 + 2(AC•CB)$

Algebra: $(a + b)^2 = a^2 + b^2 + 2ab$

Corollary 1.

∀∥gm on diagonal of a square is a square

Corollary 2.

In 2.4, if C mdpt AB, then AC•CB = AC^2 ∴ $AB^2 = 4(½AB)^2$

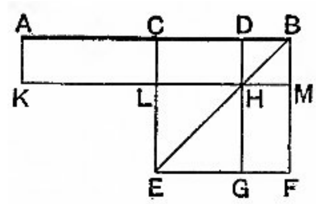
Problems

135. Theorem

∀AB cut C: if 2(AC•CB) = $AC^2 + CB^2$ then C mdpt AB

The algebra of Euclid is very powerful. Lagrange used it to determine the roots of polynomials. When analytic geometry came along, many mathematicians were able to prove theorems which were beyond the power of their grasp of pure geometry. But Newton used Euclid's pure geometry to prove the theorems of the Calculus in his Principia. Euclid is as powerful a tool as your realization of it becomes. And it is a great simplifier, removing more extraneous aspects of a problem than perhaps any other mathematic.

Proposition 5. Theorem



∀AB, C mdpt AB, ∀D∈CB:

$AD•DB + CD^2 = CB^2$

Proof

CB^2 = CEFB (1.46) Join BE.

DG∥CE × BE,EF @ H,G

MH(pr)∥CB × AK∥CE @ K and CE @ L

∥gmCH = ∥gmHF (1.43) ∴ DM + CH = HF + DM ∴ ∥gmCM = ∥gmDF

AC=CB ∴AL=CM (1.36) ∴ AL = DF ∴CH + AL = DF + CH

∴∥gmAH = ∥gmDF + ∥gmCH = gnomonCMG

DH=DB (2.4.C1) ∴AH = rect∟AD•DB ∴rect∟AH = gnomonCMG

∴LH^2 + AH = CMG + LH^2

$LH^2 = LG = CD^2$ (2.4.C1, 1.34) ∴$AD•DB + CD^2$ = CMG + LG

∴$AD•DB + CD^2 = CB^2$

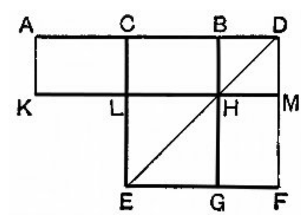
Algebra: $(a + b)(a - b) + b^2 = a^2$

Problems

136. Problem

Given: ∀AB, required C: AC•CB maximum area

Proposition 6. Theorem



∀AB, C mdpt AB, AB(pr) to D:

$AD•DB + CB^2 = CD^2$

Proof

CB^2 = CEFB (1.46) Join BE

BG∥CE × DE,EF @ H,G

MH(pr)∥CD × AK∥CE @ K and CE @ L

AC=CB (hyp) ∴rect∟AL = rect∟CH

CH = HF (1.43) ∴AL = HF ∴CM + AL = HF + CM

∴rect∟AM = gnomonCMG

DM=DB (2.4.C1) ∴AM = rect∟AD•DB

∴rect∟AD•DB = gnomonCMG ∴ LG + AD•DB = CMG + LG = CD^2

$LG = CB^2$ (2.4.C1, 1.34) ∴ $AD•DB + CB^2 = CD^2$

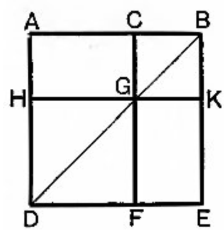
Algebra: $(a + b)(b - a) + a^2 = b^2$

Problems

137. Problem

Given: lines KL,MN: KL > MN

Required: rect∟(KL^2 - MN^2)



Proposition 7. Theorem

∀AB cut ∀C then $AB^2 + BC^2 = AC^2 + 2AB•BC$

Proof

AB^2 = ADEB. HK,CF,DE as above.

AG=GE (1.43) ∴ CK + AG = GE + CK

∴ AK=CE ∴ AK + CE = 2AK

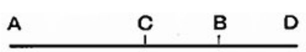
BK=BC ∴ 2AB•BC = AK + CE

∴ 2AB•BC = gnomonAKF + CK

HF = AC^2 (2.4.C1, 1.34) ∴ AC^2 + 2AB•BC = gnomonCKF + CK + HF

∴ $AC^2 + 2AB•BC = AB^2 + BC^2$

Algebra: $a^2 + b^2 = 2ab + (a - b)^2$



Proposition 8. Theorem

∀AB cut ∀C then $(AB+CB)^2 = 4(AB•CB) + AC^2$

Proof

AB(pr) to D: BD=CB

AD cut B ∴ $AD^2 = AB^2 + BD^2 + 2AB•BD$ (2.4)

BD=BC ∴ $BD^2 = BC^2$ and AB•BD = AB•BC

∴ $AD^2 = AB^2 + BC^2 + 2AB•BC$

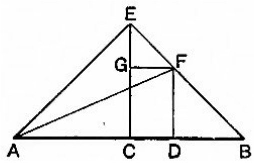
$AB^2 + BC^2 = AC^2 + 2AB•BC$ (2.7)

∴ $AD^2 = (AB+CB)^2 = AC^2 + 4AB•BC$

Algebra: $4ab + (a - b)^2 = (a + b)^2$

That last proof was not Euclid's. His proof was most often described as "cumbrous" and, by the 19th century, authors stopped using it. The next two are not Euclid's either. They are Todhunter's alternative proofs that prove Book II propositions with Book II theorems, which is something Euclid didn't do. The diagrams go with Euclid's proof, which are about four times as long and use only Book I triangles.

Proposition 9. Theorem



$\forall$AB, C mdpt AB, $\forall$D$\in$CB:

$AD^2 + DB^2 = 2(AC^2 + CD^2)$

Proof

AD cut C $\therefore AC^2 + CD^2 + 2AC \bullet CD = AD^2$ (2.4)

BC cut internally at D $\therefore BC^2 + CD^2 = 2BC \bullet CD + BD^2$ (2.7)

AC=BC $\therefore AC^2 + CD^2 = 2AC \bullet CD + BD^2$

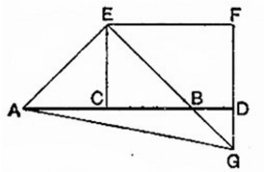
Add lines 1 and 3

$\therefore$ $2(AC^2 + CD^2) + 2AC \bullet CD = AD^2 + BD^2 + 2AC \bullet CD$

$\therefore AD^2 + DB^2 = 2(AC^2 + CD^2)$

Algebra: AC=CB=a CD=b, $(a + b)^2 + (a - b)^2 = 2a^2 + 2b^2$

Proposition 10. Theorem



$\forall$AB C mdpt AB $\forall$AB(pr) to D:

$AD^2 + DB^2 = 2(AC^2 + CD^2)$

Proof

AD cut C $\therefore AC^2 + CD^2 + 2AC \bullet CD = AD^2$ (2.4)

BC cut externally at D $\therefore BC^2 + CD^2 = 2BC \bullet CD + BD^2$ (2.7)

AC=BC $\therefore AC^2 + CD^2 = 2AC \bullet CD + BD^2$

Add lines 1 and 3

$\therefore$ $2(AC^2 + CD^2) + 2AC \bullet CD = AD^2 + BD^2 + 2AC \bullet CD$

$\therefore AD^2 + DB^2 = 2(AC^2 + CD^2)$

Algebra: AC=CB=a CD=b, $(a + b)^2 + (a - b)^2 = 2a^2 + 2b^2$

Problems

138. Problem

Given: $\forall$AB Required: C$\in$AB: $AC^2 + CB^2$ minimum

139. Theorem

$\forall$lines A,B: $(A + B)^2 + (A - B)^2 = 2(A^2 + B^2)$

140. Problem *

Given: $\forall$AB,KL

Required: C$\in$AB: $AC^2 + CB^2 = KL^2$

141. Problem

Given: $\forall$AB Required: P$\in$AB: $AP^2 = 2PB^2$

Proposition 11. Problem

Given: ∀AB

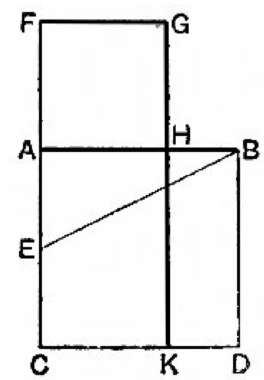
Required: H∈AB: AB•BH = AH²

Method

AB² = ABDC (1.46) E mdpt AC (1.10)

Join BE. CA(pr) to F: EF=EB (1.3)

AF² = AFGH. H required



Proof

GH(pr) × CD @ K

E mdpt AC ∴ CF•FA + AE² = EF² (2.6)

EF=EB (con) ∴ CF•FA + AE² = EB²

∠EAB ≡ ∟ ∴ CF•FA + AE² = AE² + AB² (1.47)

∴ CF•FA = AB²

FG=FA (con) ∴ rect∟FK = CF•FA

AD = AB² ∴ FK=AD ∴ FK - AK = AD - AK ∴ FH=HD (a.3)

AB=BD (con) ∴ HD = AB•BH

and FH = AH² ∴ AB•BH = AH²

Problems

142. Theorem

In diagram 2.11, if CH × BF @ L, then CL⊥BF

143. Theorem

In diagram 2.11, if CH × BE,BF @ O,L, then AO⊥CH

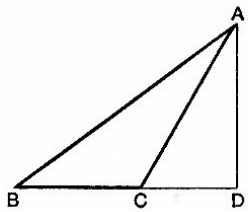
144. Theorem

In diagram 2.11, if AB•HB = AH²,

Then (AH + HB)(AH - HB) = AH•HB

In the next two propositions, we establish the relation in a triangle of an acute or obtuse angle and its opposite side. Proposition 1.47 establishes this relation for a right angle and its opposite side. In these propositions, the transition into the final lines of proof are not terribly clear in Euclid and I have not cleared them up as I can see the value of your working out the substitutions Euclid is making in the algebraic resolution of these proofs.

Proposition 12. Theorem

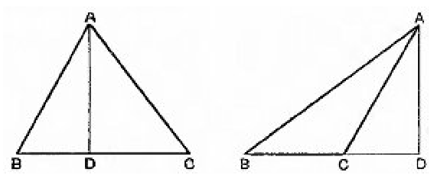


$\forall \Delta ABC$, $\forall \angle > ∟$ ($\angle C$) ,
$AD \perp BC \times BC$(pr) @ D, then
$AB^2 = AC^2 + BC^2 + 2BC \bullet CD$

Proof

BD cut C $\therefore BD^2 = BC^2 + CD^2 + 2BC \bullet CD$ (2.4)
$\therefore AD^2 + BD^2 = BC^2 + CD^2 + 2BC \bullet CD + AD^2$ (a.2)
$\angle D =$ ∟ $\therefore AB^2 = BD^2 + AD^2$ and $AC^2 = CD^2 + AD^2$ (1.47)
$\therefore AB^2 = AC^2 + BC^2 + 2BC \bullet CD$

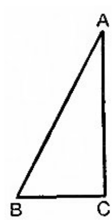
Proposition 13. Theorem



$\forall \Delta ABC$, $\forall \angle <$ ∟ ($\angle B$),
$AD \perp BC$(pr) @ D, then
$AC^2 = AB^2 + BC^2 - 2BC \bullet BD$

Proof

Case 1) AC $! \perp$ BC
BC cut D or BD cut C
$\therefore$ (in both) $BC^2 + BD^2 = CD^2 + 2BC \bullet BD$ (2.7)
$\therefore AD^2 + BC^2 + BD^2 = CD^2 + 2BC \bullet BD + AD^2$ (a.2)
$\angle D =$ ∟ $\therefore AB^2 = AD^2 + BD^2$ and $AC^2 = AD^2 + CD^2$ (1.47)
$\therefore AB^2 + BC^2 = 2BC \bullet BD + AC^2$
$\therefore AC^2 = AB^2 + BC^2 - 2BC \bullet BD$



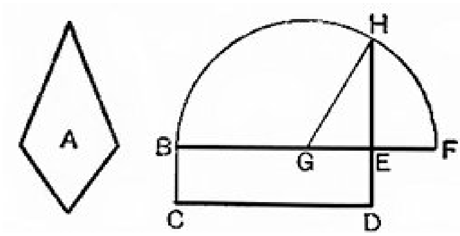
Case 2) $AC \perp BC$ $\therefore$ C, not D, is foot of $\perp$
$AB^2 = AC^2 + BC^2$ (1.47)
$\therefore AB^2 + BC^2 = AC^2 + 2BC^2$ (a.2)
$BC^2 = BC \bullet BC$ $\therefore$ $AC^2 = AB^2 + BC^2 - 2BC \bullet BC$

The power of each mathematic is the different view it gives of the form, or essence, of number. Suppose you were considering three algebraic quantities, the sum of two in relation to the third. Perhaps they are complicated expressions. Call them a,b,c. If a + b is greater than c, then the expressions come under every law of triangles under Euclid.

Proposition 14. Problem

Given: ∀n-gonA

Required: square = A



Method

∥gm BCDE = n-gonA (1.45)

If BE=ED, then BD required

Else BE(pr) to F: EF=ED. (1.3)

G mdpt BF (1.10)

semi-⊙G,GB × DE(pr) @ H. EH^2 required

Proof

Join GH.

G mdpt BF, E∈GF ∴ $GF^2 = BE \cdot EF + EG^2$ (2.5)

GF=GH (con) ∴ $GH^2 = BE \cdot EF + EG^2$

∴ $GH^2 = EG^2 + EH^2 = BE \cdot EF + EG^2$ (1.47)

∴ $EH^2 = BE \cdot EF = BE \cdot ED$ (a.3, con) = ∥gmBCDE = n-gonA

Problems

145. Theorem

∀isosΔABC, if CD alt∠C, then $BC^2 = 2AB \cdot BD$

146. Theorem

∀ΔABC: $AB^2 + AC^2 = 2(\frac{1}{2}BC)^2 + 2(AD \text{ med}\angle A)^2$

Two cases: 1) AD⊥BC 2)AD !⊥ BC

147. Theorem

∀isosΔABC: if AB(pr) to D: AB=BD, then $CD^2 = AB^2 + 2BC^2$

148. Theorem

∀∥gm: $\sum \text{side}^2 = \sum \text{diagonal}^2$

149. Theorem

⊙D,DB, diamBC, ∀A∈⊙, ∀ΔABC, $AB^2 + AC^2$ = constant

150. Theorem

∀4-gonABCD, if E,F,G,H mdpt AB,BC,CD,DA

Then $AC^2 + BD^2 = 2(EG^2 + FH^2)$

151. Theorem

∀∥gmABCD: if AC × BD @ O ⊙O,∀radius ∀P∈⊙O

Then $PA^2 + PB^2 + PC^2 + PD^2 = k$ (where k is some constant)

152. Theorem

∀4-gonABCD, if E,F mdpt AC,BD

Then $AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4EF^2$

153. Theorem

∀eqSΔABC: if AB(pr) to D: AD•DB = BC^2 then $DC^2 = 2BC^2$

154. Theorem

∀ ⊿ABC ∟A: if squareBDEC then $DA^2 + AC^2 = AE^2 + AB^2$

155. Theorem

∀ ⊿ABC ∟A: if AD alt∠A then AD^2 = BD•DC

156. Theorem

∀ ⊿ABC ∟A: if AD alt∠A then AB^2 = BC•BD, AC^2 = BC•CD

157. Theorem

∀ΔABC: if ∠B,C < ∟, BE,CF alt∠B,C then BC^2 = AB•BF + AC•CE

158. Theorem *

∀⊙O,OA: if C,D∈diamAB: CO=OD, ∀E∈⊙O, join E[C,D]

Then $EC^2 + ED^2 = AC^2 + AD^2$

159. Theorem *

∀ΔABC: D∈BC: $AB^2 + BD^2 = AC^2 + CD^2$ then mdpt AD eqD B,C

(eqD ≡ equidistant. Don't try to compute D or you will go mad.)

160. Theorem *

∀isosΔABC, ∀D∈BC then AD^2 + BD•DC = AB^2

161. Theorem *

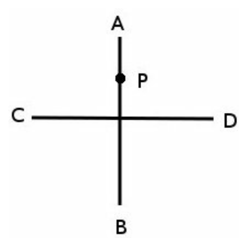
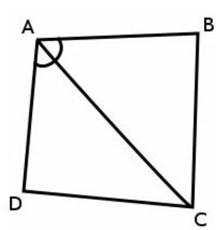
∀ ⊿ABC ∟A,∀D∈AB: DE⊥BC × BC @ E then BC•BE = BA•BD

162. Problem *

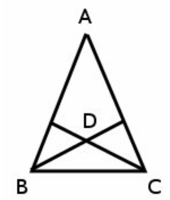
Given: ∀AB,LM: LM < ½AB

Required: P∈AB: AP•PB = LM^2

Problem Diagrams

1. Use diagram from 1.1	**2.** Draw any horizontal line AB. Draw a longer vertical line CD near it.
3. Use diagram from 1.2	4. 
5.  Make single tickmarks on each of the arcs to show that ∠CAB=∠CAD. Make tickmarks on AB, AD to show equality. Now all the data from the problem is visible in the diagram.	**6**. On a smallish base (AB), strike the apex of the eqSΔ with your compass. Without changing your compass, strike the other three apexes and fill in the lines. Carefully label as per data.

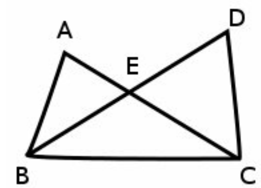
7.



8. Draw rhombus using 1.1. Then label A-D clockwise from top or left.

9. Same as for 8.

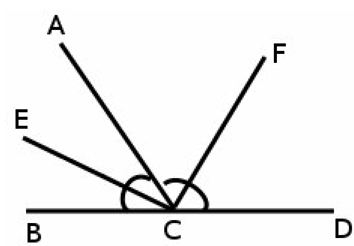
10.



AB does not have to equal DC in the diagram. Mark them as equal with tickmarks. Likewise with AC,DB. Learn to see equality with your mind.

11. The easiest way to draw this is to mark an apex A and swipe a line on your paper with the compass from A for base. Then drop a perpendicular from the apex, put D on it, and add sides.

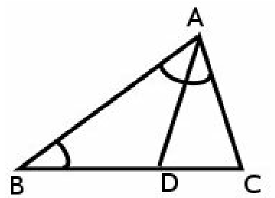
12.



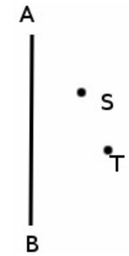
13. Diagram for proposition 1.5

14. Diagram for proposition 1.5

15.



16.



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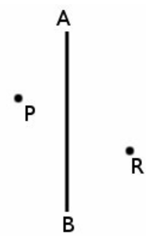

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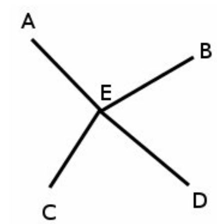

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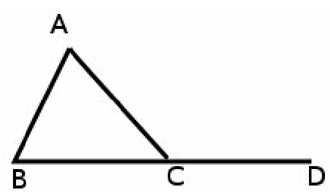

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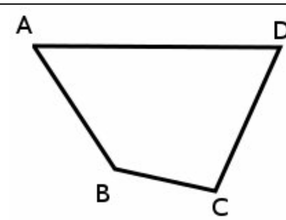

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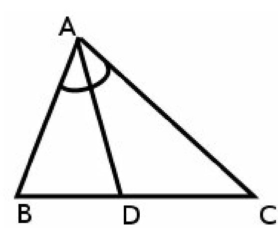

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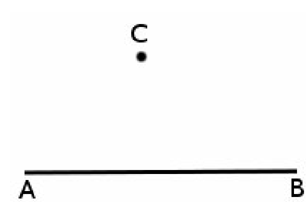
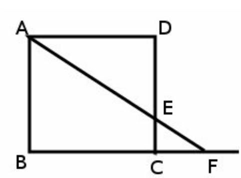
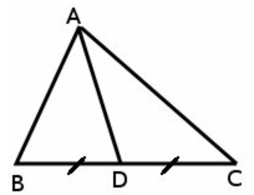
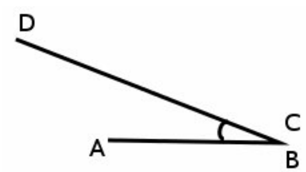
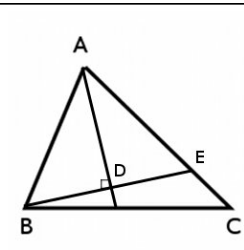
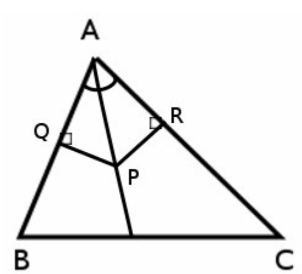
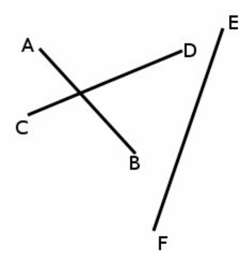
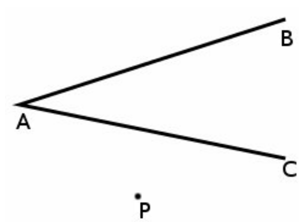
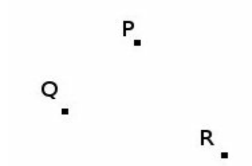


23.

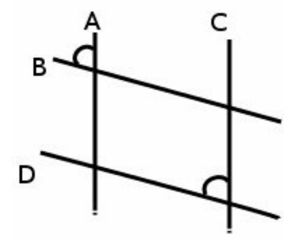


24. Use ∀Δ and for the three cases: a P in the middle of Δ, a P on one side, and a P outside.

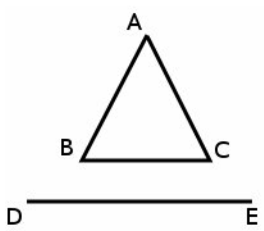
25. ∀4-gon will do.	**26.** 
27. Use ∀Δ	**28.** As for 27.
29. 	**30.** 
31. 	**32.** These lines are just examples. Any lines meeting the conditions will do. 
33. 	**34.** Again, any points meeting the conditions will do. 

35. No diagram provided. Carefully make your own.

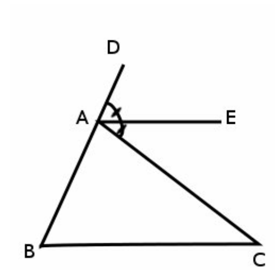
36.



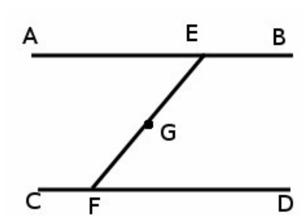
37. DE can be anywhere above or below BC



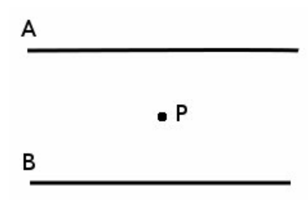
38.



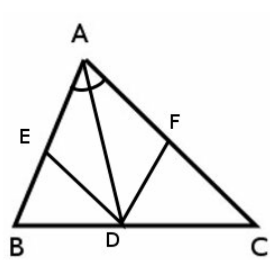
39.



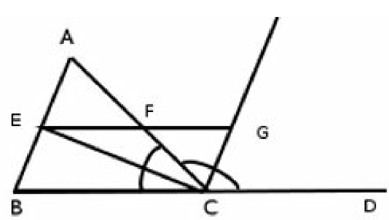
40.

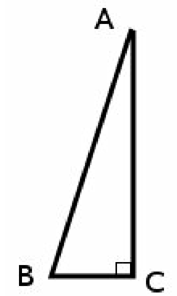
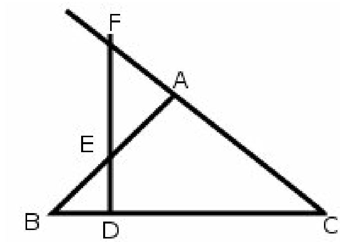
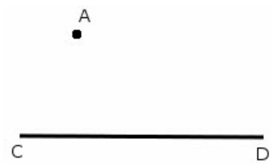
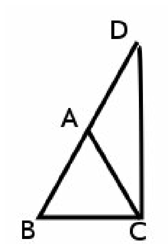
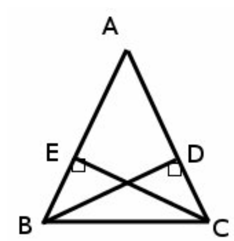
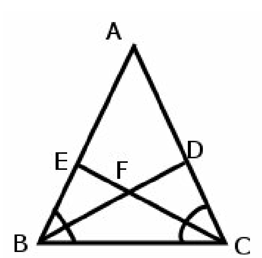
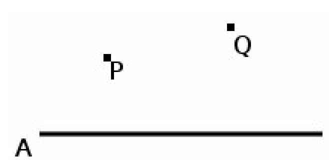
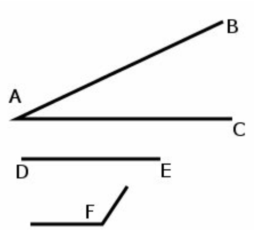
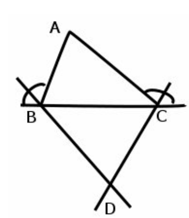


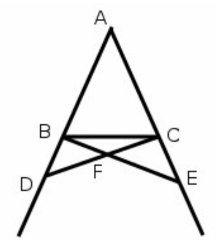
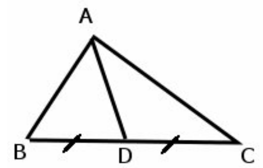
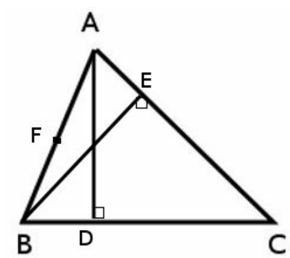
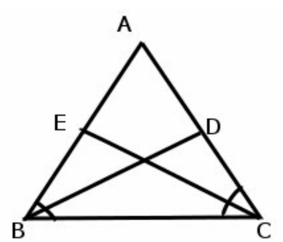
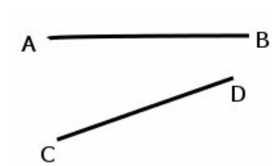
41.

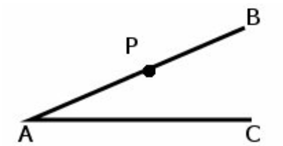
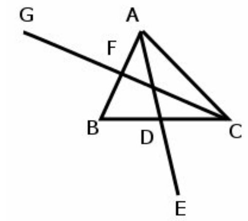
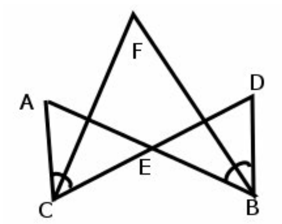
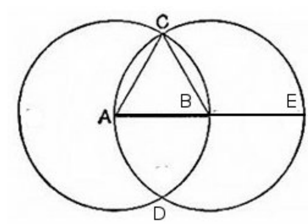


42.



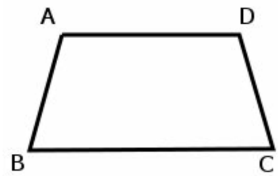
43. 	**44.** 
45. 	**46-50.** No diagrams provided.
51. 	**52.** 
53. 	**54.** 
55. 	**56.** 

57. 	**58.** 
59. 	**60.** Same as 59.
61. 	**62.** 
63-65. No diagrams provided.	**66.** Use ∀ΔABC, ∀DE Make DE long enough using your judgment. Remember that constructions are primarily logic problems.
67. As for 66. Keep the angle and perimeter separate.	**68.** No diagram provided. Don't put P in the middle. But don't crowd it against a line either.

69. 	**70.** 
71. No diagram provided.	**72.** 
73. No diagram provided.	**74.** 
75. ∀4-gon will do.	**76.** ∀4-gon and mark opposite angles as equal: $\angle A = \angle C$, $\angle B = \angle D$
77. ∀∥gm will do.	**78.** Use ∀4-gon which is not a ∥gm to prove it is a ∥gm.

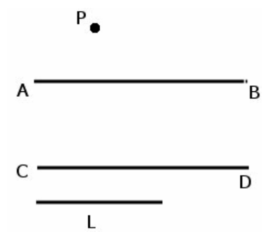
79. ∀ ‖gm will do.

80.



81. Use ∀ ΔABC

82.

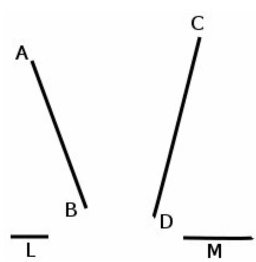


83. Use ∀ ‖gm ABCD

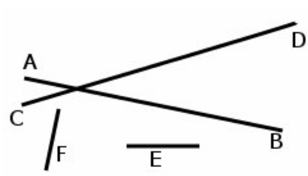
84. Use ∀ ‖gm ABCD

85. Use ∀ ‖gm ABCD

86.

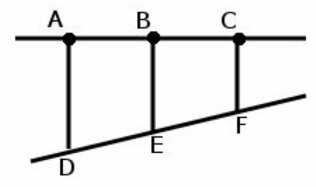


87.

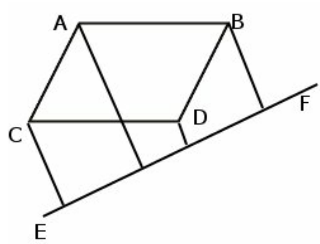


88. Use ∀ ‖gm and correctly place the eqSΔs. Labelling ABCD counter-clockwise from top left puts BC on the base and F on one side or the other of AD.

89.



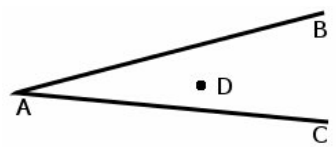
90.



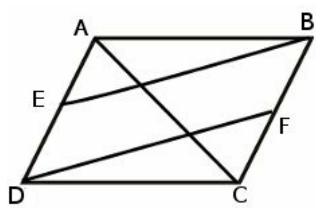
91. No diagram provided.

92. No diagram provided.

93.



94.

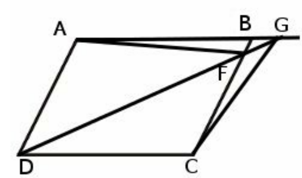


95. Use ∀4-gon, AD∥BC

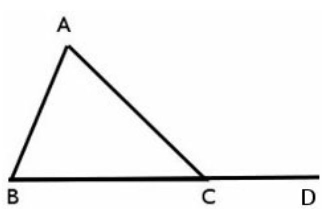
96. Use ∀Δ

97. Construct a rhombus. Join AC, BD. Bisect AB at P.

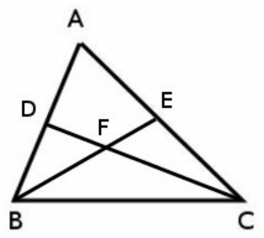
98.



99.

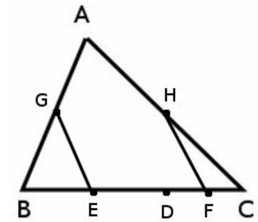


100. Use ∀ΔABC

101. Use ∀4-gon	**102.** Use ∀4-gon
103. Use ∀∥gm	**104.** Use ∀ΔABC. You can put MN anywhere. But about an inch to the left is convenient.
105. 	**106.** Use ∀ΔABC and make ∠D supplementary to ∠A, keeping the correct sides equal.
107. Use ∀∥gm	**108.** Use ∀∥gm
109. If I gave a diagram, it would give the solution.	**110.** Use ∀ΔABC
111. Let vertex A be on the left and prove for A. Make alt∠B < alt∠D.	**112.** No diagram provided.
113. Use notebook lines for ∥s	**114-116.** No diagrams provided.
117. Use ∀∥gm.	**118.** Use ∀ΔABC.

119. Use ∀ΔABC.

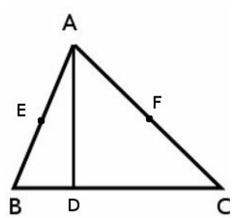
120.



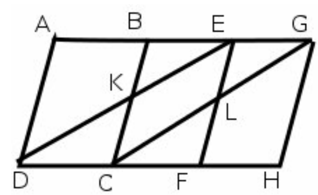
121. Use ∀4-gon.

122. Imagine your Δ. Put in D,E,F as mdpt AB,AC,BC.

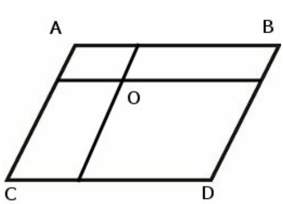
123.



124.



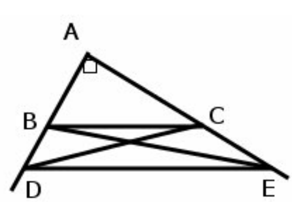
125.



126-128. No diagram provided.

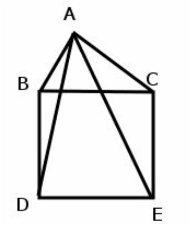
129. No diagram provided.

130.



131. rect∟ABCD. Put P a little off-center below DC.

132. No diagram provided.

133. Diagram is easy but it will mislead you unless you focus on the logic.	**134.** Make sure you label in the squares in the order given.
135. Start with the diagram of proposition 2.4.	**136-138.** No diagrams provided.
139. No diagram provided.	**140.** Diagram will resemble diagram for #29.
141. No diagram provided.	**142-144.** Diagram 2.11
145. No diagram provided.	**146-150.** No diagrams provided.
151. P outside ∥gm is easier.	**152.** No diagram provided.
153. No diagram provided. Don't try to compute BD.	**154.** 
155. No diagram provided.	**156-162.** No diagrams provided.

Problem Hints

I have mixed feelings about hints. Mostly they just make me feel stupider. I have tried to make these both helpful and consistent. But you know what they say about good intentions.

1. You only have one tool. Use it again.
2. You only have two tools. Use them.
3. One radius is fixed by BC.
4. Not only for this problem, but for the problems in general, ask yourself "What is the last tool I acquired?" and see if that tool doesn't solve the problem.
5. Same as for 4.
6. Same as 4 again.
7. Latest tool and use axiom 7.
8. Prove equal triangles.
9. Axiom 2.
10. Prove equal triangles.
11. Equal triangles, external angle.
12. Axiom 2.
13. Equal triangles.
14. Previous results or "diagram reminds you of which problem?"
15. Isosceles triangle.
16. Isosceles triangle.
17. Q is vertex of isosceles triangle.
18. Lines are 2∟. Think a little abstractly and use a.7.
19. Introduce right angles, compare, and sum.
20. A 4-gon is two triangles. And use axiom 2.
21. Use an external angle of the triangle.
22. External angles again.
23. Use both halves of the square.
24. A good diagram should say it all.
25. A 4-gon is also two pairs of two triangles.
26. Double the median.

27. Arithmetic.
28. Use prior result or "What previous problem does this look like?"
29. Where can you build an isosΔ?
30. Where can you see an isosΔ?
31. Prove equal triangles.
32. Try to see problem 31 in this one.
33. Angle bisector.
34. Use analysis, create result image, work backwards.
35. Construct what can be shown to be an isosΔ.
36. Equality is transitive. If x=y and y=z then x=z.
37. Parallel lines can imply equal angles.
38. Use analysis. Assume bisector parallel to base.
39. Proposition 1.26
40. Use previous problem and 1.15.
41. Proposition 1.26
42. Isosceles triangles.
43. Analysis and prior results with an isosceles triangle.
44. Add a parallel line.
45. It's all in parallel lines.
46. Method of problems 42, 43.
47. Pure logic, using "arithmetic" axioms.
48. Use an equilateral triangle.
49. Figure out the angles with algebra.
50. Determine the underlying angle and its relation to 2∟
51. Isosceles triangles and algebrate.
52. What two angles here equal a right angle?
53. Use first part of 1.32 by choosing ext∠.
54. Parallel lines and their angles.
55. Copy DE to AB. Supplementary angles.
56. Proposition 1.32 and algebrate.
57. Algebrate angles until you have equal base angles.
58. ∟A = ∠B + ∠C. Copy one to ∠A and think.
59. Use previous problem's results.
60. Use previous problem's results.

61. Use analysis and proposition 1.28.

62. Here there be isosceles triangles.

63. isosΔ splits out the sides. ⊙ radius = hypotenuse.

64. Use previous problem's methods a bit differently.

65. ∀ ⊿CAB ∟C ≡ ∑(2 isosΔ) sharing median CD.

66. Use 2 isosΔ and proposition 1.32.

67. In ∠FGH build ∟HGK.

68. Construct two right triangles.

69. Add isosΔ with sides = AP.

70. Show ∠FBG + ∠FBD + ∠DBE = 2∟.

71. eqSΔ: ×/2 base∠s,‖line on their × -or- isosΔ: base∠=1/4∟.

72. Show 2∟- (∠FCB+∠FBC) = ½(3d∠s of ΔACE,DBE).

73. Proposition 1.32.C1 and algebrate.

74. Show ΔCDE composed of three equal triangles.

75. Join a diagonal.

76. Proposition 1.28.

77. Consider the pairs of opposite triangles.

78. Erase 4-gon and start with the bisected diagonals.

79. Our old friend, the isosceles triangle.

80. Make a square.

81. ‖gmize ΔABC into ‖gmABCD.

82. Try using a ⊙ for magnitudes.

83. Proposition 1.29.

84. Produce two opp sides in opp directions and bisect angles.

85. Start from ‖gm self-bisected by diagonals.

86. All points distance L from AB make a line.

87. This is for ∀E,F. Put in ∀soln line, add E,F, and work backwards.

88. Make a larger, more accurate diagram than usual. Equal Δs.

89. Add a line to make equal triangles.

90. Turn diagram around in a circle until you see a previous result.

91. Consider that extreme cases and reason it out.

92. Proposition 1.33.

93. D is the vertex of a parallelogram.

94. Introduce a line so that AG,GH sides of equal triangles.

95. Turn your diagram in a circle again.
96. Make a parallelogram equal to the triangle.
97. On CD take CQ = AP.
98. Join BD and look at ΔBFG
99. Join AD and think about proposition 1.37.
100. We are still on 1.37. What is it good for?
101. Join P[AB] and use 1.37.
102. We're still on 1.37, right?
103. One diagonal of rhombus is diagonal of ‖gm.
104. More of that 1.37 stuff but MN is not one of the equal ‖s.
105. Get two equal things sharing a thing.
106. Produce a line that brings in both 1.38 and supplemental ∠s.
107. Use both equivalence and equality and rise above the details.
108. Use your last result.
109. One vertex of the triangle is intersection of diagonals.
110. Proposition 1.38 is used to show some Δ = ½ΔABC.
111. Need line parallel to AC.
112. Add a line so you can use proposition 1.39.
113. ‖gmize something.
114. Join E[BC]. Use proposition 1.41.
115. Think about problem #95.
116. Use prior results.
117. Prior result as brief method requires one line of proof.
118. ‖gmize ΔABC and use prior results.
119. ‖gmize ΔABC and use two prior results.
120. Join AD and use prior results.
121. Add the diagonals and use prior result.
122. Consider #118 and #119.
123. Use prior results.
124. You remember this diagram, right?
125. Let O ∉ AC. Proof by contradiction.
126. Side-angle-side? Hello?
127. Increase ∠A from ∟ while keeping AC constant.
128. Use the method of #127.

129. Use results #127 and #128.

130. It's all about Pythagoras.

131. Turn P[A-D] into diagonals of rectangles.

132. Use algebra and do not let diagram mislead you.

133. If $BC^2 = 4AB^2$ then BC = 2AB. What else equals ½BC?

134. Make DG the hypotenuse of a right triangle, apex G.

135. This is the reverse of 2.4.C2. Just walk it backwards.

136. Let the cut be at D in 2.5 and maximize.

137. Use the form of proposition 2.6.

138. Find a proposition to fit.

139. Proposition 2.9. And faith.

140. Go back to Book I triangle building.

141. Proposition 2.10 and the relation of a square to its diagonal.

142. Find the equal triangles.

143. ∠AOC is the sum of which two other angles?

144. Pure algebra.

145. Proposition 2.13 and algebra.

146. 1) Proposition 1.47. 2) two obvious propositions

147. Rotate diagram and check your short-term memory.

148. Use prior result.

149. And use a prior result.

150. Find the parallelogram, perhaps from a prior result.

151. It's that same previous result.

152. Use results of #146.

153. Make CD the hypotenuse of a right triangle.

154. Make DA,EA hypotenuses of right triangles. Axiom 1.

155. Think $(a + b)(a - b) = a^2 - b^2$

156. Use that last result.

157. Proposition 2.13 and algebrate.

158. Use #146 and same algebra.

159. Really, don't compute D. Use #146 on each half triangle.

160. Add AO med∠A and that algebra again in a different way.

161. Proof uses proposition 2.13 and a new algebra idea.

162. Proposition 2.5. Hypotenuse of ⊿ with side LM.

Problem Solutions

1. Method

On ∀AB, construct an eqSΔ (1.1)

Sym. Join F[AB] to create another. (see diagram 1.1)

Proof

ΔABC, ABF are eqSΔ ∴ AB=AC=BC=AF=BF (a.1)

∴4gon ABCF ≡ rhombus (d.1.33)

Notes

In our notation, we can join one point to several others, as in "Join F[AB]" which is short for "Join FA, FB." When we compare objects, we identify what kind and then list them, as in "ΔABC,ABF".

With every proposition, we should ask what tools it has given us to work with. Euclid will use 1.1's construction for 1.10 and 1.11. But all we can take away from it now, beyond an eqSΔ, is that a circle can show lines are equal if we can make them play the part of radii.

2. Method

Copy CD to A,B (1.2)

⊙A,CD × ⊙B,CD @ E,F (d.1.15)

Join E[AB] (p.1)

ΔEAB required (d.1.24)

Proof

AE,BE = CD (con)

∴AE=BE (a.1)

∴ΔEAB ≡ isosΔ on AB with sides equal to CD (d.1.24)

Notes

Just so we're clear on notation: "Copy CD to [A,B]" means "copy CD to each of A and B". "Join E[A,B]" means "Join EA, EB." The reference to "(con)" means "by construction" or "because I built it that way". These solutions beg the question, "How much do I have to reference the propositions and axioms and stuff?" Answer: "Until you know them by heart." But check the intro to the final appendix to see what that means.

3. Method

Place A ∈⊙B,BC.
Join AB and, on it, construct eqSΔ DAB (1.1)
Produce DB,DA to F,E (p.2)
⊙B,BC × DF @ D and ⊙D,BD × DE @ A (d.1.15)
Radii BC, BD required

Proof

C,D∈⊙B,BC (con)
∴BC,BD both radii ⊙B (p.3) ∴ BC=BD (d.1.15)

Notes

Even if you got this problem right, you probably didn't do it just this way. Is your way equivalent? Does it arrive at the truth? Answering these two questions is very important. I told you the truth when I said that you know all there is to know about 1.1. This last problem shows what that statement has to mean. Those of you who studied 1.2 well enough to understand it may have solved this problem easily. Those of you who, like me, rush things a bit, found yourselves thinking, "Well, how **does** this little machine work? What does the triangle do? And the first circle? And the second?" And then you solved the problem. The rest of you turned slowly in your own circle, wondering where to begin -- which is one of the big things you learn from Euclid: begin by understanding how each little bit works. When you know that, you know all there is to know. And for notation clarity, that was "Produce DB to F and DA to E" in line 3.

4. Proof

AB × CD @ O.
∀P∈AB, join P[CD] (p.1)
ΔPOC,POD: OC=OD (hyp) PO=PO (a.1) ∠POC=∠POD (hyp)
∴ΔPOC≡ΔPOD (1.4)
∴ PC=PD

Notes

∠POC=∠POD is also true by (a.11) All right angles (∟) are equal.

5. Proof

ΔBAC,DAC: AC=AC (a.1) AB=AD and ∠BAC=∠DAC (con)

∴ΔBAC ≡ ΔDAC (1.4)

∴ 1) CB=CD

and ∠ACB=∠ACD

∴ 2) AC ×/2 ∠BCD

Notes

The "and" in line 1 connects the two elements justified by (con). Not only is it important to put all the data of the problem in the diagram, two other things are important. First, the diagram should be large enough to be clearly marked and labelled without crowding. Or you **will** misread it. So make diagrams large enough. My topology professor at UT Austin, Dr. Starbird, told us to make them big enough to crawl into. I'd say three notebook-lines tall is a minimum. Second, it is **extremely** important for the diagrams not to conform to the conclusion. Let me explain. If you are proving something is a right triangle, **do not** draw a right triangle. Do not put your conclusion in the diagram at all. The other side of this is: Do not restrict the data. If your problem says "For any triangle..." and you draw an equilateral, isosceles, or right triangle, the diagram will force false implications on you.

6. Proof

Join AD, BE

ΔADC,ECB: EC=CD and BC=AC (con) ∠ACE=∠ACB=∠BCD (1.1)

∴ ∠ACE+∠ACB=∠ACB+∠BCD (a.2)

∴ ∠ECB=∠ACD

∴ΔADC ≡ ΔECB (1.4)

∴ AD=BE

Sym. AD=CF

∴ AD=BE=CF (a.1)

Notes

Watch for every instance where you can prove or solve something symmetrically (Sym.). There is no virtue in writing anything twice.

7. Proof

∠ABC=∠ACB (hyp)
∴ ∠DBC=∠DCB (a.7)
∴ DB=DC (1.6)
∴ΔDBC ≡ isosΔ (1.5, d.1.24)

Notes

I'm saying this again too early instead of too late: For every problem, draw the diagram. Put **all** the data in it. Use 1, 2, 3, ... tickmarks for equal things, numbering angles and lines separately to keep the noise down. **Then**, look at the target (isosΔ) and write down everything you know about it below the diagram. In this problem, as soon as you wrote down the first thing you knew about the target (∠DBC=∠DCB), you'd be done. Later, everything you know about the target will not be enough. But it will help.

8. Proof

Rhombus: eqS 4-gon with no ∟ (d.1.33)
Join BD (p.1)
ΔABD,CBD: AB=CD and AD=CD (1.1) BD=BD (a.1)
∴ΔABD ≡ ΔCBD (1.8) ∴ ∠A=∠C
Sym. ∠B=∠D

Notes

I started with the definition both for clarity and to remind you to always look these up until you know them. Remember to use "Sym." for identical, symmetrical arguments to save yourself effort.

9. Proof

Join AC, BD.
ΔABD,CBD: BD=BD (a.1), AB=BC and AD=DC (con)
∴ΔABD ≡ ΔCBD (1.8) ∴ ∠BDA=∠BDC
∴BD ×/2 ∠ADC (a.2)
Sym. BD ×/2 ∠ABC

Sym. AC ×/2 ∠BAD, ∠BCD

Let's make sure something is clear. When we talk about angle bisectors (XY ×/2 ∠X) or medians (XY med∠X) or altitudes (XY alt∠X), the convention is that X is on the triangle at the vertex of ∠X and Y is on the side opposite ∠X. Any time this is not true, the text will spell it out.

10. Proof

ΔABC,DBC: AB=DC and AC=BD (con), BC=BC (a.1)
∴ΔABC≡ΔDBC (1.8) and ∠ACB=∠DBC
∴ΔEBC: ∠ECB=∠EBC
∴ΔEBC≡isosΔ (1.5)

11. Proof

ΔADB,ADC: AD=AD (a.1) AB=AC and DB=DC (con)
∴ΔADB≡ΔADC (1.8) and ∠ADB=∠ADC
∴ext∠ADC=ext∠ADB*
AD(pr) × BC @ E
ΔDBE,DCE: DE=DE (a.1) DB=DC (con) ∠BDE=∠CDE*
∴ΔDBE≡ΔDCE (1.4) and BE=EC
∴ AD(pr) ×/2 BC

Notes

The asterisks here show that the latter claim comes from the former proven line. I'm holding your hand here for a bit but will stop soon. We usually put the important conclusions on their own line beginning with "∴". Then the next step in the chain starts and leads to its final "∴". Very often, all the steps are pulled together in a final line with its "∴". And this last line may have no reference because it is the result of all those internal steps. Most problems in this text require only a single chain of reasoning. At most, two.

12. Proof

∠ACB+∠ACD=2∟ (con)
∠ACE = ½ ∠ACB (con)

∠ACF = ½ ∠ACD (con)
∴ ∠ACE+∠ACF = ½ ∠BCD = ∟ (a.2)

13. Proof
∠BCE = ∠CBD (1.5)
∴ ∠CBG = ∠BCF (1.9)
∴ ΔHBC ≡ isosΔ and BH=CH (1.6)
ΔABG,ACF: AB=AC and AF=AG (con) ∠A=∠A (a.1)
∴ ΔABG ≡ ΔACF and BG=CF (1.4)
∴ BG - BH = CF - CH (a.3)
∴ FH=GH

14.Proof
ΔAHF,AHG: AH=AH (a.1) AF=AG (con) FH=GH (#13)
∴ ΔAHF ≡ ΔAHG (1.8) and ∠HAF=∠HAG
∴ AH ×/2 ∠A
Notes
We will reference prior results from problems as here with #13 being problem 13. Every useful result from a problem solved is another tool in your toolbox. Use every result you can.

15. Proof
AD ×/2 ∠A ∴ ∠BAD = ½ ∠A (1.9)
∠B = ½ ∠A (hyp)
ΔDAB: ∠ABD=∠BAD ∴ ΔDAB ≡ isosΔ (1.5) ∴ AD=BD (1.6)
Note
Here ∠A refers to the largest ∠A as opposed to ∠BAD or ∠CAD.

16. Solutions
1) Method
Join ST (p.1)
R mdpt ST (1.10) (or "bisect ST at R")
RP⊥ST x AB @ P (1.11)
Proof
ΔPRS,PRT: PR=PR (a.1) SR=ST (con) ∠SRP=∠TRP=∟ (1.11)
∴ ΔPRS ≡ ΔPRT and PS=PT (1.4)

2) If S or T on PR then SP≠TP and solution not possible.

17. Method
PS⊥AB × AB @ S (1.11)
PS(pr) to T: PS=ST (1.2)
(or "Produce PS to T such that PS = ST")
RTQ × AB @ Q. Q required
Proof
Join QP. (p.1)
ΔQST,QSP: QS=QS (a.1) PS=ST (con) ∠QSP=∠QST (1.11)
∴ΔQSP ≡ ΔQST and ∠SQP=∠SQR (1.4)
∴QS∈AB ×/2 ∠PQR

18. Proof
4 opp ∠= 4∟ (1.15.C1)
opp∠ are equal (hyp) let them be ∠P and ∠Q
∴2∠P + 2∠Q = 4∟ (hyp)
∴ ∠P + ∠Q = ∟ (a.7)
∴opp segments AED, BEC are lines

19. Method
AP⊥BC (1.11)
Proof
∠ABC < ∠APC = ∟ (1.17)
∠ACB < ∠APB = ∟ (1.17)
∴ ∠ABC + ∠ACB < 2∟ (a.2)

20. Proof
Join AC (p.1)
∠ACD > ∠DAC and ∠ACB > ∠CAB (1.18)
∴ ∠ACD + ∠ACB = ∠C > ∠A = ∠DAC + ∠CAB (a.2)
Sym. ∠B > ∠D

21. Proof
∠BDA > ∠DAC (1.16)
∠DAC = ∠DAB (hyp)

∴ BA > BD (1.19)
Sym. CA > CD

22. Proof

CD⊥AB, ∀E∈AB. Join CE.
1) ∠CDE = ∟ ∴ ∠CED < ∟ (1.17) ∴ CD < CE (1.19)
2) ∀F∈AB: DF > DE. Join CF.
∠CED > ∠CFE (1.16)
∠CEF > ∠CED (d.1.11)
∴ ∠CEF > ∠CFE ∴ CF > CE (1.19)
3) ∀G∈AB: DG=DE. Join CG
CG=CE (1.14)
∀H∈AB, H≠E,G, CH≠CE

Note

Here, as in some of Euclid's propositions, the proof starts with what supports the next step in the chain, which in turn supports the next step.

23. Proof

Join AC
∠ACD = ∠ACB > ∠AFC (1.16)
∠ACF > ∟ ∴ ∠ACF > ∠ACD (con)
∴ ∠ACF > ∠AFC
∴ AF > AC (1.19)

24. Proof

1) P in ΔABC
PA + PB > AB (1.20) Sym. other pairs > AC, BC
∴ 2∑P[ABC] > perimeter (a.4)
∴ ∑P[ABC] > ½ perimeter (a.7)
2) P∈AB. Join PC
PA + PC > AC and PC + PB > BC (1.20)
PA + PB = AB (con)
∴ 2∑P[ABC] > perimeter (a.4)
∴ ∑P[ABC] > ½ perimeter (a.7)

3) P outside Δ

Same demonstration as part **1)**

25. Proof

AB + AD or BC + CD > BD (1.20)

∴ ∑sides > 2BD Sym. ∑sides > 2AC (a.1)

∴2∑sides > 2∑diagonals (a.1)

∴ ∑sides > ∑diagonals (a.7)

26. Proof

1) ∠ADB = ∠ADC = ∟

∟ > ∠ABD, ACD (1.17)

∴AB > AD and AC > AD (1.19)

∴ AB + AC > 2AD (a.2)

2) ∠ADB = ∠ABD

∴AB=AD (1.6)

∠ADC > ∠ACD (1.13) ∴ AC > AD

∴ AB + AC > 2AD (a.2)

3) ∠ADB < ∠ABD

AD(pr) to E: DE=DA. Join BG

ΔADC,EDB: AD=DE and BD=DC (con) ∠D=∠D (1.15)

∴BE=AC (1.4) ∴AB + BE > AE (1.20)

BE=AC and AE = 2AD

∴ AB + AC > 2AD (a.2)

Notes

Join CE here and ABCD ≡ ‖gm. Parallelograms make their full appearance in proposition 1.34. And then all of their properties can be used to solve triangle problems. In other words, given ΔABC, we **‖gmize** it into ‖gm ABCD and reap the benefits. Keep this in mind.

27. Proof

Let ∠A = ∠B + ∠C

Copy ∠C to ∠CAD: AD × BC @ D (1.23)

∴ ∠DAB = ∠A - ∠C = ∠B (con)

∴ΔDAC: ∠A = ∠C

∴ΔDAC ≡ isosΔ (1.6)
∴ΔDAB: ∠A = ∠B
∴ΔDAB ≡ isosΔ (1.6)

28. Proof

AD med∠A
∴ ΔADB,ADC ≡ isosΔ (#27)
∴AD=DB and AD=DC (1.6)
AD + DC = BC ∴ BC = 2AD

29. Method

Copy ∠B to B∈AB (1.23)
Copy CD to B @ ∠B (1.2)
Join DA. (p.1)
Copy ∠ADE to ∠DAE @ A (1.23)
ΔEAB required

Proof

ΔEDA: ∠D = ∠A (con)
∴ AE = ED (1.6)
and DE + EC = DC (a.2)
(And yes, the hint was part of the solution.)

30. Proof

ΔADB,ADE: AD=AD (a.1) ∠DAB = ∠DAE = ½∠A (con)
∠ADB = ∠ADE = ∟ (con)
∴ΔADB ≡ ΔADE and BD=BE (1.26)

31. Proof

AD ×/2 ∠A, PQ,PR⊥AB,AC
ΔPRA,PQA: ∠PAQ = ∠PAR
∠PQA = ∠ PRA = ∟ (con)
PA = PA (a.1)
∴ ΔPRA ≡ ΔPQA (1.26) and PQ = PR

32. Proof

In problem 30, let GH = AD and the result follows.

Notes

Make the effort to see how the two problems are symmetrical.

33. Method and Proof

Using analysis, from P draw PEF: AE = AF
Then ∆AEF ≡ isos∆
AD ×/2 ∠A (1.9)
PEF⊥AD × AD @ D (1.12)
∆ADE,ADF: AD=AD (a.1)
∠ADE = ∠ADF = ∟
and ∠DAE = ∠DAF (con)
∴∆ADE ≡ ∆ADF and AE=AF (1.26)

Notes

In analysis, draw the diagram. Then add the solution, From there you can reason your way from both ends.

34. Method

Join QR (p.1)
QR ×/2 @ O (1.10)
Join OP. OP(pr) required

Proof

Add QS⊥OP @ S, RT⊥OT (OPT or OTP)
∆OQS,ORT: ∠S = ∠T = ∟ and QO=OR (con)
∠TOR = ∠SOQ (1.15)
∴∆OQS ≡ ∆ORT and RT=QS (1.26)

35. Proof

Let ∆ABC,DEF be similarly oriented.
Produce CD to G: CB=BG. Join AG.
∆ABG,DEF: ∠ABG = ∟ (1.13)
∴ ∠ABG = ∠DEF
AB=DE and BG=EF (con)
∴∆ABG ≡ ∆DEF and AG=DF (1.26)
AC=DF (hyp)

∴ AG=AC and ∠ACG = ∠AGC (1.6)

∴ ΔABG ≡ ΔABC (1.26)

∴ΔABC = ΔDEF (a.1)

36. Proof

B∥D ∴ ∠(A with B) = ∠(A with D) (1.29)

A∥C ∴ ∠(D with A) = ∠(D with C) (1.29)

∴ ∠(A with B) = ∠(C with D)

Notes

Often, if you can rise up a step in abstraction, the proofs become shorter and simpler.

37. Proof

isosΔABC and DE∥BC

If DE !× AB,AC, produce AB,AC × DE @ D,E

DE∥BC ∴ ∠BDE = ∠ABC (1.29)

Sym. ∠CED = ∠ACB

∠ABC = ∠ACB (1.4) ∴ ∠CED = ∠BDE (a.1)

38. Proof

BA(pr) to D. AE ×/2 ext∠A (∠DAC)

Assume AE∥BC

∠DAE = ∠ABC and ∠EAC = ∠ACB (1.29)

∠DAE = ∠EAC (1.9) ∴ ∠ABC = ∠ACB

∴ΔABC ≡ isosΔ

39. Proof

∀H∈AB, join HGL (L∈CD) then HL·|·(AB,CD)

ΔEGH,FGL: EG=GF (con) ∠G = ∠G (1.15) ∠GEH = ∠GFL (1.29)

∴ΔEGH ≡ ΔFGL and for ∀H: GH=GL

40. Proof

CPD,EPF C,E∈AC, D,F∈BD

ΔPFD,PEC: CP=PD and EP=PF (#39) ∠CPE = ∠DPF (1.15)

∴ΔPFD ≡ ΔPEC and FD=EC (1.4)

41. Proof

ΔEAD,FAD: AD=AD (a.1) $\angle$FAD = $\angle$ADE and $\angle$EAD = $\angle$ADF (1.29)
$\angle$FAD = $\angle$EAD (con) $\therefore$ $\angle$EDA = $\angle$FDA
$\therefore$ ΔEAD $\equiv$ ΔFAD and DE=DF (1.26)

42. Proof

EF$\parallel$BC $\therefore$ $\angle$BCE = $\angle$CEF (1.29)
$\angle$BCE = $\angle$ECF (con) $\therefore$ $\angle$FEC = $\angle$FCE
$\therefore$ ΔFEC $\equiv$ isosΔ and EF=FC (1.6)
Sym. $\angle$FCG = $\angle$FGC
$\therefore$ ΔFCG $\equiv$ isosΔ and FC=FG
$\therefore$ EF=FG (a.1)

43. Method

BE ×/2 $\angle$B × AC @ E
ED$\perp$AC x AB @ D
D required

Proof

$\angle$EBC = $\angle$DEB (1.29)
$\angle$EBC = $\angle$DBE (con)
$\therefore$ $\angle$DEB = $\angle$DBE (a.1)
$\therefore$ ΔDBE $\equiv$ isosΔ (1.6)
$\therefore$ DB=DE (1.6)

44. Proof

GAH$\perp$BC × BC @ G
$\therefore$ $\angle$DEB = $\angle$AEF and $\angle$CAG = $\angle$FAH (1.15)
$\angle$CAG = $\angle$BAG (con)
$\therefore$ $\angle$BAG = $\angle$AEF and $\angle$FAH = $\angle$AFE (1.29)
$\therefore$ $\angle$AFE = $\angle$AEF (a.1)
$\therefore$ ΔAEF $\equiv$ isosΔ (1.6)

45. Method

AF$\parallel$CD (1.31)
Copy $\angle$E to A (1.23)
Produce $\angle$E to B$\in$CD

B required

Proof

∠ABC= ∠FAB (1.29) = ∠E and B∈CD (con)

46. Method

BE ×/2 ∠B (1.9) × AC @ E

ED‖BC × AB @ D

D,E required

Proof

ΔDBE ≡ isosΔ (#42) ∴ DE=DB

Sym. (using ×/2∠C) DE=EC

47. Proof

ΔABC: **1)** ∠A = ∠B + ∠C

∴ 2∟ = ∠A + ∠B + ∠C = 2∠A (1.32, a.2)

∴ ∠A = ∟ (a.3)

2) ∠A > ∠B + ∠C

∴ 2∟ = ∠A + ∠B + ∠C = ∟ + (∠A - ∟) + ∠B + ∠C

∴ ∠A > ∟ (obtuse)

3) Sym. if ∠A < ∠B + ∠C then ∠A acute

48. Method

∀line, construct eqSΔ (1.1)

Construct ∟ (1.11)

Copy ∀∠ of eqSΔ into ∟ (1.23)

Bisect result. Bisection required

Proof

∠ of eqSΔ = 2/3∟ (1.32)

Bisected = 1/3∟ (a.3)

49. Method

∀BC construct eqSΔABC (1.1)

Let ×/2 ∠A = ∠E (1.9)

∀EF, copy ∠E to E,F (1.23)

∠E × ∠F @ D and ΔDEF required

Proof

eqSΔ, ∀∠ = 2/3∟ (1.1, 1.32)

∴ ∠E,F = 1/3 2∟ (1.9, a.3)

∴ ∠D = 2∟ - 2/3∠ = 6/3∟ - 2/3∟ = 4/3∟ = 4∠E,F

50. Method/Proof (analysis)

½ apex = ∠D

∴ ∠A + ∠B + ∠C = 8∠D = 2∟ (1.32, a.2)

∴ ∠D = 1/4∟ (a.3)

Construct ∟ (1.11) and ×/2 for ∠E = ½∟ (1.9)

∀AB, copy ∠E to A,B (1.23)

Copy ×/2∠E = ∠D (1.9) onto ∠E @ A,B (1.9, 1.23)

∠A × ∠B @ F and ΔFAB required

Then ∠A = ∠B = 3∠D and ∠F = 2∠D

51. Proof

ΔABC,ACD ≡ isosΔ (con, 1.6)

∴ ∠DCB = ∠ACB + ∠ACD (1.32)

∠ACB = ∠B and ∠ACD = ∠ADC (1.5)

∴ 2∟ = ∠DCB + ∠B + ∠D (1.32)

∴ 2∟ = 2∠DCB (a.2)

∴ ∟ = ∠DCB and ΔDCB ≡ ⊿

52. Proof

AF ×/2 ∠A ∴ ΔBAF ≡ ΔCAF (1.4) ∴ ∠AFB = ∟

ΔBAF,BCE: ∠ABF = ∠EBC (a.1) ∠AFB = ∠CEB = ∟ (con)

∴ ∠BAF = ∠BCE = ½∠A (1.32)

Sym. ∠CAF = ∠CBD = ½∠A

∴ ∠CBD + ∠ECB = ∠A

53. Proof

∠B = ∠C (con)

∴ ∠DBC + ∠ECB = ∠B,C (con, a.2)

∴ ∠BFC = 2∟ - ∠B (1.32)

∴ ∠BFC = ext∠B,C

54. Method

∀EF, construct eqSΔDEF (1.1)

Construct line on P∥A (1.31)

Copy ∠D to P (1.23) away from Q

Produce P on ∠D to R∈A

Sym. create QS

RP × SQ @ T and ΔTRS required

Proof

∠D = ∠PRS = ∠QSR (1.29, con)

∴ ∠D = ∠RTS (1.32)

55. Method

Copy DE to AB (1.2) and ×/2 ∠F (1.9)

Copy ½∠F to E (1.23)

∠AEQ(pr) × AC @ Q (p.1)

Copy ½∠F to ∠EQP

PQ required

Proof

ΔPQE: ∠Q = 2∟- ∠F (1.32), ∠E,Q = ½∠F

∴ ΔPQE ≡ isosΔ and PQ=PE (1.6)

∠EPQ = 2∟- ∠F

∴ ∠APQ = ∠F

56. Proof

∠BCD = ½(∠A + ∠B) (1.32)

∠CBD = ½(∠A + ∠C)

∴ ∠D = 2∟- ∠A - ½(∠B + ∠C) (1.32)

∠B + ∠C = 2∟- ∠A (1.32)

∴ ∠D = 2∟ - ∠A - ∟ + ½∠A (a.3)

∴ ∠D = ∟- ½∠A (a.1)

∴ ∠D + ½∠A = ∟ (a.2)

57. Proof

BE × CD @ F

ΔFBC ≡ isosΔ #1 (con)

ΔFCE: ∠BCE = 2∟- 3∠BCD (1.13), ∠CDE = ∠BCD (con)

∴ ∠CEB = 2∠BCD (1.32)

∠DFE = 2∟- 2∠BCD (1.32)

∴ ∠EFD = 2∠BCD (1.32)

∴ ∠CEB = ∠BCD

∴ΔCFE ≡ isosΔ #2 (1.6)

Sym. ΔBFD ≡ isosΔ #3

Note

In my diagram, ΔBEA looks more isos than ΔCFE. Do not rely on or try to justify visual judgment. Rely on the relations of the diagram and on correct algebra for the angles.

58. Proof

Copy ∠B to A × BC @ D (1.23)

∠A = ∟ ∴ ∠A = ∠B + ∠C (1.32)

∴ ∠DAB = ∠A - ∠B = ∠C and ∠B = ∠BAD

∴ΔDAB ≡ isosΔ and BD=AD (1.6)

Sym. ΔDAC ≡ isosΔ and DC=AD

∴BD=DC and AD = ½BC

Note

This theorem is extremely useful.

59. Proof

DF = ½AB (#58)

EF = ½AB (#58)

∴DF=EF (a.1)

60. Proof

×/2 AB @ F. Join CF,DE (1.10,p.1), CF × DE @ G

ΔFEG,FDG: FE=FD (#59) FG=FG (a.1) ∠FGE = ∠FGD (con)

∴ΔFEG ≡ ΔFDG and DG=GE

61. Proof

ΔBCD,CBE: ∠BCD = ∠CBE and ∠DBC = ∠ECB (con) BC=BC (a.1)
∴ΔBCD ≡ ΔCBE and CD=BE (1.26)
∴ AD=AE (a.3)
∴ ∠ADE = ∠AED (1.6)
∠ABC = ∠ACB (con)
∴ ∠AEB = ∠ABC (1.32)
∴ DE∥BC (1.28)

62. Proof

AB × CD @ E
∠ABD = ∠CDB (hyp) ∴ EB=ED (1.5)
∴ EA=EC and ∠EAC = ∠ECA (a.2, 1.6)
∴ ∠EBD + ∠EDB = ∠EAC + ∠ECA (1.32)
∴ ∠EBD = ∠EAC (a.1)
∴ AC∥BD (1.28)

Note

Another case of relying on logic with an inaccurate diagram. I can't see any isosΔ in my diagram.

63. Method

Let AD = ∑sides
Produce DE: ∠ADE = ½∟ (1.9,11,23)
⊙A,hypotenuse × DE @ B
BC⊥AD (1.11)
ΔABC required

Proof

∠ACB = ∟ (con)
AB = hypotenuse (con)
∠BCD = ∟ and ∠CDB = ½∟
∴ ∠CBD = ½∟ (1.32)
∴CB=CD and CB + CA = ∑sides

Note

Minimum hypotenuse must be AB⊥DE.

64. Method

AD = ~(sides) (difference of sides)

Produce DE: ∠ADE = ½∟ (1.9,11,23)

⊙A,hypotenuse × DE @ B

BC⊥DA(pr)

ΔABC required

Proof

∠ACB = ∟ and BA = hypotenuse (con)

∠ACB =∟ and ∠D = ½∟ ∴ ∠CBD = ½∟ (1.32)

∴BC = CD (1.6) and AD = BC - AC

Note

Hypotenuse must be bigger than ~(sides).

I really enjoy Euclid construction problems. But I rarely solve them. They are harder than theorems because they include no diagram. You're left to stare into the darkness as you grope about for a place to start. They also require more mastery of the propositions in the way that fine work requires more mastery of one's tools. In these last two, the use of a circle is almost startling. Do not be discouraged if you can't solve them. Almost no one can solve very many of these. Just try hard and then go study the solution before your head explodes.

65. Method

D mdpt AB (1.9)

DE⊥AB: DE = altitude (1.11)

FEG∥AB (1.31)

⊙D,DE × FG @ C

ΔCAB required

Proof

∠ACD = ∠CAD and ∠BCD = ∠CBD (1.5)

∴ ∠ACB = ∠CAB + ∠CBA (a.2)

∴ ∠ACB = ∟ (1.32) and CH⊥AB = DE = altitude

66. Method

∠LDE = ½∠ABC (1.9,23)

∠MED = ½∠ACB (1.9,23)

DL × EM @ F

∠DFG = ∠FDE × DE @ G (1.23)

∠EFH = ∠FED × DE @ H (1.23)

ΔFGH required

Proof

FG=DG, FH=HE (1.6)

∴ perimeter ΔFGH = DE

∠FGH = ∠FDG + ∠DFG (1.32) = 2∠FDG = ∠ABC

Sym. ∠FHG = ∠ACB

∴ΔABC,FGH eq∠ (1.32)

To follow the proofs, it becomes necessary to build them as you go. And there is no point in reading them if you cannot realize the importance of each step. Make the effort.

67. Method

GK⊥GH: GF inside ∠HGK (1.11)

DL: ∠EDL = ½∠FGH (1.9)

EM: ∠DEM = ½∠FGK (1.9)

DL × EM @ C

∠DCA = ∠CDE and ∠ECB = ∠DEC (1.23)

ΔABC required

Proof

∠CAB = ∠ACD + ∠ADC = ∠FGH (1.32)

Sym. ∠CBA = ∠FGK

∴ ∠A + ∠B = ∟ = ∠C (con, 1.32)

AC=AD and CB=CE (con)

∴AC + AB + BC = perimeter (a.2)

68. Method

QPR⊥AB × AB,CD @ Q,R (1.11)

Copy PR to QSB, PQ to RTD (1.3)

PS,PT required

Proof

ΔSQP ≡ ΔPRT (1.4)

∴ PS=PT, ∠RPT = ∠QSP

∴ ∠RPT + ∠RTP = ∟ (1.32)

∴ ∠RTP + ∠QSP = ∟ (a.1)

∴ ∠SPT = ∟ (1.13) and PS⊥PT

Note

If P !·|·(AB,CD) then QS opposite side of RT works.

69. Method

AD ∈ AC: AD=AP

Join DP

ADQ: DQ=DP

∴ ∠APQ = 3∠AQP

Proof

DP=DQ

∴ ∠DPQ = ∠DQP (1.5)

∠ADP = ∠DPQ + ∠DQP (1.32)

∴ ∠ADP = 2∠DQP (1.13,32)

∴ ∠APD = 2∠DQP (1.5)

∴ ∠APD + ∠DPQ = ∠APQ = 3∠AQP (a.2)

70. Theorem

ΔACD,EBD: AD=DE and CD=DB (con) ∠D = ∠D (1.15)

∴ΔACD ≡ ΔEBD and ∠C = ∠DBE (1.4)

Sym. ∠A = ∠FBG

∴ ∠FBG + ∠B + ∠DBE = ∠A + ∠B + ∠C = 2∟ (1.32)

∴GBE colinear

71. Method (eqSΔ)

eqSΔCAB (1.1)

×/2∠A × ×/2∠B @ D (1.9)

DE,DF∥CA,CB × AB @ E,F (1.31)

AE = EF = FB

Proof

∠EDA = ∠DAC (1.29) and ∠DAE = ∠DAC (con)

∴ ∠EDA = ∠EAD (a.1) ∴ AD=DE (1.5)

Sym. DF=FB

∠DEF = ∠CAB and ∠DFE = ∠CBA (1.29)

∴ ∠EDF = ∠ACB (1.32) ∴ΔDEF ≡ eq∠Δ ∴ eqSΔ (1.6)

∴ DE=EF=FD ∴ AE=EF=FB

Method/Proof (isosΔ)

Bisect ∟ twice and copy 1/4∟ to A,B

Then ∠C is 3/2∟. Trisect ∠C with ½∟.

Then you have two overlapping ⊿s.

Their medians trisect the base.

Supply your own references to supporting propositions.

72. Proof

Join CB

∠AEC = ∠B + ∠D (1.32) = ∠DEB (1.15)

∴ ∠ECB + ∠EBC = ½(∠AEC + ∠DEB) (con)

∴ ∠ECF + ∠EBF = ½(∠ECA + ∠EBD) (con)

∴ ∠ECB+∠EBC+∠ECF+∠EBF = ½(∠AEC+∠DEB+∠ECA+∠EBD)

∴ (2∟ - LHS) = ∠CFB and (2∟ - RHS) = ½(∠EAC + ∠EDB)

Note

LHS, RHS are "left-hand side," "right-hand side" of any equation.

73. Solution

∑int∠ + 4∟ = n2∟ (1.32.C1)

8-gon has 8 sides, 8 angles.

∴ 8∠ + 4∟ = 16∟ ∴ 8∠ = 12∟

∴ ∠ = 12/8∟= 3/2∟

74. Proof
Join BD
∆BCD,BDE,BEC:
BD=BE=BC = radius ⊙B
∠CBA = 2/3∟ (1.32) ∴ ∠CBE = 4/3∟ (1.13)
Sym. ∠DBE,DBC = 4/3∟
∴ ∆BCD ≡ ∆BDE ≡ ∆BEC (1.4)
∴ ∠C = ∠D = ∠E (a.2) and ∆CDE ≡ eqS∆

75. Proof
4-gonABCD: AD=BC, AB=CD
Join BD (p1)
∆ABD ≡ ∆CBD (1.8)
∴ ∠BDA = ∠DBC ∴ AD∥BC (1.27)
Sym. AB∥CD ∴ 4-gon ≡ ∥gm (1.34)

76. Proof
4-gonABCD: ∠A = ∠C, ∠B = ∠D
∴ ∠A + ∠B = ∠C + ∠D (a.2)
∠A + ∠B + ∠C + ∠D = 4∟ (1.32.C1)
∴ ∠A + ∠B = 2∟ (a.3) ∴ AD∥BC (1.28)
Sym. AB∥CD ∴ 4-gon ≡ ∥gm (1.34)

77. Proof
AC × BD @ E
∆EAD,ECB: AD=BC (con) ∠E = ∠E (1.15) ∠EBC = ∠EDA (1.29)
∴ ∆EAD ≡ ∆ECB and AE=EC (1.26) Sym. BE=ED.
∴ AC,BD ×/2 e.o.

78. Proof
AC,BD ×/2 e.o. @ E (hyp)
Join 4 vertices to create 4-gon
Then opp ∆s are equivalent (1.26) ∴ opp sides and ∠s equal and opp sides∥by equal angles (1.27)
∴ 4-gon ≡ ∥gm (1.34)

You are perfectly justified in abbreviating what has already been established. Your only real concerns are clarity and correctness.

79. Proof

∥gmABCD: BD ×/2 ∠B,D (hyp) and ∠B = ∠D (1.34)
∠A = ∠C (1.34)
∴ΔADB,CBD isosΔ on same base (1.6,24)
∴All sides are equal. (1.8)

Note

I don't know that 1.24 is necessary. But Todhunter cites it.

80. Proof

4-gonABCD: AD∥BC, AB=CD
AE,DF⊥AD × BC @ E,F (1.11) ∴AE=DF, AD=DF (1.33) AB=CD (hyp)
∴ ∠ABE = ∠DCF and ∠EAB = ∠FDC (1.26)
squareADEF: ∠A,D,E,F = ∟ (1.29)
∴ ∠ABC + ∠ADC = ∟ - ∠EAB + ∟ + ∠FDC (a.2,3, 1.34)
∠EAB = ∠FDC ∴ ∠ABC + ∠ADC = 2∟
Sym. ∠BAD + ∠BCD = 2∟

Note

Line 5: If we ∥gmize ΔFDC into ∥gmFDGC, then ∠FCG = ∟
∠FCD = ∟FCG - ∠DCF. But by 1.34 ∠DCF = ∠FDC and
∠FCG = ∠EFD

81. Proof

∥gmize ΔABC into ∥gmABCD ∴ AC,BD ×/2 e.o. (1.34)
∀CE, E∈AB: EF∥BC × CD @ F (1.31) Join BF
BF × AC @ G ∴ EC,BF ×/2 e.o. in ∥gmEFCB (1.34)
But ∀BF: BG < BF (a.8)

82. Method

∀E∈AB, ⨀E,L × CD @ F (d.1.15) Join EF.
PH∥EF × AB,CD @ G,H PH required

Proof

EGHF≡∥gm (con, d.1.30)
∴GH = EF = ∟

83. Proof

∥gm ABCD: ×/2∠A × ×/2∠B @ E

∠EAB + ∠EBA = ½(∠DAB + ∠ABC) (con)

∠DAB + ∠ABC = 2∟ (1.29)

∴ ∠EAB + ∠EBA = ∟ (a.7)

∴ ∠E = ∟ (1.32)

84. Proof

∥gm ABCD: produce DA,BC

AE ×/2 ∠A × BC(pr) @ E, CF ×/2 ∠C × DA @ F

∠A = ∠C (1.34) ∴ ∠EAD = ∠FCB (1.7)

AD∥BC ∴ AE∥CF (1.29)

If ABCD ≡ rectangle (rect∟) then AC ×/2∠A,C

∴ AE,CF would coincide.

Note

It is perfectly legitimate for the last two lines to merely "state the case" so long as the case is clear.

85. Proof

∥gmABCD: AC=BD and AC × BD @ E

∥gm self-bisected by diagonals (1.34)

But AC=BD ∴ AE=EC=BE=ED (hyp, a.7)

∴ ∀4 internalΔ ▻ isosΔ (d.1.24)

∴ ∀8 internal∠ equal (1.5)

∴ ∀4 ∥gm ∠ equal (a.6)

Note

Here again, we can simply state the case without proving equal triangles in detail -- because they are obvious, so long as you correctly understand "∀" as "all."

86. 1) Method

AE,CF ⊥ AB,CD equal to L,M (1.11,3)

EG,FH∥AB,CD (1.31)

EG × FH @ P required

Proof

Perpendiculars from P equal AE,GF (con)

These equal L,M (1.34)

2) Number of such points

If AB×CD, there are two such points, one either side of intersection.

If AB∥CD, there are none, unless the distance between AB,CD = L+M and then there are infinitely many such points.

87. Method

∀G∈CD, produce GH∥F towards AB (1.31)

On GKH make GK=E (1.3)

KL∥CD × AB @ L (1.31)

LM∥GK required

Proof

KLMG ≡ ∥gm and GK∥F (con) ∴ LM∥F (1.34)

GK=E ∴ LM=E (1.34)

88. Proof

ΔABC,EBF: AB=EB, BC=BF, and ∠FBC = ∠ABE = 2/3∟ (con)

∴ ∠ABF + ∠FBC = ∠ABF + ∠ABE (a.2)

∴ ∠ABC = ∠EBF

∴ ΔABC ≡ ΔEBF and EF=AC (1.4)

Sym. GF=BD

Note

Diagrams should be mostly whitespace: lines and labels dominated by emptiness. And the more intricate diagrams are, the larger and more accurate they need to be. There is no point in rushing the creation of a diagram in a problem that will require a more than usually patient effort in your solution. You want to express the same patient thoughtfulness throughout.

89. Proof

GEH∥AB × AD,BE,CF @ G,E,H

ΔEGD,EFH: ∠E = ∠E (1.15) ∠G = ∠H (con)

ABEG, BCHE ≡ ∥gm ∴ GE=AB=BC=EH (con, 1.34)

∴ΔEGD ≡ ΔEFH and DG=FH (1.26)

∴AD = BE + GD and CF = BE - FH ∴ AD + CF = 2BE

Note

This theorem is stupidly useful. Just as you watch for potential isosΔ, keep your eyes open for this pattern of a line pivoting from its mdpt, making equal Δs.

90. Proof

AC × BD @ P ∴P mdpt AC,BD (1.34)

PQ⊥EF (1.12)

∴ ⊥on B + ⊥on D = 2PQ = ⊥on A + ⊥on C (#89)

91. Proof

Consider the extreme cases. If the angle is zero, the diagonal and sides coincide and equal ½(∑(opp sides)). If the angle is 2∟, the diagonal is zero. ∴As the angle increases from zero to 2∟, the diagonal diminishes from ½(∑(opp sides)) to zero.

Note

This is not a Euclidean proof. With Euclid, you could show that the hypothesis is true for two static ∥gms in a Euclidean way. But that does not handle the extreme cases.

92. Proof

∀6-gonABCDEF: diags AD,BE,CF

Consider ∀2 diags: AD,CF. Join AC,FD.

AF=CD and AF∥CD (hyp) ∴ AC=DF and AC∥DF (1.33)

∴AFDC ≡ ∥gm (d.1.30) and AD,CF ×/2 e.o. @ G

Sym. AD,BE ×/2 e.o. @ G

G mdpt AD ∴G mdpt BE

∴All diagonals concur @ G

By now, you should be recognizing the use of transitive relations. The simplest is if A=B and B=C then A=C. That last proof uses bisection at G in the same way. Note also that you can usually get the middle bit of such proofs using symmetry.

93. Method

DE∥AB × AC @ E (1.31)
F∈EC: EF=AE (1.3)
FDG × AB @ G required

Proof

EH∥FG × AB @ H
ΔAEH,EFD: AE=EF (con) ∠AEH = ∠EFD and ∠EAH = ∠FED (1.29)
∴EH=FD (1.26) and EH=DG (1.34) ∴FD=DG

94. Proof

GK∥AD × DF @ K
ED=GK, ED∥BF ∴ EB=DF, EB∥DF (1.33)
∴EGKD ≡ ∥gm and GK=ED ∴ GK=AE (hyp)
ΔAEG,GKH: AE=GK (proven)
∠EAG = ∠KGH and ∠EGA = ∠KHG (1.29)
∴ AG=GH (1.26) Sym. CH=GH ∴ BE,DF ×/3 AC

95. Proof

E mdpt CD. Produce BC.
FEG∥AB x AD,BC @ F,G (1.31)
ΔFED = ΔCEG (#89)
∴ ∥gm ABGF = 4-gon ABCD

96. Proof

AG∥BC toward C (1.31)
FEG∥AB × BC,AG @ F,G (1.31)
∴ABFG ≡ ∥gm and DE ×/2 ABFG (con)
ADEG ≡ ∥gm and AE ×/2 ADEG (1.34)
∴ΔADE = 1/4∥gm ABFG
ΔADE,EFC: ∠E = ∠E (1.15) AE=EC and GE=EF (con)
∴ΔADE ≡ ΔEFC (1.4) ∴ΔADE = 1/4ΔABC

97. Method

Rhombus ABCD, P mdpt AB

Add CQ=AP (1.3) Join AC,PQ. AC × PQ @ R

SRT⊥AD × AD,BC @ S,T

Rhombus PSQT required

Proof

ΔAPR,CQR: AP=CQ (con) ∠ARP = ∠CRQ (1.15) ∠RAP = ∠RCQ (1.29)

∴ PR=QR and AR=CR (1.26)

ΔPRS,QRS: PR=QR RS=RS ∠PRS = ∠QRS = ∟

∴ PS=QS (1.4)

ΔCRT,ARS: AR=CR ∠ARS = ∠CRT (1.15) ∠ASR = ∠CTR (1.29)

∴ RS=RT (1.26)

ΔSRP,TRP: RS=RT RP=RP ∠SRP = ∠TRP = ∟

∴ SP=TP (1.4)

Sym. TQ=SQ=TP ∴ PSQT ≡ rhombus

Note

You will notice that the asterisked problems require more than one chain of proof.

98. Proof

Join BD

ΔBCG = ΔBDG (1.37)

ΔBDG = ΔBFD + ΔBFG = ΔBFA + ΔBFG (1.37)

ΔBCG = ΔCFG + ΔBFG

∴ ΔBFA = ΔCFG

Note

When you need equality of areas, 1.37 is often applicable. Look for where to add the line like BD

99. Method

Join AD CE∥AD × AB @ E

ΔEBD required

Proof

ΔECD = ΔECA (1.37)

∴ ΔEBC + ΔECD = ΔAEC + ΔEBC (a.2) ∴ ΔEBD = ΔABC

100. Method

Join AD CE∥AD × BA(pr) @ E

Join DE ΔDEB required

Proof

ΔABC = ΔABD + ΔDAC

ΔDEB = ΔABD + ΔDAE

ΔDAE = ΔDAC (1.37) ∴ ΔABC = ΔDEB

101. Method

Join P[AB]   CE,DF∥BP,AP

EPF∥AB x CE,DF @ E,F

4-gon ABEF required

Proof

ΔPEB = ΔPCB and ΔPFA = ΔPDA (1.37) ∴ ABCD = ABEF

Note

When you add the first two∥lines, you know there will be an E and F. The next line defines E,F.

102. Method

Join P[AB]

CM,DN∥PB,PA × AB(pr) @ M,N

ΔPMN required

Proof

ΔPBC = ΔPBM and ΔPAD = ΔPAN (1.37) ∴ ΔPMN = ABCD

103. Method

O mdpt AC

DE∥AC (1.31)

OE⊥AC × DE @ E

EO(pr) to F: EO=OF

Rhombus AFCE required

Proof

ΔACD = ΔACE and ΔACF = ΔACB (1.37)

All 4 Δs equivalent by AC⊥EF = ∟, AO=OC, EO=OF (1.26)

∴ AF=FC=CE=EA ∴ AFCE ≡ rhombus

104. Problem

AC(pr) × MN @ D Join BD
CE∥BD × BA(pr) @ E
ΔAED required

Proof

ΔCED = ΔCEB (1.37)
∴ΔCED - ΔCEA = ΔCEB - ΔCEA (a.3)
∴ΔAED = ΔABC

105. Proof

ΔBDC and ΔABE = ½ΔABC (1.38) ∴ΔBDC = ΔABE
∴ΔBDC - ΔDFB = ΔABE - ΔDFB (a.3)
∴ΔBFC = 4-gonADFE

106. Proof

CA(pr) to G: CA=AG Join BG
ΔABG,DEF: AB=DE (hyp) AG=AC=DF (con)
∠GAB = 2∟ - ∠BAC = ∠EDF ∴ΔABG≡ΔDEF (1.4)
AG=AC and ΔABC,ABG share apex B ∴ΔABC = ΔABG (1.38)
∴ΔABC = ΔDEF (a.1)

107. Proof

∥gmABCD: AC × BD @ E   FDG∥AC
Then oppΔ equivalent (1.4 or 1.8)
And adjΔ equal (1.38)
∴All 4 Δs equal

Note

We use two pairs opp Δs: (EAB,ECD) (EAD,EBC) Sym for adjΔ.

108. Proof

AC × BD @ O
ECF,GAH∥BD
ΔAOD = ΔCOD (#107)
∴ΔAOP = ΔCOP (1.38)
∴ΔAOD - ΔAOP = ΔCOD - ΔCOP (a.3)
∴ΔPAD = ΔPCD

109. Proof

4-gon ABCD: AC × BD @ E

Take any of 4 intΔ of 4-gon, say DEC.

Add EDF,ECG: ED=DF, EC=CG

Join FG and ∥gmize ΔEFG to EFGH

ΔECF = ΔFCG (#107) ΔEDC = ΔFDC (1.38)

∴ΔEDC = 1/4ΔFEG (#96)

Sym. ∀int Δs of 4-gon = 1/4 constructed Δ

And all 4 constructed Δs are equal (1.38) ∴Δ = 4-gon

110. Method

D mdpt BC   Join D[AP]

AE∥DP × BC @ E and EP required

Proof

ΔPAD = ΔPED (1.37)

ΔPDC + ΔPAD = ΔPED + ΔPDC (a.2) ∴ ΔPCE = ΔACD

BD=DC ∴ΔACD = ½ΔABC (1.38)

∴ΔPCE = ½ΔABC and PE ×/2 ΔABC

Note

My "nearer A than C" ensured this variant of the solution. How would the method change if P were nearer C? Or if P mdpt C?

111. Method

Join AC, BD E mdpt BD Join E[AC]

EG∥AC × BC @ G

AG required

Proof

ΔAEC = ΔAGC (1.37)

ΔABC + ΔAEC = ΔAGC + ΔABC

∴ABCE = ABCG

ΔABE,CBE = ½ΔABD,CBD (1.38)

∴ABCE = ABCG = ½ABCD

Note

If alt∠B > alt∠D then EG is above AC and ADCE = ADCG.

If altitudes equal, 4-gon bisected by AC.

112. Proof

Join CB

ΔAEC = ΔBED (1.37, a.3) ∴ ΔCEB + ΔAEC = ΔBED + ΔCEB (a.2)

∴ΔACB = ΔDCB and both on CB ∴ AD∥CB (1.39)

113. Proof

On BC, ΔABC over ΔDBC   Join AD   BC × AD @ G

∥gmize ΔBAD to ∥gm AGDEBF

ΔABC = ΔDBC (hyp) ∴ ∥s BC,FA = ∥s BC,DE (1.40, con)

∴BG ×/2 ∥gmADEF ∴ AG = GD

Note

Clearly, there are other ways to show this result, such as proving ΔABG = ΔDBG.

114. Proof

ΔBEC = ½∥gmABCD (1.41) ∴ΔBEC = FEDC (hyp)

∴ΔBEC - ΔFEC = FEDC - ΔFEC (a.3)

∴ΔEBF = ΔCED

115. Proof

FEG∥AB × AD,BC @ F,G

ABCD = ∥gmABGF (#95)

∴ΔAEB = ½ABGF (1.41) = ½ABCD

116. Proof

∥gmABCD, O,G,H mdpt BD,AD,BC   Join GH

∀E∈AD, EOF × BC @ F

ΔDOG = ΔBOH ∴ΔDOE + ΔEOG = ΔBOF + ΔFOH (#95, a.2)

∴EFCD = ΔDBC ∴EFCD = ½ABCD (1.34,41)

117. Method

AC × BD @ O

PO produced to sides required

Proof

∀line on O ×/2 ∥gm (#116)

118. Proof

‖gmize ΔABC to ‖gmADCB E,F mdpt AB,AC
AC × BD @ F (1.34) EF × DC @ G
∴ EG ×/2 ‖gm (#116)
E mdpt AB ∴ G mdpt DC (#89)
∴ EF‖BC

119. Proof

ΔABC: D,E mdpt AB,AC.
‖gmize ΔABC to ‖gmAFCB
DE(pr) × CF @ G ∴ DE = EG (#116) = ½DG
DE‖BC (#118) ∴ DE = ½BC

120. Proof

Join AD
EG,FH‖AD (#118)
∴ EG,FH = ½AD (#119) ∴ EG=FH

121. Proof

4-gonABCD: E,F,G,H mdpt AB,BC,CD,DA
EF,GH‖AC and EH,FG‖BD (#118) ∴ EFGH ≡ ‖gm

122. Method

Add lines on D,E,F‖EF,DF,DE
∴ lines are ADB,AEC,BFC
ΔABC required

Proof

‖gm DEFB,DECF: ΔDEF = ½ each ‖gm (1.34)
D,E mdpts AB,AC and DE = ½BC (#119) and DE‖BC (#118)
Sym. for other pairs of sides

123. Proof

1) EF × AD @ G ≡ mdpt AD and AD ⊥ EF (con, #118)
∴ ΔAEG = ΔDEG and ΔAFG = ΔDFG (1.4) ∴ ∠BAC = ∠FDE
2) ΔAEF = 1/4ΔABC (#96)
∴ AFDE = 2ΔAEF = ½ΔABC

124. Proof

DE ×/2 ∥gmAEFD and BC (1.34, con)

ΔEDA: BK∥AD (con)

∴ΔEBK = 1/4ΔEAD (#96)

ΔEBK,DCK: BK=KC (#116) ∠K = ∠K (1.15) BE=DC (con)

∴ΔEBK ≡ ΔDCK (1.4)

∴ΔEBK = 1/4 each ∥gm (#96)

Sym. ΔCLF = 1/4 each ∥gm

∴ ∥KELC = ½ each ∥gm

125. Proof

Assume O∉AC

EOF∥BC × AB,DC @ E,F

AC × EF @ G: G·|·(O,F)

Line on G∥AB ∴ ∥gmGB = ∥gmGD (1.43)

∴ ∥gmOB < ∥gmOD ↴ (OB=OD by hyp)

∴O∈AC

Note

If O·|·(G,F) letters change but proof is the same.

126. Proof

ΔCBD,CAF: CD=AC and CF=BC (con) ∠DCB = ∠ACF = ∟+∠C

∴ΔCBD ≡ ΔCAF (1.4)∴AF=BD

127. Proof

AD⊥AB: AD=AC Join BD

BD > BC (1.24)

$BD^2 = BA^2 + AD^2$ (1.47) ∴ $BC^2 < BA^2 + AD^2$

AD=AC ∴ $BC^2 < BA^2 + AC^2$

128. Proof

AD⊥AB: AD=AC Join BD

BD < BC (1.24)

$BD^2 = BA^2 + AD^2$ (1.47) ∴ $BC^2 > BA^2 + AD^2$

AD=AC ∴ $BC^2 > BA^2 + AC^2$

129. Proof

1) Converse 127

$\Delta ABC: BC^2 < AB^2 + AC^2$

$\angle A \neq ∟$ (1.47) and $\angle A$ not obtuse (#127) $\therefore \angle A$ acute

2) Converse 128

$\Delta ABC: BC^2 > AB^2 + AC^2$

$\angle A \neq ∟$ (1.47) and $\angle A$ not acute (#128) $\therefore \angle A$ obtuse

Note

This is proof by exhaustion where you exclude all other possibilities. If something can be A, B, or C, then to prove it is A, we show it cannot be B or C.

130. Proof

$BE^2 = AB^2 + AE^2$ and $CD^2 = AD^2 + AC^2$ (1.47)

$\therefore BE^2 + CD^2 = AB^2 + AE^2 + AD^2 + AC^2$ (a.2)

$AB^2 + AC^2 = BC^2$ and $AD^2 + AE^2 = DE^2$ (1.47)

$\therefore BE^2 + CD^2 = BC^2 + DE^2$

Note

We have been justifying most lines of proofs with references to propositions and previous results. From this point, we justify only the less obvious. If a line of a proof puzzles you, justify it.

131. Proof

$PK \parallel AD \times AB,CD$ @ K,L (1.31)

$PM \parallel AB \times AD,BC$ @ M,N (1.31)

$\therefore AK=DL=MP$ and $KB=LC=PN$ and $DM=LP=CN$ (1.34)

$\therefore PA^2 + PC^2 = AM^2 + PM^2 + CN^2 + PN^2$ (1.47)

$\therefore PA^2 + PC^2 = BN^2 + PN^2 + DM^2 + PM^2$ (a.1)

$\therefore PA^2 + PC^2 = PB^2 + PD^2$

132. Proof

$4BE^2 = 4AB^2 + 4AE^2$ and $4CF^2 = 4AF^2 + 4AC^2$ (1.47)

$\therefore 4(BE^2 + CF^2) = 4(AB^2 + AE^2 + AF^2 + AC^2)$

$\therefore 4(BE^2 + CF^2) = 4(BC^2 + AE^2 + AF^2)$

$\therefore 4(BE^2 + CF^2) = 4BC^2 + AC^2 + AB^2 = 5BC^2$

133. Proof

$BC^2 = AB^2 + AC^2$ (1.47) ∴ $BC^2 = 4AB^2$ ∴ BC = 2AB

BC = 2DC (#58) ∴ AC=DC=AD ∴ ΔADC ≡ eqSΔ

∴ ∠DAC = 2/3∟ ∴ ∠BAE = 1/3∟

ΔCEA,DEA: CA=DA ∴ ∠ADC = ∠ACD

∠CEA,DEA = ∟ ∴ ∠CAE = ∠DAE (1.32)

∴ ∠DAE = 1/3∟ ∴ ∠BAE = ∠EAD = ∠DAC

134. Proof

DM⊥GB(pr)

∴ ∠DBM + ∠MBC = ∟ and ∠CBA + ∠MBC = ∟ ∴ ∠DBM = ∠CBA

ΔDBM,CBA: DB=CB, ∠DBM = ∠CBA, ∟DMB = ∟CAB

∴ BM=BA and DM=CA (1.26)

∴ GM = 2AB ∴ $GM^2 = 4AB^2$

$DG^2 = GM^2 + DM^2$ (1.47) $= 4AB^2 + AC^2$

Sym. $EJ^2 = 4AC^2 + AB^2$

∴ $DG^2 + EJ^2 = 5BC^2$

135. Proof

$AB^2 = AC^2 + CB^2 + 2(AC•CB)$ (2.4)

Or AB^2 = ‖gm (sides = AC) + ‖gm (sides = CB) + 2 complements

Let the complements equal the two squares.

∴ $2(AC•CB) = AC^2 + CB^2 = ½AB^2$ (2.4.C2)

∴ $AC•CB = ½(AC^2 + CB^2) = 1/4AB^2$

∴ $AC^2 = CB^2$ ∴ C mdpt AB

Note

This is a case where one knows all the elements of the proof but may have no idea how to lay them out in a mathematical way. This was one way.

136. Method/Proof

C mdpt AB, D∈CB ∴ $AD•DB + CD^2 = AC^2$ (2.5)

∴ $AD•DB = AC^2 - CD^2$

∴ AD•DB maximum when CD = 0

∴ C mdpt AB required

137. Method/Proof

C mdpt AB, AB(pr) to D: $AD \bullet DB + CB^2 = CD^2$ (2.6)

$\therefore AD \bullet DB = CD^2 - CB^2$ (a.3)

AC=MN. AC(pr) to D: CD=KL.

$B \notin CD$: CB=AC $\therefore AD \bullet DB = KL^2 - MN^2$

Note

In Book II, we use the propositions mathematically and diagrams only as needed. Here, the form of 2.6 is its own proof.

138. Method/Proof

C mdpt AB, $D \in CB$: $AD^2 + DB^2 = 2(AC^2 + CD^2)$ (2.9)

$\therefore AD^2 + DB^2$ minimum when $CD^2 = 0$ $\therefore$ when C=D

$\therefore$ C mdpt AB required

139. Proof

C mdpt AB, AB(pr) to D: $AD^2 + DB^2 = 2(AC^2 + CD^2)$ (2.10)

If AC=A, CD=B, and CB=AC $\therefore AD = A + B$, $DB = A - B$

$\therefore AD^2 + DB^2 = 2(AC^2 + CD^2) = (A + B)^2 + (A - B)^2 = 2(A^2 + B^2)$

140. Method

$\angle ABD = \frac{1}{2}\llcorner$ ⊙A,KL × BD @ E $EC \perp AB$ C required

Proof

$\angle ECB = \llcorner$ and $\angle ABD = \frac{1}{2}\llcorner$ $\therefore \angle CEB = \frac{1}{2}\llcorner$ (1.32) $\therefore$ CE=CB (2.9)

KL=AE (con) $\therefore AE^2 = KL^2 = AC^2 + CB^2$

Note

If you construct the diagram **accurately**, so that CE=CB, Todhunter writes, "For it may be shown as in 2.9 that CE is equal to CB." If you can't see it, 1.6 will do for equality. Food for thought.

141. Method

AB(pr) to C: AC = diagonal of AB^2

$D \in BA$: BD=DC $\therefore AD^2 = 2DB^2$ $\therefore$ D required

Proof

B mdpt CD, CD(pr) to A: $CA^2 + DA^2 = 2(BA^2 + BD^2)$ (2.10)

$CA^2 = BA^2 + BA^2$ (1.47, con)

$\therefore AD^2 = 2DB^2$ (a.3)

142. Proof

ΔHAC,FAB: HA=FA and AC=AB (con) ∠HCA = ∠FAB = ∟

∴ ∠HCA = ∠FBA (or LBH)

∠LHB = ∠AHC (1.15) ∴ ∠HLB = ∠HAC (1.32)

∴ ∠HLB = ∟ ∴ CL⊥BF

143. Proof

EF=EB ∴ ∠EBF = ∠EFB (1.5) or ∠OBL = ∠CFL

∴ ∠FCL = ∠LOB (1.32, #142) ∴ ∠ECO = ∠LOB

∴ ∠ECO = ∠EOC (1.15) ∴ EO=EC (1.6) = EA (con)

∴ ∠EOC + ∠ECO = ∠ECO + ∠EAO

∴ ∠AOC = ∠ACO + ∠CAO = ∟ (1.32)

Note

If your diagram is not accurate, none of this will appear true.

144. Proof

(AH+HB)•(AH-HB) = AH^2 - HB^2 (algebra) = AB•HB - HB^2 (hyp)

= (AH+HB)HB - HB^2 = AH•HB + HB^2 - HB^2 = AH•HB

Note

If you are in this for the long run, you get good mileage from realizing the geometry of (A+B)(A-B) = A^2 - B^2

145. Proof

BC^2 = AB^2 + AC^2 - 2AB•AD (2.13)

= $2AB^2$ - 2AB•AD (d.1.24)

= 2(AB^2 - AB•AD) = 2AB(AB - AD) = 2AB•BD

146. Proof

1) AD⊥BC

AB^2 = AD^2 + BD^2 and AC^2 = AD^2 + DC^2 (1.47)

BD=DC ∴ AB^2 + AC^2 = 2(AD^2 + BD^2)

2) AD !⊥ BC

AE alt∠A

AB^2 = AD^2 + BD^2 + 2BD•DE (2.12)

AC^2 = AD^2 + CD^2 - 2DC•DE (2.13)

BD=DC ∴ AB^2 + AC^2 = 2(AD^2 + BD^2)

Make sure you understand that last result. Most of the remaining problems use it. Ask yourself why.

147. Proof

$AC^2 + CD^2 = 2AB^2 + 2BC^2$ (#146)

$AB=AC \therefore AB^2 = AC^2$

$\therefore CD^2 = 2AB^2 - AB^2 + 2BC^2 = AB^2 + 2BC^2$

148. Proof

AC × BD @ E

$AB^2 + BC^2 = 2BE^2 + 2AE^2$ (#146)

$AD^2 + CD^2 = 2DE^2 + 2CE^2$ (#146)

BE=DE and AE=CE (1.34)

$\therefore \sum side^2 = 4BE^2 + 4AE^2$ (a.2) $= (2BE)^2 + (2AE)^2 = BD^2 + AC^2$ (1.34)

149. Proof

∀A: med∠A ≡ AD ∈⊙D,DA

$\therefore AB^2 + AC^2 = 2DC^2 + 2DA^2$ (#146)

DC = AD = radius ⊙D,DA (d.1.15)

$\therefore AB^2 + AC^2 = 4(radius)^2$

150. Proof

Join EF,FG,GH,HE ∴ EFGH ≡ ∥gm (#118)

½BD = EH and ½AC = HG (#119)

$\therefore AC^2 + BD^2 = 4HG^2 + 4EH^2$

$= 2(HG^2 + EF^2 + EH^2 + FG^2)$

$= 2(EG^2 + FH^2)$ (#148)

Note

Line 4: 4 × one side = 2 × 2 equal sides

151. Proof

$PA^2 + PC^2 = 2(PO^2 + AO^2)$ (#146)

$PB^2 + PD^2 = 2(PO^2 + DO^2)$ (#146)

∀⊙O,PO: AO,DO constant = half diags in ∥gm

$\therefore \sum P[A\text{-}D]^2 = k$

152. Proof

$AB^2 + BC^2 = 2(AE^2 + BE^2)$ (#146)

$AD^2 + DC^2 = 2(AE^2 + DE^2)$ (#146)

$\therefore \sum side^2 = 4AC^2 + 2(BE^2 + DE^2)$

$BE^2 + DE^2 = 2(BF^2 + DF^2)$ (#146)

$\therefore 2(BE^2 + DE^2) = 4(BF^2 + DF^2) = BD^2 + 4EF^2$

$\therefore \sum side^2 = \sum diag^2 + 4EF^2$

153. Proof

$CE \perp AB \therefore \Delta CEA \equiv \Delta CEB$ (1.26)

$DC^2 = CB^2 + DB^2 + 2DB \bullet BE$ (2.12) and $BA=2BE$

$\therefore DC^2 = CB^2 + DB^2 + DB \bullet BA = CB^2 + AD \bullet DB$ (2.3)

$= BC^2 + BC^2 = 2BC^2$

154. Proof

$DF \perp AB$(pr) $EG \perp AC$(pr)

$\angle DBF + \angle CBA = ∟$ (1.13)

$\angle ACB + \angle CBA = ∟$ (1.32)

$\therefore \angle DBF = \angle ACB$

$\Delta DBF, ACB$: $\angle DBF = \angle ACB$ $\angle DFB = \angle BAC = ∟$ $DB=BC$

$\therefore \Delta DBF \equiv \Delta ACB$

$DA^2 = DB^2 + BA^2 + 2BA \bullet BF$ (2.12)

$\therefore DA^2 = BC^2 + BA^2 + 2BA \bullet AC$

$\therefore AC^2 + DA^2 = BC^2 + BA^2 + 2BA \bullet AC + AC^2$

Sym. $EA^2 + AB^2 = BC^2 + BA^2 + 2BA \bullet AC + AC^2$

$\therefore DA^2 + AC^2 = AB^2 + EA^2$

155. Proof

E mdpt BC $\therefore$ $BD = BE - DE$ and $CD = BE + DE$

$\therefore BD \bullet CD = BE^2 - DE^2$

$AE=BE$ (#89) $\therefore$ $AE^2 - DE^2 = AD^2$ (1.47) $\therefore AD^2 = BD \bullet CD$

156. Proof

$AC^2 = AD^2 + DC^2$ (1.47) $= BD \bullet DC + AD^2$ (#155) $= BC \bullet CD$ (2.3)

Sym. $AB^2 = BC \bullet BD$

157. Proof

$AC^2 + 2AB \bullet BF = AB^2 + BC^2$ (2.13)

$AB^2 + 2AC \bullet CE = AC^2 + BC^2$ (2.13)

$\therefore AC^2 + AB^2 + 2AB \bullet BF + 2AC \bullet CE = AB^2 + AC^2 + 2BC^2$ (a.2)

$\therefore 2AB \bullet BF + 2AC \bullet CE = 2BC^2 \;\; \therefore \;\; AB \bullet BF + AC \bullet CE = BC^2$

158. Proof

$EC^2 + ED^2 = 2(DE^2 + OC^2) = 2(AO^2 + OC^2)$

$AD = AO + OC$ and $AC = AO - OC$

$\therefore AD^2 + AC^2 = 2(AO^2 + AC^2)$ (algebra)

$\therefore EC^2 + ED^2 = AO^2 + AC^2$

159. Proof

O mdpt AD

$AB^2 + BD^2 = 2(BO^2 + OD^2)$ (#146)

$AC^2 + CD^2 = 2(CO^2 + OD^2)$ (#146)

$AB^2 + BD^2 = AC^2 + CD^2$ (hyp) $\therefore \;\; BO^2 + OD^2 = CO^2 + OD^2$

$\therefore BO=CO \;\; \therefore$ O mdpt AD

160. Proof

AO med∠A and D·|·(B,O)

$AB^2 = BO^2 + AO^2$ and $DA^2 = DO^2 + AO^2$ (1.47)

$\therefore AB^2 = AD^2 + (BO^2 - DO^2) = AD^2 + (B+O)(B-O) = AD^2 + CD \bullet CB$

Note

Sym. with letter changes if D·|·(C,O)

161. Proof

$CD^2 + 2BC \bullet BE = BC^2 + BD^2$ (2.13)

$CD^2 + 2BA \bullet BD = BC^2 + BD^2$ (2.13)

$\therefore \;\; BC \bullet BE = BA \bullet BD$

162. Method

C mdpt AB   LO⊥LM   ⊙M,AC × LO @ N

D∈AB: CD=LN   D required

Proof

$AD \bullet DB = AC^2 - CD^2$ (2.5) $= MN^2 - LN^2$ (con) $= LM^2$ (1.47)

Notation

Labelling is done top to bottom, left to right; or clockwise from top-left apex of non-triangular figure. Labelling in propositions follows that of the original 1867 diagrams.

Operators

intersect, cut	×
bisect, bisector	×/2
trisect	×/3
at	@
parallel	‖
between	·|·
A between B and C	A·|·(B,C)
perpendicular	⊥
AB perpendicular to CD	AB⊥CD
equivalent, equal in every way	≡
equal in magnitude	=
on	∈
not on	∉
equilateral (equal sides)	eqS
equiangular	eq∠
equidistant	eqD
absolute difference	~
|a-b|	~(a,b) or a~b
summation	∑
A+B+C+D	∑[A-D]

Points

on or endpoints of lines	A, B, C, ...
considered in themselves	P, R, S, ..
as center of a figure	O

Lines

by endpoints	AB
creation from points	Join AB
Join AB, AC, AD	Join A[B-D]
mid-point	mdpt
P mdpt AB, Q mdpt CD	P,Q mdpt AB,CD

Angles

angle	∠
interior angle	int∠
exterior angle	ext∠
alternate angle	alt∠
opposite angle	opp∠
right angle	∟

Triangles

triangle	Δ
right triangle	⊿
∀triangle	ΔABC
equilateral triangle	eqSΔ
equiangular triangle	eq∠Δ
isosceles triangle	isosΔ
CF bisector of angle C	CF ×/2 ∠C
AD median on angle A	AD med∠A
BE altitude on angle B	BE alt∠B

Circles

circle	⊙
create by center and radius	⊙A,AB
as existing circle	⊙A
as defined by three points	⊙ABC
touching center	on center
on circumference	∈⊙
in circle's whitespace	in ⊙

Polygons

polygon	n-gon
by number of sides (4+)	4-gon
parallelogram	∥gm
rectangle	rect∟
rectangle, sides AB,CD	AB•CD
square on line AB	AB^2

Logic

therefore	∴
symmetrically	Sym.
by hypothesis	(hyp)
by construction	(con)
contradiction	↴
any, every, each, all	∀
exists, exists only one	∃, ∃!
not, not equivalent	! !≡

Euclid's Axioms, Postulates, and Definitions

All of the following are from Loney's last edition of Todhunter's Euclid. Their numbering differs slightly from another version of Todhunter's. And looking around, there is no conclusive numbering. All are close. Beyond that, you will find that there is a bit of back and forth between axioms and postulates from text to text as well. Corollaries date from the 17thC and can differ from text to text. **The numbering of the propositions are Euclid's and are the same in all Euclid texts.**

Euclid's Axioms

a.1 Things equal to the same thing are also equal to one another.
a.2 Things added to equals make equals.
a.3 Things taken from equals leave equals.
a.6 Things twice the same thing are equal to each other.
a.7 Things half of the same thing are equal to each other.
a.8 The whole is greater than its part.
a.9 Magnitudes which can be made to coincide are equal.
a.10 Two lines cannot enclose a space. They must have 0, 1, or all points in common.
a.11 All right angles are equal.
a.12 If a line cut two other lines such that, on one side of the first, the other two make angles summing to less than two right angles, the lines, extended on that side, must intersect.

Euclid's Postulates

p.1. A line may be drawn between any two points.
p.2. A line may be indefinitely extended.
p.3. Any point and any line from it may be used to construct a circle.

Euclid's Definitions
Book I

d.1.1 A **point** is position without magnitude.
d.1.2 A **line** is length without breadth.
d.1.3 The **extremities** and **intersections** of lines are points.
d.1.5 A **surface** is length and breadth.
d.1.6 The **boundaries** of surfaces are lines.
d.1.7 A **plane** is a surface such that, for any two points, their line lies entirely on the surface.
d.1.8 A **plane angle** is the inclination of two lines to one another which meet on the plane.
d.1.9 A **plane rectilinear angle** is the plane angle of two straight lines which meet at their **vertex**.
d.1.10 When a line meets another so that the two angles created by the former on one side of the latter are equal, these are **right angles** and the lines are **perpendicular**.
d.1.11 An **obtuse angle** is greater than a right angle.
d.1.12 An **acute angle** is less than a right angle.
d.1.13 A **plane figure** is any shape enclosed by lines, which are its perimeter.
d.1.15 A **circle** is a plane figure bounded by its **circumference** which is equidistant from its **center**.
d.1.20 A **triangle** is bounded by three straight lines. Any of its angular points can be its **apex** which is opposite its **base**.
d.1.22 A **polygon** or **n-gon** is a plane figure with n lines for sides. A figure with 4 sides is a 4-gon or "quadrilateral."
d.1.23 An **equilateral triangle** has three equal sides.
d.1.24 An **isosceles triangle** has two equal sides.
d.1.29 **Parallel lines** are coplanar lines which cannot be produced to intersect.
d.1.30 A **parallelogram** is a 4-gon of opposing parallel sides
d.1.31 A **square** is an eqS 4-gon with one right angle.
d.1.33 A **rhombus** is an eqS 4-gon with no right angles

Euclid Definitions

Book II

d.2.1 ∀ rectangle ABCD is **contained** by any two adjacent sides. In our notation, this is "rectangle ABCD ≡ AB•AD".

d.2.2 In a ∥gm, there are two internal ∥gms on a diagonal and two complements. The complements combined with either internal ∥gm is a **gnomon**.

d.2.3 ∀ AB produced in both directions: if we choose a point (cut) between A and B, we divide AB **internally**. If we choose a point to either side, outside of AB, we divide AB **externally**.