A KEY

TO THE

TREATISE ON ALGEBRA

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PREFACE.

At the urgent solicitation of a large number of teachers, I have at length consented to publish a Key to my Treatise on Algebra. It will probably be found convenient to many teachers who are just commencing the business of instruction, and also to those who are obliged to give instruction in a large number of departments, and consequently have but little time to devote to the preparation for any single subject. I have hitherto declined to publish this Key, from the apprehension that it might fall into the hands of young pupils, and thus be the means of defeating the main object to be secured by the study of Algebra. To most persons who are pursuing a course of scientific studies, the principal advantage to be anticipated from the study of Algebra is mental discipline; and a student can only hope to attain this object by the effort to overcome difficulties in reliance upon his own resources. The student who begins the study of Algebra with the determination to work out every thing for himself without assistance, soon acquires confidence in his own powers, and is daily becoming better prepared to encounter future difficulties; while the individual who resorts to a Key for assistance as soon as he encounters some slight difficulty, is sure to lose confidence in his own ability, and acquires a habit of shrinking from severe effort, which probably will not be confined to mathematical subjects. It should, therefore, be the aim of every teacher to prevent his pupils from having access to the Key; and each pupil should be fully aware that, if he depend upon a Key for assistance in preparing for his recitations, although he may seem to have attained a present advantage, he will in the end sink very much below his companion who relies entirely upon his own resources in contending with mathematical difficulties.

ELIAS LOOMIS.

KEY

TO

LOOMIS'S TREATISE ON ALGEBRA.

CHAPTER I.

Art. 36, Page 18.

Ex. 2. $\frac{3}{x+4}=2b-8.$

Ex. 3. $\frac{6x-4}{3}=\frac{5}{a+b}.$

Ex. 4. $\frac{3x}{4}+5=\frac{3b}{7}-17.$

Ex. 5. $\frac{6x+5}{9}+\frac{2x+4}{3}=abc.$

Ex. 6. $\frac{a+b}{cd}>4(m+n+x+y).$

Art. 38, Page 20.

Ex. 4. $4+\frac{40}{\sqrt{48+16}}=4+\frac{40}{8}=9.$

Ex. 5. $\sqrt{25-24}+\sqrt{48+16}=\sqrt{1}+\sqrt{64}=1+8=9.$

Ex. 6. $3\sqrt{4}+12\sqrt{12+5+8}=6+60=66.$

Ex. 7. $(3\sqrt{4}+12)\sqrt{12+5+8}=18\sqrt{25}=90.$

Ex. 8. $\frac{36}{6}-\frac{15}{5}+\frac{12}{6}-\frac{30}{10}=6-3+2-3=2.$

Ex. 9. $\frac{36\times25\times16\times64+3\times6\times5\times4\times8+2}{6\times5\times4\times8+1}=\frac{921600+2880+2}{960+1}=962.$

Ex. 13. $\sqrt{10+6}-\sqrt[4]{10+6}=\sqrt{16}-\sqrt[4]{16}=4-2=2.$

Ex. 14. $5(16+36)+4\times3\times9=260+108=368.$

Ex. 15. $81-108+63-18=18.$

Ex. 16. $\sqrt{5\times2+5\times3}+2+3=\sqrt{25}+2+3=10.$

Ex. 17. $\frac{16+36-25}{4-6+5}+\frac{144}{4+12}-\frac{324-105+1}{6+4}=\frac{27}{3}+\frac{144}{16}-\frac{220}{10}$
$=9+9-22=-4.$

Ex. 18. $4\left\{\frac{3}{2}+\frac{5}{4}+\frac{18}{6}\right\}=4\times\frac{23}{4}=23.$

Ex. 19. $\frac{216-64}{16+24+36}=\frac{152}{76}=2.$

CHAPTER II.

Art. 40, Page 21.

Ex. 3. $12b+15x.$

Ex. 4. $15a-11x^2.$

Ex. 5. $20a+12y^2.$

Art. 41, Page 22.

Ex. 3. $6ay-19.$

Ex. 4. $3a^2x.$

Ex. 5. $-5a^2-11b.$

Art. 42, Page 23.

Ex. 8. $ax-5.$

Ex. 9. $10x^2+9.$

Ex. 10. $9a^3x^2+6ax.$

Ex. 11. $7a^2b^2c^2.$

Ex. 12. $6ax^4-4x.$

Art. 44, Page 24.

Ex. 4. $(2+a+b)x+(3+b+3m)xy.$

Ex. 5. $(m+3a+4b)x+(n-2+a)y.$

Ex. 6. $(4m+2a+b)\sqrt{x}+2+y.$

Ex. 7. $(3a+4b+m)x^2+(2b-a-n)x+7.$

Ex. 8. $(2a+3m+4)x^4+(3b-n-a)x^3-4.$

Ex. 9. $(a+b+c)mx^3+(b-a-1)nx^2+(c+a+3b)x$.
Ex. 10. $(2a-c)\sqrt{x}$.

CHAPTER III.

Art. 46, Page 26.

Ex. 12. $3b+m-4x$.
Ex. 13. $x^3-x^2y+7xy^2-y^3$.
Ex. 14. $2n$.
Ex. 15. $2m+2n+2x$.
Ex. 16. $7a^2+a-17$.
Ex. 17. $5m^3-5m^2+m-4$.
Ex. 18. $-10x^4+20x^3-6$.
Ex. 19. $x^2+a^2x+5ax+a^2$.
Ex. 20. $9abx-ax-7mn$.

Art. 48, Page 27.

Ex. 4. $(4m-2a)\sqrt{x}+4$.
Ex. 5. $(2a-3m)x^4+(3b+n)x^3-9$.
Ex. 6. $(a-b)mx^3+(b+a)nx^2+(c-a)x$.
Ex. 7. $(1-c)m$.
Ex. 8. $1+(3a-1)x^2+(5a+3)ax^3+(7a+5)a^2x^4$.

Art. 50, Page 29.

Ex. 4. $3a+2b-3m-3x^2$.
Ex. 5. $7a$.
Ex. 6. $7a-5b$.
Ex. 7. $7a^2xy-8bx^2y+11cxy^3-12y^5$.
Ex. 8. $-2ax^2-22a^2x^2+17a^2x+4a^4$.

CHAPTER IV.

Art. 59, Page 34.

Ex. 5. $63a^6xy$.
Ex. 6. $132a^8b^8c^8$.
Ex. 8. $72a^{m+n}$.
Ex. 10. $108a^{2m}$.
Ex. 15. $28a^4b^6c^3$.
Ex. 16. $105a^{m+2}b^3x^2$.
Ex. 17. $a^5b^4c^3d^2x$.
Ex. 18. $a^{12}b^8c^5x$.

Art. 61, Page 36.

Ex. 4. $a^3-3a^2b+3ab^2-b^3$.
Ex. 5. $3a^3-14a^2+13a-20$.
Ex. 7. $2a^4+a^3b-6a^2b^2-2a^2+17ab-12$.
Ex. 8. a^4-b^4.
Ex. 9. $a^2+(m+n)ab+mnb^2$.
Ex. 10. $9a^2-4b^2x^2+12bx^3-9x^4$.
Ex. 11. $x^3-19x-30$.
Ex. 12. $x^4-8x^3-11x^2+198x-360$.
Ex. 13. a^6-b^6.
Ex. 14. $abc+(ab+ac+bc)x+(a+b+c)x^2+x^3$.
Ex. 15. a^5-b^5.
Ex. 16. $a^5+3a^2-22a-24$.
Ex. 17. x^6-2x^3+1.
Ex. 18. $196a^6x^2-36a^4b^2x^2+12a^2bx^3-x^4$.
Ex. 19. $x^5-x^4y-x^4y^2+x^3y^3-xy^5$.
Ex. 20. $12x^4+11x^3y+7x^2-56x^2y^2+107xy-45$.

Art. 65, Page 38.

Ex. 2. $9a-7b+7c$.
Ex. 3. $115+14a-14b-12c$.
Ex. 4. $65a-17b+17c$.
Ex. 5. $23a-51b+36c$.
Ex. 6. $-54b+106c$.

Art. 66, Page 39.

Ex. 1. $9a^2+6ab+b^2$.
Ex. 2. $9a^2+18ab+9b^2$.
Ex. 3. $25a^2+30ab+9b^2$.
Ex. 4. $25a^4+20a^2b+4b^2$.
Ex. 5. $25a^6+10a^3b+b^2$.
Ex. 6. $25a^4+70a^3b+49a^2b^2$.
Ex. 7. $25a^6+80a^5b+64a^4b^2$.
Ex. 8. $4a^2+2a+\frac{1}{4}$.
Ex. 9. $1+\frac{2}{3}+\frac{1}{9}=\frac{16}{9}$.
Ex. 10. $9+\frac{6}{5}+\frac{1}{25}=\frac{256}{25}$.

ART. 67, PAGE 39.

Ex. 1. $4a^2-12ab+9b^2$.
Ex. 2. $25a^2-40ab+16b^2$.
Ex. 3. $36a^4-12a^2x+x^2$.
Ex. 4. $36a^4-36a^2x+9x^2$.
Ex. 5. $x^2-xy+\frac{y^2}{4}$.
Ex. 6. $49a^4-168a^3b+144a^2b^2$.
Ex. 7. $49a^4b^4-168a^3b^3+144a^2b^2$.
Ex. 8. $4a^6-20a^3+25$.
Ex. 9. $4-\frac{4}{3}+\frac{1}{9}=\frac{25}{9}$.
Ex. 10. $16-\frac{8}{5}+\frac{1}{25}=\frac{361}{25}$.

ART. 69, PAGE 40.

Ex. 1. $9a^2-4b^2$.
Ex. 2. $49a^2b^2-x^2$.
Ex. 3. $64a^2-49b^2c^2$.
Ex. 4. $25a^4-36b^6$.
Ex. 5. $16a^4-9m^2x^2$.
Ex. 6. $9a^4b^2-a^6$.
Ex. 7. m^2-1.
Ex. 8. $16-\frac{1}{9}=\frac{143}{9}$.

CHAPTER V.

ART. 74, PAGE 42.

Ex. 5. $-4a^2bcd$.
Ex. 6. $-5a^2b^3cd$.
Ex. 7. $50a^6b^7$.
Ex. 8. $-16abc^4x^2$.
Ex. 9. $-2a^5$.
Ex. 10. $6a^3b^3$.

ART. 78, PAGE 44.

Ex. 3. $-40a^2b^2-60ab+17$.
Ex. 4. $-3b+\frac{2x^2}{a}-\frac{cd^2}{a}$.
Ex. 5. $-4x^3+7x^2+3x-15$.
Ex. 6. $2x^2y^4-4axy^4+5a^2x^3y$.
Ex. 7. $x-x^2+x^3-x^4$.
Ex. 8. $-3y^3+4ay^2-5a^2y+7a^3$.

Art. 80, Page 47.

Ex. 4. $a^4+4a^3x+12a^2x^2+16ax^3+16x^4$.
Ex. 6. $16x^4-8x^3y+4x^2y^2-2xy^3+y^4$.
Ex. 7. x^2-xy+y^2.
Ex. 8. $x^2+3x+5+\frac{24x+12}{x^2-2x-3}$.
Ex. 9. $a^3-2a^2b+2ab^2-b^3$.
Ex. 10. $x^4+2x^3+3x^2+2x+1$.
Ex. 11. x^2-xy+y^2.
Ex. 13. $3x^4+3x^2y^2+3y^4$.
Ex. 15. x^4-5x^2+4.
Ex. 16. $a^2-2ab+3b^2$.
Ex. 17. $x^2-xy+y^2+x+y+1$.
Ex. 18. $ab+bc-ac$.
Ex. 19. a^2+ab+b^2.
Ex. 20. $a^3+a^2b+ab^2+b^3$.

Art. 86, Page 50.

Ex. 4. $7a^2b^2(a-b-c)$.
Ex. 5. $4abc(2a+3b-4c)$.
Ex. 6. $5ab^2c(2mx-y+1)$.

Art. 87, Page 50.

Ex. 3. $(a-3b)(a-3b)$.
Ex. 4. $(3a-4b)(3a-4b)$.
Ex. 5. $(5a^2-6b^3)(5a^2-6b^3)$.
Ex. 6. $(2mn-1)(2mn-1)$.
Ex. 7. $(7a^2b^2-12ab)(7a^2b^2-12ab)$.
Ex. 8. $n(n+1)(n+1)$.
Ex. 9. $(4a^2b-3mx)(4a^2b-3mx)$.
Ex. 10. $m^2n^2(m+n)(m+n)$.

Art. 88, Page 51.

Ex. 2. $(3ab+4ac)(3ab-4ac)$.
Ex. 3. $ax(a^2+3x)(a^2-3x)$.

Ex. 4. $(a^2+b^2)(a+b)(a-b)$.
Ex. 5. $(a^2-ab+b^2)(a^2+ab+b^2)(a+b)(a-b)$.
Ex. 6. $(a^4+b^4)(a^2+b^2)(a+b)(a-b)$.
Ex. 7. $(1+\frac{1}{5})(1-\frac{1}{5})$.
Ex. 8. $(2+\frac{1}{7})(2-\frac{1}{7})$.

Art. 89, Page 51.

Ex. 2. $(a^2-ab+b^2)(a+b)$.
Ex. 3. $(a^2-ab+b^2)(a^2+ab+b^2)(a+b)(a-b)$.
Ex. 4. $(a^2+2ab+4b^2)(a-2b)$.
Ex. 5. $(4a^2+2a+1)(2a-1)$.
Ex. 6. $8(a^2+ab+b^2)(a-b)$.
Ex. 7. $(1-3b+9b^2)(1+3b)$.
Ex. 8. $(4a^2-6ab+9b^2)(2a+3b)$.
Ex. 9. $(a^8+b^8)(a^4+b^4)(a^2+b^2)(a+b)(a-b)$.

CHAPTER VI.

Art. 93, Page 53.

Ex. 4. $7amx^2$.
Ex. 5. $a-b$.
Ex. 6. $3x-1$.
Ex. 7. $2a-3b$.

Art. 97, Page 57.

Ex. 4.

$$\begin{array}{r|l} a^2-3ab+2b^2 & a^2-ab-2b^2 \\ \cline{2-2} \underline{a^2-ab-2b^2} & 1 \\ -2ab+4b^2 & \end{array}$$

Suppress the factor $-2b$.

$$\begin{array}{r|l} a^2-ab-2b^2 & a-2b \\ \cline{2-2} \underline{a^2-2ab} & a+b \\ ab-2b^2 & \\ ab-2b^2 & \end{array}$$

Hence $a-2b$ is the greatest common divisor.

Ex. 5.

$$\begin{array}{r|l} a^3-a^2b+3ab^2-3b^3 & a^2-5ab+4b^2 \\ \cline{2-2} \underline{a^3-5a^2b+4ab^2} & a+4b \\ 4a^2b-ab^2-3b^3 & \\ \underline{4a^2b-20ab^2+16b^3} & \\ 19ab^2-19b^3 & \end{array}$$

Suppress the factor $19b^2$.

$$\begin{array}{r|l} a^2-5ab+4b^2 & a-b \\ a^2-\ ab\quad\ \ & \overline{a-4b} \\ \hline -4ab+4b^2 & \\ -4ab+4b^2 & \end{array}$$

Hence $a-b$ is the greatest common divisor.

Ex. 6.

$$\begin{array}{r|l} 6x^3+\ \ x^2-44x+21 & 3x^3-13x^2+23x-21 \\ 6x^3-26x^2+46x-42 & \overline{2} \\ \hline 27x^2-90x+63 & \end{array}$$

Suppress the factor 9.

$$\begin{array}{r|l} 3x^3-13x^2+23x-21 & 3x^2-10x+7 \\ 3x^3-10x^2+\ 7x\qquad & \overline{x-1} \\ \hline -\ 3x^2+16x-21 & \\ -\ 3x^2+10x-\ 7 & \\ \hline 6x-14 & \end{array}$$

Suppress the factor 2.

$$\begin{array}{r|l} 3x^2-10x+7 & 3x-7 \\ 3x^2-\ 7x\quad\ & \overline{x-1} \\ \hline -\ 3x+7 & \\ -\ 3x+7 & \end{array}$$

Hence $3x-7$ is the greatest common divisor.

Ex. 7.

$$\begin{array}{r|l} x^4-7x^3+\ 8x^2+28x-48 & x^3-8x^2+19x-14 \\ x^4-8x^3+19x^2-14x\qquad & \overline{x+1} \\ \hline x^3-11x^2+42x-48 & \\ x^3-\ 8x^2+19x-14 & \\ \hline -\ 3x^2+23x-34 & \end{array}$$

Suppress the factor -1, and multiply the second polynomial by 3.

$$\begin{array}{r|l} 3x^3-24x^2+57x-42 & 3x^2-23x+34 \\ 3x^3-23x^2+34x\qquad & \overline{x,\ -1} \\ \hline -\ \ x^2+23x-42 & \end{array}$$

Multiply again by 3.

$$\begin{array}{r} -3x^2+69x-12b \\ -3x^2-23x-34 \\ \hline 46x-92 \end{array}$$

Suppress the factor 46.

$$\begin{array}{r|l} 3x^2-23x+34 & x-2 \\ \cline{2-2} 3x^2-\ \ 6x & 3x-17 \\ \cline{1-1} -17x+34 & \\ -17x+34 & \end{array}$$

Hence $x-2$ is the greatest common divisor.

ART. 98, PAGE 58.

Ex. 1. The quantities are $3m^2\times a^2$, $3m^2\times 2b^2$, and $3m^2\times 4mx$.
Hence $3m^2$ is the greatest common divisor.

Ex. 2.

$$\begin{array}{r|l} 4x^3-21x^2+15x+20 & x^2-6x+8 \\ \cline{2-2} 4x^3-24x^2+32x & 4x+3 \\ \cline{1-1} 3x^2-17x+20 & \\ 3x^2-18x+24 & \\ \cline{1-1} x-\ \ 4 & \end{array}$$

$$\begin{array}{r|l} x^2-6x+8 & x-4 \\ \cline{2-2} x^2-4x & x-2 \\ \cline{1-1} -2x+8 & \\ -2x+8 & \end{array}$$

$$\begin{array}{r|l} x^2-\ \ x-12 & x-4 \\ \cline{2-2} x^2-4x & x+3 \\ \cline{1-1} 3x-12 & \\ 3x-12 & \end{array}$$

Hence $x-4$ is the greatest common divisor.

Ex. 3.

$$\begin{array}{r|l} 4x^3-6x^2-4x+3 & 2x^3+x^2+x-1 \\ \cline{2-2} 4x^3+2x^2+2x-2 & 2 \\ \cline{1-1} -8x^2-6x+5 & \end{array}$$

Reject the factor -1, and multiply the last divisor by 4.

$$\begin{array}{r|l} 8x^3+4x^2+4x-4 & 8x^2+6x-5 \\ \cline{2-2} 8x^3+6x^2-5x & x,\ -1 \\ \cline{1-1} -2x^2+9x-4 & \end{array}$$

Multiply by 4.

$$\begin{array}{r} -8x^2+36x-16 \\ -8x^2-\ \ 6x+\ \ 5 \\ \hline 42x-21 \end{array}$$

Reject the factor 21.

$$\begin{array}{r|l} 8x^2+6x-5 & 2x-1 \\ \cline{2-2} 8x^2-4x & 4x+5 \\ \hline 10x-5 & \\ 10x-5 & \end{array}$$

$$\begin{array}{r|l} 6x^4+x^3-x & 2x-1 \\ \cline{2-2} 6x^4-3x^3 & 3x^3+2x^2+x \\ \hline 4x^3-x & \\ 4x^3-2x^2 & \\ \hline 2x^2-x & \\ 2x^2-x & \end{array}$$

Hence $2x-1$ is the greatest common divisor.

Ex. 4.

$$\begin{array}{r|l} 3x^3+5x^2-x+2 & x^3+x^2-x+2 \\ \cline{2-2} 3x^3+3x^2-3x+6 & 3 \\ \hline 2x^2+2x-4 & \end{array}$$

Suppress the factor 2.

$$\begin{array}{r|l} x^3+x^2-x+2 & x^2+x-2 \\ \cline{2-2} x^3+x^2-2x & x \\ \hline x+2 & \end{array}$$

$$\begin{array}{r|l} x^2+x-2 & x+2 \\ \cline{2-2} x^2+2x & x-1 \\ \hline -x-2 & \\ -x-2 & \end{array}$$

$$\begin{array}{r|l} 4x^4+9x^3+2x^2-2x-4 & x+2 \\ \cline{2-2} 4x^4+8x^3 & 4x^3+x^2-2 \\ \hline x^3+2x^2-2x-4 & \\ x^3+2x^2 & \\ \hline -2x-4 & \\ -2x-4 & \end{array}$$

Hence $x+2$ is the greatest common divisor.

ART. 102, PAGE 60.

Ex. 4. The quantities may be written

$$5aabb,\ 2\times 5abbb, \text{ and } 2abx.$$

Hence the least common multiple is $2\times 5a^2b^3x$

Ex. 5. The quantities may be written

$$3abb,\ 2\times 2axx,\ 5bbx, \text{ and } 2\times 3aaxx.$$

Hence the least common multiple is $2\times 2\times 3\times 5aabbxx$.

Ex. 6. The quantities may be written

$$(x-2)(x-1), \text{ and } (x-1)(x+1).$$

Hence the least common multiple is $(x-2)(x-1)(x+1)$.

Ex. 7. The quantities may be written

$$(a^2-ab+b^2)(a+b)x, \text{ and } 5(a+b)(a-b).$$

Hence the least common multiple is

$$5x(a+b)(a-b)(a^2-ab+b^2).$$

ART. 103, PAGE 60.

Ex. 2. The quantities may be written

$$(x^2+x+1)(x-1), \text{ and } (x+2)(x-1).$$

The greatest common divisor is $x-1$.

Hence the least common multiple is

$$\frac{(x^3-1)(x^2+x-2)}{x+1}=(x^3-1)(x+2).$$

Ex. 3. The greatest common divisor is $x-1$.

Hence the least common multiple is

$$\frac{(x^3-9x^2+23x-15)(x^2-8x+7)}{x-1}=(x^3-9x^2+23x-15)(x-7).$$

Ex. 4. The quantities may be written

$$(a+3)(a-1),\ (a+1)(a-1), \text{ and } a-1.$$

Hence the least common multiple is

$$(a+3)(a+1)(a-1).$$

Ex. 5. The quantities may be written

$$4a^2+1,\ (2a+1)(2a-1), \text{ and } 2a-1.$$

Hence the least common multiple is

$$(4a^2+1)(2a+1)(2a-1).$$

Ex. 6. The quantities may be written

$$a(a+1)(a-1),\ (a^2-a+1)(a+1), \text{ and } (a^2+a+1)(a-1).$$

Hence the least common multiple is

$$a(a^2-a+1)(a^2+a+1)(a+1)(a-1).$$

Ex. 7. The quantities may be written

$$(x+2a)^2, (x-2a)^2, \text{ and } (x+2a)(x-2a).$$

Hence the least common multiple is

$$(x+2a)^2(x-2a)^2.$$

CHAPTER VII.

Art. 112, Page 65.

Ex. 3. Cancel the factor $2a-b$.

Ex. 4. Cancel the factor x^2-a^2.

Ex. 5. Cancel the factor x^2-8x-3.

Ex. 6. Cancel the factor $3x+3y$.

Ex. 7. Cancel the factor $a-b$, and we obtain $\frac{a+b}{a-b}$, *Ans.*

Ex. 8. Cancel the factor $a-x$, and we obtain $\frac{a^2+ax+x^2}{a-x}$, *Ans.*

Ex. 9. Cancel the factor $x+4$, and we obtain $\frac{x-4}{x-5}$, *Ans.*

Ex. 10. Cancel the factor x^2-5x+6, and we obtain $\frac{3x-1}{2x-1}$, *Ans.*

Ex. 11. Cancel the factor $2x^2-3x+2$, and we obtain $\frac{3x+2}{x+1}$, *Ans.*

Ex. 12. Cancel the factor x^2+3x-1, and we obtain $\frac{2x+3}{3x-4}$, *Ans.*

Art. 113, Page 65.

Ex. 1. $a-x$, *Ans.*

Ex. 2. $a-\frac{2a^2}{b}$, *Ans.*

Ex. 4. x^2+xy+y^2.

Ex. 5. $2x-1+\frac{3}{5x}$.

Ex. 6. $b^2-2+\frac{7a^2b}{8}$.

Ex. 7. $x-3$.

Ex. 8. $a+b$.

Art. 114, Page 66.

Ex. 3. $\frac{17x-7}{3x}$.

Ex. 4. $\frac{x-1}{a}$.

Ex. 5. $\frac{10x^2+6x-3}{5x}$.

Ex. 6. $\frac{7a^2-4b^2-8c^2}{a^2-b^2}$.

Art. 115, Page 67.

Ex. 2. $\frac{36cx}{24ac}, \frac{16ab}{24ac}$, and $\frac{6acd}{24ac}$.

Ex. 3. $\frac{45}{60}, \frac{40x}{60}$, and $\frac{60a+48x}{60}$.

Ex. 4. $\frac{7a^2-7ax}{14a-14x}, \frac{6ax-6x^2}{14a-14x}$, and $\frac{14a+14x}{14a-14x}$.

Ex. 5. $\frac{5x+5x^2}{15+15x}, \frac{3x^2+6x+3}{15+15x}$, and $\frac{15-15x}{15+15x}$.

Art. 116, Page 69.

Ex. 8. $\frac{a}{x^3}, \frac{mx}{x^3}$, and $\frac{nx^2}{x^3}$.

Ex. 9. $\frac{567a}{504m}, \frac{98b}{504m}, \frac{198a}{504m}$, and $\frac{882(a+b)}{504m}$.

Ex. 10. $\frac{8x^2-2}{4x^3-x}, \frac{6x^2+3x}{4x^3-x}$, and $\frac{2x^2-3x}{4x^3-x}$.

Art. 117, Page 70.

Ex. 3. $\frac{a^2+b^2}{a^2-b^2}$.

Ex. 4. $\frac{60x^3+6x^2+3ax+8a}{12x^3}$.

Ex. 8. *Ans.* a.

Ex. 9. *Ans.* a.

Ex. 10. *Ans.* Unity.

Ex. 11. $\frac{61y^4-25y^2-13}{28y^4-27y^2+5}$.

Ex. 13. *Ans.* $\frac{3x^4-2x^3+2x^2-1}{1-x^4}$.

Art. 118, Page 71.

Ex. 2. $\frac{39x}{35}$.

Ex. 3. $\frac{62a-33x}{21}$.

Ex. 6. $\frac{4bcx+2bx+cx-2ab}{2bc}$.

Ex. 7. b.

Ex. 10. $\frac{(a-b)^2}{4ab}$.

Ex. 11. $\frac{14x^2-43xy+88y^2}{77xy-55y^2}$.

Ex. 12. $\frac{a^2}{a^2-x^2}$.

Art. 119, Page 72.

Ex. 2. $\frac{x^2+ax}{a^2+ab}$.

Ex. 3. $\frac{ab+bx}{x}$.

Ex. 6. $\frac{4x^3}{21}$.

Ex. 7. $9ax$.

Ex. 9. $\frac{2a}{81m^2ny}$.

Ex. 10. $\frac{mnx}{a^{11}b^7c^{11}}$.

Ex. 12. $\frac{1}{11}$.

Ex. 13. $\frac{693bdn+396ab^2cm-770amn}{1092dmn}$.

Ex. 16. $\frac{ax}{a^2-x^2}$.

Ex. 19. The multiplicand reduces to $\frac{4b(a-b)}{a^2-b^2}$.

Art. 120, Page 74.

Ex. 2. $-a$.

Ex. 3. Unity.

Ex. 4. a^{n-m}.

Ex. 5. a^{-m-n}.

Ex. 6. $(a-b)^2$.

ART. 121, PAGE 75.

Ex. 2. $\frac{ad}{2bc}$.

Ex. 3. $\frac{2x}{a^2-ax+x^2}$.

Ex. 4. $\frac{x+1}{4x}$.

Ex. 11. $\frac{20n}{3}$.

Ex. 12.

$$\begin{array}{l} \frac{x}{y^2}-\frac{1}{y}+\frac{1}{x}\Big)\frac{x^2}{y^3}+\frac{1}{x}\Big(\frac{x}{y}+1=\text{quotient.} \\ \qquad\qquad\quad \frac{x^2}{y^3}-\frac{x}{y^2}+\frac{1}{y} \\ \hline \qquad\qquad\qquad \frac{x}{y^2}-\frac{1}{y}+\frac{1}{x} \end{array}$$

Ex. 13. Reducing the divisor and dividend to a common denominator, and rejecting the common denominator $(x+y)y$, we have $\frac{(x+2y)y+(x+y)x}{(x+2y)y+(x+2y)x-xy}$, which equals unity.

Ex. 14.

$$\begin{array}{l} x+\frac{1}{x}\Big)x^2+2+\frac{1}{x^2}\Big(x+\frac{1}{x}=\text{quotient.} \\ \qquad\qquad x^2+1 \\ \hline \qquad\qquad 1+\frac{1}{x^2} \end{array}$$

Ex. 15. Multiplying the dividend by $a+b-c$, we have

$$a^3+a^2b-a^2c-ab^2-b^3-b^2c-ac^2+c^3-2abc+bc^2,$$

and dividing this product by $a+b+c$, we obtain

$$a^2-b^2+c^2-2ac.$$

ART. 122, PAGE 76.

Ex. 2. $-a^3$.

Ex. 3. a^4.

Ex. 4. $-3a^{n+3}$.

Ex. 5. b^{-n}.

Ex. 6. $-3x^{-3}y^{-6}$.

Ex. 7. $(x-y)^2$.

Art. 124, Page 78.

Ex. 2. $\dfrac{-m}{a-m}\div\dfrac{m}{a+m}=\dfrac{-m}{a-m}\times\dfrac{a+m}{m}.$

Ex. 3. $\dfrac{24}{35}\div\dfrac{12}{5}=\dfrac{24}{35}\times\dfrac{5}{12}=\dfrac{2}{7}.$

Ex. 4. $\dfrac{22abc}{39mnx}\times\dfrac{3mx}{11ab}=\dfrac{2c}{13n}$

Ex. 5. $\dfrac{(a+b)(c-d)+(a-b)(c+d)}{(c+d)(c-d)}\times\dfrac{(c+d)(c-d)}{(a+b)(c+d)+(a-b)(c-d)}$

$$=\frac{2ac-2bd}{2ac+2bd}.$$

Ex. 6. $\dfrac{(a+x)^2+(a-x)^2}{a^2-x^2}\times\dfrac{a^2-x^2}{(a+x)^2-(a-x)^2}=\dfrac{2a^2+2x^2}{4ax}.$

Ex. 7. $\dfrac{m^2-mn+n^2}{n}\times\dfrac{mn}{m-n}\times\dfrac{m^2-n^2}{m^3+n^3}=\dfrac{m^3+n^3}{n}\times\dfrac{mn}{m-n}\times\dfrac{m-n}{m^3+n^3}=m.$

Ex. 8. $\dfrac{a}{b+\dfrac{c}{\dfrac{dn+m}{n}}}=\dfrac{a}{b+\dfrac{cn}{dn+m}}=\dfrac{a}{\dfrac{bdn+bm+cn}{dn+m}}=\dfrac{adn+am}{bdn+bm+cn}.$

CHAPTER VIII.

Art. 137, Page 83.

Ex. 4. $24x+12x+8x=480.$

Ex. 6. $24x-9x=16.$

Art. 140, Page 86.

Ex. 10. Multiply by 16.

$$336+3x-11=10x-10+776-56x;$$
$$49x=441\cdot$$
$$x=9.$$

Ex. 11. Multiply by 12.

$$36x-3x+12-48=20x+56-1\cdot$$
$$13x=91;$$
$$x=7.$$

Ex. 15. $$51-9x-20x-10=75-90x+35x+70;$$
$$26x=104;$$
$$x=4.$$

Ex. 16. $$x-\frac{3x-3}{5}+4=\frac{20-x}{2}-\frac{6x-8}{7}+\frac{4x-4}{5};$$
$$70x-42x+42+280=700-35x-60x+80+56x-56;$$
$$67x=402;$$
$$x=6.$$

Ex. 17. $$28x^2-13x-176-21x-168=28x^2-77x;$$
$$43x=344;$$
$$x=8.$$

Ex. 18. $$36x^2+60x+21+63x-117=36x^2+90x+36;$$
$$33x=132;$$
$$x=4.$$

Ex. 19. Multiply by 60.
$$50ab+48ac-40cx=45ac+120ab-360cx;$$
$$320cx=70ab-3ac;$$
$$x=\frac{70ab-3ac}{320c}.$$

Ex. 20. Multiply by 60.
$$30x-10a+15x-3a=20x-5a;$$
$$25x=8a;$$
$$x=\frac{8a}{25}.$$

Ex. 21. $$63x+81-36x+8x-4=252;$$
$$35x=175;$$
$$x=5.$$

Ex. 22. Multiply by 12.
$$6x-6+4x-8=3x+9+2x+8+12;$$
$$5x=43;$$
$$x=8\tfrac{3}{5}.$$

Ex. 23. Multiply by $36x$.
$$36x-24+144x-30=252x-32+360x-33;$$
$$432x=11;$$
$$x=\frac{11}{432}.$$

Ex. 24. $42x^2-743x+606-35x+175=42x^2-707x;$

$$71x=781;$$
$$x=11.$$

Ex. 25. Uniting three terms,

$$\frac{9x+4}{5x-48}=\frac{15x+96-11x-13-4x+19}{51}=2;$$
$$9x+4=10x-96.$$
$$x=100.$$

Ex. 26. $x^2+5x+6=(x+2)(x+3).$

Hence $4x+12+7x+14=37;$

$$11x=11;$$
$$x=1.$$

Ex. 27. $x^3-18x^2+104x-192-x^3+16x^2-76x+96$

$$=x^3-14x^2+56x-64-x^3+12x^2-44x+48;$$
$$16x=80;$$
$$x=5.$$

Ex. 28. $b^2x-a^2x+abx+a^2x=a^2b-a^3;$

$$b^2x+abx=a^2(b-a);$$
$$x=\frac{a^2(b-a)}{b(b+a)}.$$

Ex. 29. $x^2+x-\frac{15}{4}+\frac{3}{4}=x^2+2x-15;$

$$x-3=2x-15;$$
$$x=12.$$

Ex. 30. Uniting two terms,

$$\frac{2-\frac{x}{3}}{x+1}+\frac{16x+4\frac{1}{5}}{3x+2}=5.$$

Multiplying by the denominators,

$$5\tfrac{1}{3}x-x^2+4+16x^2+20\tfrac{1}{5}x+4\tfrac{1}{5}=15x^2+25x+10;$$
$$5\tfrac{1}{3}x+8\tfrac{1}{5}+20\tfrac{1}{5}x=25x+10;$$
$$80x+123+303x=375x+150;$$
$$8x=27;$$
$$x=3\tfrac{3}{8}.$$

ART. 144, PAGE 91.

Prob. 8. Let x denote the first part.

Then will mx " the second part,

and nx " the third part;

and we shall have $x+mx+nx=a$.

Hence $$x=\frac{a}{1+m+n}.$$

In order that the parts may be expressed in whole numbers, the number a must be exactly divisible by $1+m+n$.

Prob. 9. Let x denote the price of one book.

Then $$15x-10x=25.$$

Hence $$x=5.$$

Prob. 10. Let x denote the required number.

Then $$mx-nx=a.$$

Hence $$x=\frac{a}{m-n}.$$

Prob. 11. Let x denote C's share.

Then will $3x$ " B's share,

and $6x$ " A's share;

and we shall have $10x=1000$.

Hence $$x=100.$$

Prob. 12. Let x denote the first part.

Then will mx " the second part,

and mnx " the third part;

and we have $x+mx+mnx=a$.

Hence $$x=\frac{a}{1+m+mn}.$$

Prob. 13. Let x denote the number of gallons of brandy.

Then will $x+10$ " " " wine,

and $2x+10$ " " " water;

and we have $4x+20=120$.

Hence $x=25$ gallons of brandy.

Prob. 14. Let x denote the first part.

Then will $x+m$ " the second part,

and $2x+m$ " the third part;

and we have $4x+2m=a$.

Hence $$x=\frac{a-2m}{4}.$$

Also, $$x+m=\frac{a-2m+4m}{4}=\frac{a+2m}{4},$$

and $$\frac{a-2m}{4}+\frac{a+2m}{4}=\frac{a}{2}=\text{the third part.}$$

Prob. 15. Let x denote the wages of the fourth.

Then will $x+4$ " " the third,

$x+7$ " " the second,

and $x+9$ " " the first;

and we have $4x+20=32$.

Hence $x=3$.

Prob. 16. Let x denote the first part.

Then will $x+m$ " the second part,

$x+m+n$ " the third part,

and $x+m+n+p$ " the fourth part;

and we have $4x+3m+2n+p=a$.

Hence $$x=\frac{a-3m-2n-p}{4}.$$

Also, $$x+m=\frac{a-3m-2n-p+4m}{4};$$

$$x+m+n=\frac{a+m-2n-p+4n}{4};$$

and $$x+m+n+p=\frac{a+m+2n-p+4p}{4}.$$

Prob. 18. Let x denote one part.

Then will $a-x$ " the other part;

and we have $x+b=m(a-x)=ma-mx$.

Hence $$x=\frac{ma-b}{m+1},$$

and $$a-x=\frac{ma+a-ma+b}{m+1}=\frac{a+b}{m+1}.$$

Prob. 20. Let x denote the required number of hours.
Then will nx " the distance traveled by one,
and mx " " " by the other;
and we have $mx-nx=a$.

Hence $x=\frac{a}{m-n}$.

Prob. 21. Let x denote the less part.
Then will $197-x$ " the greater part;
and we have $4(197-x)=5x+50$.
Hence $9x=738$;
$x=82$.

Prob. 22. Let x denote the less part.
Then will $a-x$ " the greater part;
and we have $m(a-x)=nx+b$.

Hence $x=\frac{ma-b}{m+n}$,

and $a-x=\frac{ma+na-ma+b}{m+n}=\frac{na+b}{m+n}$.

Prob. 24. Let mx denote the first part.
Then will nx " the second part;
and we have $mx+nx=a$.

Hence $x=\frac{a}{m+n}$.

Prob. 26. Let mnx denote the required number.
Then $nx-mx=a$,

and $x=\frac{a}{n-m}$.

Prob. 27. Let x denote the miles traveled in coach.

Then will $\frac{x}{9}$ " the hours spent in riding,

and $\frac{x}{3}$ " the hours spent in walking back;

and we have $\frac{x}{9}+\frac{x}{3}=8$.

Hence $x+3x=72$;
$x=18$.

Prob. 28. Let x denote the number of miles traveled in coach.

Then will $\frac{x}{m}$ " the hours spent in riding,

and $\frac{x}{n}$ " the hours spent in walking back;

and we have $$\frac{x}{m}+\frac{x}{n}=a.$$

Hence $$nx+mx=amn,$$

and $$x=\frac{amn}{m+n}.$$

Prob. 29. Let x denote the number who received 9 cts. each.

Then will $12-x$ " " " 7 cts.;

and we have $9x+7(12-x)=100$.

Hence $2x=16$,

$x=8$ who received 9 cents,

and $12-x=4$ " 7 cents.

Verification. $8\times 9+4\times 7=100$.

Prob. 30. Let x denote the first part.

Then will $a-x$ " the second part;

and we have $mx+n(a-x)=b$,

or $mx+na-nx=b$.

Hence $$x=\frac{b-na}{m-n},$$

and $$a-x=\frac{ma-na-b+na}{m-n}=\frac{ma-b}{m-n}.$$

Prob. 31. Let x denote the days before the first conjunction.

Then will x " the degrees the sun advances,

and $13x$ " " the moon advances;

and we have $13x-x=60$.

Hence $x=5$ days, to first conjunction.

If x denote the days before the second conjunction, we shall have $13x-x=360+60=420$.

Hence $x=35$ days to second conjunction.

In the same manner we find 65 days to the third conjunction, and so on.

Prob. 32. Let x denote the days before the *first* meeting.

Then will nx " the miles traveled by one,

and mx " the miles traveled by the other;

and we have $mx-nx=b.$

Hence $$x=\frac{b}{m-n}\text{ days.}$$

If x denote the days before the *second* meeting, we shall have

$$mx-nx=a+b.$$

Hence $$x=\frac{a+b}{m-n}\text{ days.}$$

If x denote the days before the *third* meeting, we shall find

$$x=\frac{2a+b}{m-n}\text{ days.}$$

Prob. 33. Let x denote one of the parts.

Then will $12-x$ " the other part;

and we have $144-24x+x^2-x^2=48.$

Hence $$24x=96,$$

$$x=4,$$

and $$12-x=8.$$

Verification. $64-16=48.$

Prob. 34. Let x denote one of the parts.

Then will $a-x$ " the other part;

and we have $a^2-2ax+x^2-x^2=b.$

Hence $$2ax=a^2-b,$$

$$x=\frac{a^2-b}{2a},$$

and $$a-x=\frac{2a^2-a^2+b}{2a}=\frac{a^2+b}{2a}.$$

Prob. 35. The given ratios are $2:3,$

and $4:5,$

which may be represented by the numbers $8:12:15.$

Let $8x$ denote A's share.

Then will $12x$ " B's share,

and $15x$ " C's share;

and we have $35x=21{,}000.$

Whence $x=600,$

and A's share is 4800, B's share is 7200, and C's share is 9000.

Verification. $4800+7200+9000=21,000.$

Also, $4800:7200::2:3,$

and $7200:9000::4:5.$

Prob. 36. Let mx denote the first part.

Then will nx " the second part,

and px " the third part;

and we have $mx+nx+px=a.$

Hence $$x=\frac{a}{m+n+p}.$$

Prob. 37. Let x denote the pounds of tea at 72 cents.

Then will $80-x$ " " " at 40 cents;

and the value of the tea will be denoted by

$$72x+40(80-x).$$

This is required to be equal to 80×60, or 4800.

Hence $72x+3200-40x=4800,$

and $x=50$ pounds.

Prob. 38. Let x denote the pounds at a cents.

Then will $n-x$ " " at b cents;

and we have $ax+b(n-x)=nc.$

Hence $ax-bx=nc-nb,$

and $$x=\frac{n(c-b)}{a-b}.$$

Also, $$n-x=\frac{na-nb-nc+nb}{a-b}=\frac{n(a-c)}{a-b}.$$

Prob. 39. Let x denote the required number of days.

Then will $\frac{x}{6}$ " the portion of the work done by A,

$\frac{x}{8}$ " " " done by B,

and $\frac{x}{24}$ " " " done by C;

and we have $\frac{x}{6}+\frac{x}{8}+\frac{x}{24}=$ the whole; i.e., $=1$.

Hence $4x+3x+x=24;$

$$x=3 \text{ days}.$$

Verification. A does $\frac{3}{6}$ of the whole work,

B does $\frac{3}{8}$ of the work,

C does $\frac{3}{24}$ of the work;

and $$\frac{3}{6}+\frac{3}{8}+\frac{3}{24}=1.$$

Prob. 40. Let x denote the required number of days.

Then $$\frac{x}{a}+\frac{x}{b}+\frac{x}{c}=1.$$

Hence $$bcx+acx+abx=abc,$$

and $$x=\frac{abc}{ab+ac+bc}.$$

Prob. 42. Let x denote the required time.

Then $$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{2}{x}.$$

Hence $$bcx+acx+abx=2abc,$$

and $$x=\frac{2abc}{ab+ac+bc}.$$

Prob. 43. Let x denote the number of pieces of the first kind.
Then will $8-x$ " " " of the second kind.

Also, $\frac{x}{20}$ " the value of the pieces of the first kind,

and $\frac{8-x}{4}$ " " " of the second kind.

Hence $$\frac{x}{20}+\frac{8-x}{4}=1.$$

Whence $$x+40-5x=20,$$

and $$x=5.$$

Verification. 5 pieces at 5 cents each amount to 25 cents,
3 " 25 " " 75 cents;
25 cents + 75 cents = one dollar.

Prob. 44. Let x denote the number of pieces of the first kind.
Then will $c-x$ " " " of the second kind.

Hence
$$\frac{x}{a}+\frac{c-x}{b}=1,$$
$$bx+ac-ax=ab,$$
$$x=\frac{a(c-b)}{a-b}.$$

Also, $$c-x=\frac{ac-bc-ac+ab}{a-b}=\frac{ab-bc}{a-b}=\frac{b(a-c)}{a-b}.$$

Prob. 46. Let $x-m$ denote the first part.
Then will $x+m$ " the second part,
$\frac{x}{m}$ " the third part,
mx " the fourth part.

Hence
$$2x+\frac{x}{m}+mx=a,$$
$$2mx+x+m^2x=ma,$$
$$x=\frac{ma}{m^2+2m+1}=\frac{ma}{(m+1)^2}.$$

Prob. 47. Let x denote the original stock.
Then will $x-500$ " sum not expended the first year,
$\frac{x-500}{3}$ " his gain the first year,
$\frac{4}{3}(x-500)$ " sum he had at end of first year;
$\frac{4x-2000}{3}-500$ " sum he traded with the second year,
$\frac{4}{3}\cdot\frac{4x-3500}{3}$ " sum he had at end of second year;
$\frac{16x-14,000}{9}-500$ " sum he traded with the third year,
$\frac{4}{3}\cdot\frac{16x-18,500}{9}$ " sum he had at end of third year.

Hence
$$\frac{4}{3}\cdot\frac{16x-18,500}{9}=2x,$$
$$32x-37,000=27x,$$
$$x=7400.$$

Prob. 48. Let x denote the original stock.

Then will $\frac{4}{3}(x-a)$ = sum he had at end of first year,

$\frac{4}{3}\cdot\frac{4x-7a}{3}$ = sum he had at end of second year,

$\frac{4}{3}\cdot\frac{16x-37a}{9}$ = sum he had at end of third year.

Hence
$$\frac{4}{3}\cdot\frac{16x-37a}{9}=2x,$$
$$64x-148a=54x,$$
$$x=\frac{148a}{10}=\frac{74a}{5}.$$

Prob. 49. Let x denote the required number of years.

Then will $54+x$ " the age of the father at that time,

$9+x$ " " the son at that time.

Hence
$$54+x=4(9+x),$$
$$x=6.$$

Prob. 50. Let x denote the required number of years.

Then will
$$a+x=n(b+x),$$
$$nx-x=a-nb,$$
$$x=\frac{a-nb}{n-1}.$$

CHAPTER IX.

Art. 151, Page 102.

Ex. 1. Given
$$\begin{cases} 11x+3y=100, & (1) \\ 4x-7y=4. & (2) \end{cases}$$

Multiply (1) by 7, $77x+21y=700.$ (3)

Multiply (2) by 3, $12x-21y=12.$ (4)

Add (4) to (3), $89x=712.$

Hence $x=8.$

Substitute this value in equation (1),
$$88+3y=100.$$

Whence $y=4.$

Verification. $88+12=100,$
$32-28=4.$

Ex. 2. Clearing of fractions, we have

$3x+2y=42,$ (1)
$2x+3y=48.$ (2)
Multiply (1) by 2, $6x+4y=84.$ (3)
Multiply (2) by 3, $6x+9y=144.$ (4)
Subtract (3) from (4), $5y=60.$
Whence $y=12.$
Substitute in (2) $2x+36=48.$
Hence $x=6.$

Ex. 3. Clearing of fractions, and transposing, we have

$x+24y=91,$ (1)
$40x+y=763.$ (2)
Multiply (1) by 40, $40x+960y=3640.$ (3)
Subtract (2) from (3), $959y=2877.$
Whence $y=3.$
Substitute in (1) $x+72=91.$
Hence $x=19.$

Verification. $\frac{21}{3}+24=31,$

$\frac{8}{4}+190=192.$

Ex. 4. Clearing of fractions, and transposing, we have

$48y-17x=155,$ (1)
$2y+30x=160.$ (2)
Multiply (2) by 24, $48y+720x=3840.$ (3)
Subtract (1) from (3), $737x=3685.$
Whence $x=5.$
Substitute in (2) $2y+150=160.$
$y=5.$

Verification. $10-\frac{8}{4}=7+\frac{5}{5},$

$20-\frac{3}{3}=24\frac{1}{2}-\frac{11}{2}.$

Ex. 5.

$$\text{Given}\begin{cases}\frac{a}{x}+\frac{b}{y}=m, & (1)\\ \frac{c}{x}+\frac{d}{y}=n. & (2)\end{cases}$$

Multiply (1) by c, $\frac{ac}{x}+\frac{bc}{y}=mc.$ (3)

Multiply (2) by a, $\frac{ac}{x}+\frac{ad}{y}=na.$ (4)

Subtract (4) from (3), $\frac{bc}{y}-\frac{ad}{y}=mc-na.$ (5)

Whence $y=\frac{bc-ad}{mc-na}.$

Multiply (1) by d, $\frac{ad}{x}+\frac{bd}{y}=md.$ (6)

Multiply (2) by b, $\frac{bc}{x}+\frac{bd}{y}=nb.$ (7)

Subtract (6) from (7), $\frac{bc}{x}-\frac{ad}{x}=nb-md.$

Whence $x=\frac{bc-ad}{nb-md}.$

Ex. 6.

$$\text{Given}\begin{cases}5x-7y=20, & (1)\\ 9x-11y=44. & (2)\end{cases}$$

Multiply (1) by 9, $45x-63y=180.$ (3)
Multiply (2) by 5, $45x-55y=220.$ (4)
Subtract (3) from (4), $8y=40.$
Whence $y=5.$
Substitute in (1) $5x-35=20.$
$x=11.$

Ex. 7.

$$\text{Given}\begin{cases}17x-13y=144, & (1)\\ 23x+19y=890. & (2)\end{cases}$$

Multiply (1) by 19, $323x-247y=2{,}736.$ (3)
Multiply (2) by 13, $299x+247y=11{,}570.$ (4)
Add (4) to (3), $622x=14{,}306.$
$x=23.$

Substitute in (1) $391-13y=144.$

$$13y=247,$$

$$y=19.$$

Ex. 8.

$$\text{Given}\begin{cases}\dfrac{1}{x}=m-\dfrac{1}{y}, & (1)\\[2mm] \dfrac{1}{y}=\dfrac{1}{x}-n. & (2)\end{cases}$$

Substitute (2) in (1), $\dfrac{1}{x}=m-\dfrac{1}{x}+n.$

Hence $\dfrac{2}{x}=m+n,$

or $x=\dfrac{2}{m+n}.$

Substitute in (2), $\dfrac{1}{y}=\dfrac{m+n}{2}-n=\dfrac{m-n}{2}.$

Whence $y=\dfrac{2}{m-n}.$

Ex. 9. Clearing the equations of fractions, and transposing, we have

$$4x-60y=-183, \quad (1)$$

$$12x-60y=-165. \quad (2)$$

Subtract (1) from (2), $8x = 18.$

Whence $x=2\frac{1}{4}.$

Substitute in (1) $9-60y=-183.$

$$60y=192,$$

$$y=3\frac{1}{5}.$$

Ex. 10. Clearing of fractions, we have

$$x+a+n(y-b)=2na, \quad (1)$$

$$a(x+a)+(y-b)=a+na^2. \quad (2)$$

Multiply (1) by a, $a(x+a)+an(y-b)=2na^2.$ (3)

Subtract (2) from (3), $(an-1)(y-b)=na^2-a.$ (4)

Divide (4) by $an-1$, $y-b=a.$

Whence $y=a+b.$

Substitute in (1) $x+a+na=2na.$

Whence $x=na-a.$

Ex. 11. Given $\begin{cases} 1209\frac{1}{3} = 60x + 77y, & (1) \\ 152\frac{1}{3} = -24x + 35y. & (2) \end{cases}$

Multiply (1) by 2, $2418\frac{2}{3} = 120x + 154y.$ (3)

Multiply (2) by 5, $761\frac{2}{3} = -120x + 175y.$ (4)

Add (4) to (3), $3180\frac{1}{3} = 329y.$

Hence $y = 9\frac{2}{3}.$

Substitute in (2) $152\frac{1}{3} = -24x + 338\frac{1}{3}.$

Hence $x = 7\frac{3}{4}.$

Ex. 12. Clearing of fractions, and transposing, we have

$3x - 77y = -151,$ (1)

$19x - 34y = -49.$ (2)

Multiply (1) by 19, $57x - 1463y = -2869.$ (3)

Multiply (2) by 3, $57x - 102y = -147.$ (4)

Subtract (4) from (3), $-1361y = -2722.$

Whence $y = 2.$

Substitute in (1) $3x - 154 = -151.$

Whence $x = 1.$

Ex. 13. Clearing of fractions, and transposing, we have

$y - x = 2,$ (1)

$y + x = 12.$ (2)

Add (1) to (2), $2y = 14,$

$y = 7.$

Substitute in (2) $x = 5.$

Ex. 14. Clearing of fractions, and transposing, we have

$55x - 59y = -87,$ (1)

$105x - 101y = -73.$ (2)

Multiply (1) by 21, $1155x - 1239y = -1827.$ (3)

Multiply (2) by 11, $1155x - 1111y = -803.$ (4)

Subtract (4) from (3), $128y = 1024.$

Whence $y = 8.$

Substitute in (1) $55x - 472 = -87.$

Whence $55x = 385,$

$x = 7.$

Ex. 15. Given $\begin{cases} x^2-y^2=a, & (1) \\ x-y=b. & (2) \end{cases}$

Divide (1) by (2), $x+y=\frac{a}{b}$. (3)

Add (2) to (3), $2x=\frac{a}{b}+b=\frac{a+b^2}{b}$.

Whence $x=\frac{a+b^2}{2b}$.

Substitute in (2), $y=x-b=\frac{a+b^2-2b^2}{2b}=\frac{a-b^2}{2b}$.

Ex. 16. Clearing of fractions, and transposing, we have

$20x+15y=145,$ (1)

$9x+\ \ y=\ 25.$ (2)

Multiply (2) by 15, $135x+15y=375.$ (3)

Subtract (1) from (3), $115x\ \ =230.$

Whence $x=2.$

Substitute in (2) $18+y=25.$

Whence $y=7.$

Ex. 17. Clearing of fractions, and transposing, we have

$15x-14y=\ 17,$ (1)

$24x+\ 7y=\ 86.$ (2)

Multiply (2) by 2, $48x+14y=172.$ (3)

Add (1) to (3), $63x\ \ =189.$

Whence $x=3.$

Substitute in (2) $72+7y=86.$

Whence $7y=14,$

$y=2.$

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Ex. 1. Given $\begin{cases} 2x+\ 4y-\ 3z=22, & (1) \\ 4x-\ 2y+\ 5z=18, & (2) \\ 6x+\ 7y-\ \ z=63. & (3) \end{cases}$

Multiply (1) by 2, $4x+\ 8y-\ 6z=44.$ (4)

Subtract (2) from (4), $10y-11z=26.$ (5)

Multiply (1) by 3, $6x+12y-\ 9z=66.$ (6)

Subtract (3) from (6), $5y-\ 8z=\ 3.$ (7)

Multiply (7) by 2, $10y-16z=\ 6.$ (8)

Subtract (8) from (5), $5z=20.$
Hence $z=4.$
Substitute in (7) $5y-32=3.$
Hence $y=7.$
Substitute in (1) $2x+28-12=22.$
Hence $x=3.$

Ex. 2.

$$\text{Given}\begin{cases} x+y=a, & (1)\\ x+z=b, & (2)\\ y+z=c. & (3)\end{cases}$$

Add (1), (2), and (3), $x+y+z=\dfrac{a+b+c}{2}.$ (4)

Subtract (3) from (4), $x=\dfrac{a+b-c}{2}.$

Subtract (2) from (4), $y=\dfrac{a-b+c}{2}.$

Subtract (1) from (4), $z=\dfrac{b+c-a}{2}.$

Ex. 3. Clearing the third equation of fractions, we have

$x+y+z=29,$ (1)
$x+2y+3z=62,$ (2)
$6x+4y+3z=120.$ (3)
Multiply (1) by 6, $6x+6y+6z=174.$ (4)
Subtract (3) from (4), $2y+3z=54.$ (5)
Substitute (5) in (2), $x=8.$
Subtract (2) from (3), $5x+2y=58.$ (6)
Substitute in (6) $40+2y=58.$
Hence $y=9.$
Substitute in (1) $8+9+z=29.$
Hence $z=12.$

Ex. 4. Clearing of fractions, we have

$6x+3y+2z=192,$ (1)
$20x+15y+12z=900,$ (2)
$15x+12y+10z=720.$ (3)
Multiply (1) by 6, $36x+18y+12z=1152.$ (4)
Subtract (2) from (4), $16x+3y=252.$ (5)

Multiply (1) by 5, $30x+15y+10z=960.$ (6)
Subtract (3) from (6), $15x+3y=240.$ (7)
Subtract (7) from (5), $x=12.$
Substitute in (7) $180+3y=240.$
Hence $y=20.$
Substitute in (1) $72+60+2z=192.$
Hence $z=30.$

Ex. 5.

Given $\begin{cases} x+y-z=1320, & (1) \\ x-y+z=654, & (2) \\ -x+y+z=-12. & (3) \end{cases}$

Add (1) to (2), $2x=1974.$
Hence $x=987.$
Add (1) to (3), $2y=1308.$
Hence $y=654.$
Add (2) to (3), $2z=642.$
Hence $z=321.$

Ex. 6. Clearing of fractions, we have

$$x-y+z=6, \quad (1)$$
$$14x-19y+22z=128, \quad (2)$$
$$21x-19y+22z=142. \quad (3)$$

Subtract (2) from (3), $7x=14.$
Hence $x=2.$
Multiply (1) by 19, $19x-19y+19z=114.$ (4)
Subtract (4) from (3), $2x+3z=28.$ (5)
Substitute in (5) $4+3z=28.$
Hence $z=8.$
Substitute in (1) $2-y+8=6.$
Hence $y=4.$

Ex. 7. Clearing of fractions, we have

$$63x+45y+35z=81{,}270, \quad (1)$$
$$45x+35y+63z=95{,}760, \quad (2)$$
$$35x+63y+45z=93{,}240. \quad (3)$$

Add (1), (2), and (3), $143x+143y+143z=270{,}270.$ (4)
Hence $x+y+z=1{,}890.$ (5)
Multiply (5) by 35, $35x+35y+35z=66{,}150.$ (6)
Subtract (6) from (3), $28y+10z=27{,}090.$ (7)

Multiply (5) by 45, $45x+45y+45z=85,050.$ (8)
Subtract (8) from (2), $-10y+18z=10,710.$ (9)
Multiply (7) by 9, $252y+90z=243,810.$ (10)
Multiply (9) by 5, $-50y+90z=53,550.$ (11)
Subtract (11) from (10), $302y=190,260.$
Hence $y=630.$
Substitute in (7) $17,640+10z=27,090.$
Hence $z=945.$
Substitute in (5) $x+630+945=1890.$
Hence $x=315.$

Ex. 8.

$$\text{Given}\begin{cases}\dfrac{1}{x}+\dfrac{1}{y}=a, & (1)\\[2mm] \dfrac{1}{x}+\dfrac{1}{z}=b, & (2)\\[2mm] \dfrac{1}{y}+\dfrac{1}{z}=c. & (3)\end{cases}$$

Add (1), (2), and (3), $\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=\dfrac{a+b+c}{2}.$ (4)

Subtract (3) from (4), $\dfrac{1}{x}=\dfrac{a+b-c}{2}.$

Hence $x=\dfrac{2}{a+b-c}.$

Subtract (2) from (4), $\dfrac{1}{y}=\dfrac{a+c-b}{2}.$

Hence $y=\dfrac{2}{a+c-b}.$

Subtract (1) from (4), $\dfrac{1}{z}=\dfrac{b+c-a}{2}.$

Hence $z=\dfrac{2}{b+c-a}.$

Ex. 9.

$$\text{Given}\begin{cases}\dfrac{1}{x}+\dfrac{1}{y}-\dfrac{1}{z}=a, & (1)\\[2mm] \dfrac{1}{x}-\dfrac{1}{y}+\dfrac{1}{z}=b, & (2)\\[2mm] -\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=c. & (3)\end{cases}$$

Add (1), (2), and (3), $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=a+b+c.$ (4)

Subtract (3) from (4), $\frac{2}{x}=a+b.$

Hence $x=\frac{2}{a+b}.$

Subtract (2) from (4), $\frac{2}{y}=a+c.$

Hence $y=\frac{2}{a+c}.$

Subtract (1) from (4), $\frac{2}{z}=b+c.$

Hence $z=\frac{2}{b+c}.$

Ex. 10. Put $2x+3y=a$, $3x+4z=b$, $5y+9z=c$. Then we have

$$\frac{12}{a}-\frac{7\frac{1}{2}}{b}=1, \quad (1)$$

$$\frac{30}{b}+\frac{37}{c}=3, \quad (2)$$

$$\frac{222}{c}-\frac{8}{a}=5. \quad (3)$$

Multiply (1) by 4, $\frac{48}{a}-\frac{30}{b}=4.$ (4)

Add (4) to (2), $\frac{48}{a}+\frac{37}{c}=7.$ (5)

Multiply (3) by 6, $-\frac{48}{a}+\frac{1332}{c}=30.$ (6)

Add (5) to (6), $\frac{1369}{c}=37.$

Hence $c=37.$

Substitute in (3) $6-\frac{8}{a}=5.$

Hence $a=8.$

Substitute in (2) $\frac{30}{b}+1=3.$

Hence $b=15.$

We have then $2x+3y=8,$ (7)

$3x+4z=15,$ (8)

$5y+9z=37.$ (9)

Multiply (7) by 3, $6x+9y=24.$ (10)

Multiply (8) by 2, $6x+8z=30.$ (11)

Subtract (10) from (11), $-9y+8z=6.$ (12)

Multiply (12) by 5, $-45y+40z=30.$ (13)

Multiply (9) by 9, $45y+81z=333.$ (14)

Add (13) and (14), $121z=363.$

Hence $z=3.$

Substitute in (8). $3x+12=15.$

Hence $x=1.$

Substitute in (7) $2+3y=8.$

Hence $y=2.$

Ex. 11.

Given $\begin{cases} 2x+5y-7z=-288, & (1) \\ 5x-y+3z=227, & (2) \\ 7x+6y+z=297. & (3) \end{cases}$

Multiply (3) by 3, $21x+18y+3z=891.$ (4)

Subtract (2) from (4), $16x+19y=664.$ (5)

Multiply (3) by 7, $49x+42y+7z=2{,}079.$ (6)

Add (6) to (1), $51x+47y=1{,}791.$ (7)

Multiply (7) by 19, $969x+893y=34{,}029.$ (8)

Multiply (5) by 47, $752x+893y=31{,}208.$ (9)

Subtract (9) from (8), $217x=2821.$

Hence $x=13.$

Substitute in (5) $208+19y=664.$

Hence $y=24.$

Substitute in (3) $91+144+z=297.$

Hence $z=62.$

Ex. 12, Clearing of fractions, we have

$35x+21y+30z=6{,}090,$ (1)

$15x+2y+4z=912,$ (2)

$20x+8v+15z=3{,}160,$ (3)

$v+y+z=248.$ (4)

Multiply (4) by 8, $8v+8y+8z=1{,}984.$ (5)

Subtract (5) from (3), $20x-8y+7z=1{,}176.$ (6)

Multiply (2) by 4,	$60x+8y+16z=3,648.$	(7)
Add (6) to (7),	$80x+23z=4,824.$	(8)
Multiply (2) by 21,	$315x+42y+84z=19,152.$	(9)
Multiply (1) by 2,	$70x+42y+60z=12,180.$	(10)
Subtract (10) from (9),	$245x+24z=6,972.$	(11)
Subtract (8) from (11),	$165x+z=2,148.$	(12)
Multiply (12) by 24,	$3960x+24z=51,552.$	(13)
Subtract (11) from (13),	$3715x=44,580.$	
Hence	$x=12.$	
Substitute in (12)	$1980+z=2148.$	
Hence	$z=168.$	
Substitute in (2)	$180+2y+672=912.$	
Hence	$y=30.$	
Substitute in (4)	$v+30+168=248.$	
Hence	$v=50.$	

Ex. 13.

$$\text{Given}\begin{cases} 7x-2z+3u=17, & (1)\\ 4y-2z+v=11, & (2)\\ 5y-3x-2u=8, & (3)\\ 4y-3u+2v=9, & (4)\\ 3z+8u=33. & (5)\end{cases}$$

Multiply (2) by 2,	$8y-4z+2v=22.$	(6)
Subtract (4) from (6),	$4y-4z+3u=13.$	(7)
Multiply (1) by 3,	$21x-6z+9u=51.$	(8)
Multiply (3) by 7,	$-21x+35y-14u=56.$	(9)
Add (8) to (9),	$35y-6z-5u=107.$	(10)
Multiply (7) by 35,	$140y-140z+105u=455.$	(11)
Multiply (10) by 4,	$140y-24z-20u=428.$	(12)
Subtract (12) from (11),	$-116z+125u=27.$	(13)
Multiply (13) by 3,	$-348z+375u=81.$	(14)
Multiply (5) by 116,	$348z+928u=3828.$	(15)
Add (14) to (15),	$1303u=3909.$	
Hence	$u=3.$	
Substitute in (5)	$3z+24=33.$	
Hence	$z=3.$	
Substitute in (1)	$7x-6+9=17.$	
Hence	$x=2.$	

Substitute in (3) $5y-6-6=8.$
Hence $y=4.$
Substitute in (4) $16-9+2v=9.$
Hence $v=1.$

Ex. 14. The sum of the six equations divided by 5 gives

$$x+y+z+t+u+v=33. \qquad (7)$$

Subtract (1) from (7), $v=8.$
Subtract (2) from (7), $t=7.$
Subtract (3) from (7), $u=6$, etc.

Prob. 1. Let x denote the first number, and y the second.

Then we have $x+4y=29,$ (1)
and $6x+y=36.$ (2)
Multiply (2) by 4, $24x+4y=144.$ (3)
Subtract (1) from (3), $23x=115.$
Hence $x=5,$
$y=6.$

Prob. 2. Let x denote A's money, and y denote B's money.

Then $x+36=3y;$
also, $y-5=\frac{x}{2}.$
Hence $x=42$, and $y=26.$

Prob. 3. Let x denote the price of a pound of tea, and y that of sugar. Then $x+3y=6,$ (1)

$$\frac{11x}{10}+\frac{3}{2}\cdot 3y=7. \qquad (2)$$

Multiply (2) by 10, $11x+45y=70.$ (3)
Multiply (1) by 11, $11x+33y=66.$ (4)
Subtract (4) from (3), $12y=4.$
Hence $y=\frac{1}{3}$ shilling, or 4 pence;
also, $x=5.$

Prob. 4. Let x denote the numerator, and y the denominator of the fraction. Then

$$\frac{x+4}{y}=\frac{1}{2}, \qquad (1)$$

$$\frac{x}{y+7}=\frac{1}{5}. \qquad (2)$$

Clearing of fractions, $2x+8=y,$ (3)

$5x \quad =y+7.$ (4)

Subtract (3) from (4), $3x-8=7.$

Hence $x=5,$

and $y=18.$

Prob. 5. Let x denote the sum of money, and y denote the rate per cent.

Then $\frac{xy}{100}$ will denote the interest for one year,

$\frac{2xy}{300}$ " " for eight months.

Hence $$x+\frac{2xy}{300}=1488, \quad (1)$$

$$x+\frac{5xy}{400}=1530. \quad (2)$$

Subtract (1) from (2), $$\frac{7xy}{1200}= \quad 42.$$

Hence $xy=7200.$

Substitute in (1) $x+48=1488.$

Hence $x=1440,$

$$y=\frac{7200}{1440}=5 \text{ per cent.}$$

Prob. 6. $\frac{xy}{1200}$ will denote the interest for one month.

Hence $$x+\frac{mxy}{1200}=a, \quad (1)$$

$$x+\frac{nxy}{1200}=b, \quad (2)$$

Subtract (1) from (2), $(n-m)\frac{xy}{1200}=b-a.$

Hence $$xy=\frac{1200(b-a,}{n-m}.$$

Substitute in (1) $$x+\frac{m(b-a)}{n-m}=a.$$

Clearing of fractions,

$$(n-m)x+mb-ma=na-ma.$$

Hence $$x=\frac{na-mb}{n-m},$$

$$y=\frac{1200(b-a)}{n-m}\times\frac{n-m}{na-mb}=\frac{1200(b-a)}{na-mb}.$$

Prob. 7. Let x denote the left-hand digit, and y the right-hand digit.

Then $10x+y$ will denote the required number

Hence $$\frac{10x+y}{x+y}=4, \qquad (1)$$

$$\frac{10y+x}{y-x+2}=14. \qquad (2)$$

Clearing of fractions, and reducing, we have

$$2x=y, \qquad (3)$$

$$15x-4y=28. \qquad (4)$$

Substituting (3) in (4), $15x-8x=28.$

Hence $x=4,$

and $y=8.$

Prob. 8. Let x denote the number of apples, and y the number of pears. Then $\frac{x}{4}$ will denote the cost of the apples, and $\frac{y}{5}$ the cost of the pears.

Hence $$\frac{x}{4}+\frac{y}{5}=30, \qquad (1)$$

$$\frac{x}{8}+\frac{y}{15}=13. \qquad (2)$$

Clearing of fractions, $5x+4y=600, \qquad (3)$

$$15x+8y=1560. \qquad (4)$$

Multiply (3) by 2, $10x+8y=1200. \qquad (5)$

Subtract (5) from (4), $5x=360.$

Hence $x=72,$

and $y=60.$

Prob. 9. Let x denote the amount of the fortune.

The first receives $300+\frac{x-300}{6}$, or $\frac{1500+x}{6}$.

After the second has received 600, there will remain

$$x-\frac{1500+x}{6}-600, \text{ or } \frac{5x-5100}{6}.$$

The second receives $600+\frac{5x-5100}{36}$, or $\frac{16,500+5x}{36}$.

Hence $$\frac{1500+x}{6}=\frac{16,500+5x}{36},$$

or $$9000+6x=16,500+5x.$$

Hence $$x=7500 \text{ dollars.}$$

The first receives $\frac{1500+x}{6}=1500$ dollars.

$$\frac{7500}{1500}=5, \text{ the number of children.}$$

Prob. 10. The first receives $a+\frac{x-a}{n}$, or $\frac{an+x-a}{n}$.

After the second has received $2a$, there will remain

$$x-\frac{an+x-a}{n}-2a, \text{ or } \frac{nx-3an-x+a}{n}.$$

The second receives

$$2a+\frac{(n-1)x-(3n-1)a}{n^2}, \text{ which equals } \frac{an+x-a}{n}.$$

Hence $$x=an^2-2an+a=a(n-1)^2.$$

The first receives

$$\frac{an+x-a}{n}, \text{ which equals } \frac{an^2-an}{n}=a(n-1).$$

Hence $$\frac{a(n-1)^2}{a(n-1)}=n-1, \text{ the number of persons.}$$

Prob. 11. Let x denote the cost of a quart of the poorer wine, and y that of the better.

Then $$9x+7y=16\times55=880, \quad (1)$$
$$3x+5y=8\times58=464. \quad (2)$$

Multiply (2) by 3, $$9x+15y=1392.$$

Subtract (1) from (3), $$8y=512.$$

Hence $$y=64,$$
$$x=48.$$

Prob. 12. Let x and y denote the two debts.

Then $$\frac{4x}{11}+\frac{y}{6}+30=530, \qquad (1)$$

$$\frac{3x}{11}+\frac{5y}{18}-10=420. \qquad (2)$$

Clearing of fractions, $24x+11y=33{,}000$, (3)

$$54x+55y=85{,}140. \qquad (4)$$

Multiply (3) by 5, $120x+55y=165{,}000$. (5)

Subtract (4) from (5), $66x=79{,}860$.

Hence $x=1210$.

Substitute in (1) $440+\frac{y}{6}=500$.

Hence $y=360$.

Prob. 13. Let x denote the time required if all worked together.

A and B together would perform $\frac{1}{12}$ of the work in one day.
A and C " " $\frac{1}{15}$ " "
B and C " " $\frac{1}{20}$ " "

Hence $$\tfrac{1}{12}+\tfrac{1}{15}+\tfrac{1}{20}=\frac{2}{x},$$

$$5x+4x+3x=120.$$

Hence $x=10$.

A can do in one day $\frac{1}{10}-\frac{1}{20}=\frac{1}{20}$.
B " " $\frac{1}{10}-\frac{1}{15}=\frac{1}{30}$.
C " " $\frac{1}{10}-\frac{1}{12}=\frac{1}{60}$.

Hence A could perform the whole work in 20 days, B in 30 days, and C in 60 days.

Prob. 14. $$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{2}{x}.$$

Hence

$x=\frac{2abc}{ab+ac+bc}$, the time required if all worked together.

In one day A can do $\frac{ab+ac+bc}{2abc}-\frac{1}{c}$, or $\frac{ac+bc-ab}{2abc}$.

" B can do $\frac{ab+ac+bc}{2abc}-\frac{1}{b}$, or $\frac{ab+bc-ac}{2abc}$.

" C can do $\frac{ab+ac+bc}{2abc}-\frac{1}{a}$, or $\frac{ab+ac-bc}{2abc}$.

Prob. 15. Let x denote the gallons contained in the first cask, and y the gallons in the second cask.

After the first operation the gallons in the two casks will be denoted by $x-y$ and $2y$.

After the second operation the contents will be

$$2x-2y \text{ and } 2y-x+y, \text{ or } 3y-x.$$

After the third operation the contents will be

$$2x-2y-3y+x \text{ and } 6y-2x.$$

Hence $$3x-5y=6y-2x,$$

$$5x=11y,$$

and $$y=\frac{5x}{11}.$$

But $$3x-5y=a.$$

By substitution, $$3x-\frac{25x}{11}=a.$$

Hence $$x=\frac{11a}{8},$$

and $$y=\frac{5a}{8}.$$

Prob. 16. Let x denote the number of working days, and y the number of idle days.

Then px denotes the number of pence he earns,

qy " " " forfeits.

We have $$px-qy=a, \quad (1)$$

$$x+y=n. \quad (2)$$

Multiply (2) by q, $$qx+qy=nq. \quad (3)$$

Add (1) to (3), $$px+qx=nq+a. \quad (4)$$

Hence $$x=\frac{nq+a}{p+q},$$

and $$y=n-\frac{nq+a}{p+q}=\frac{np-a}{p+q}.$$

Prob. 17. Let x denote the left-hand digit, and y the right-hand digit. Then $$\frac{10x+y}{x+y}=4, \quad (1)$$

$$\frac{10x+y}{xy}=3. \quad (2)$$

From Eq. (1), $y=2x.$ (3)
From Eq. (2), $10x+y=3xy.$ (4)
Substitute (3) in (4), $12x=6x^2.$
Hence $x=2,$
and $y=4.$
The required number is 24.

Prob. 18. Let x denote the age of the father, and y that of the younger son. $x+2=2(2y+4+4),$ (1)
$x-6=6(2y+4-12).$ (2)
From Eq. (1), $x=4y+14.$
From Eq. (2), $x=12y-42.$
Hence $4y+14=12y-42.$
Therefore $y=7,$
and $x=42.$

Prob. 19. Let x denote the second part, and y the third part. Then $96-x-y$ will denote the first part.

$$\frac{96-x-y}{x}=2+\frac{3}{x}, \quad (1)$$

$$\frac{x}{y}=4+\frac{5}{y}. \quad (2)$$

From (1) we have $3x+y=93.$ (3)
From (2) we have $x-4y=5.$ (4)
Multiply (4) by 3, $3x-12y=15.$ (5)
Subtract (5) from (3), $13y=78.$
Hence $y=6,$
$x=29,$
and $96-29-6=61.$

Prob. 20. The total number of apples was $128\times 7=896.$

After the first distribution the number of apples in the first basket was doubled at each distribution. Hence, after the sixth distribution it contained 64 apples; after the fifth, 32; fourth, 16; third, 8; second, 4; and after the first distribution, 2 apples.

Let x denote the number of apples in the first basket at first. Then the other baskets contained $896-x$ apples.

Hence the number of apples in the first basket after the first distribution is denoted by

$$x-(896-x), \text{ which equals } 2.$$

Hence $$x=449.$$

For a like reason, the number of apples in the second basket after the second distribution was 4. Let y denote the number of apples in the second basket at first. Then $2y$ denotes the number of apples after the first distribution, and $896-2y$ will denote the apples in the other baskets. Hence, after the second distribution this basket will contain

$$2y-(896-2y), \text{ which equals } 4.$$

Hence $$y=225.$$

So, also, after the third distribution the third basket contained 8 apples. If we put z to denote the number of apples in this basket at first, we shall have

$$4z-(896-4z)=8.$$

Hence $$z=113.$$

We may proceed in the same manner with the other baskets.

We notice, however, that these numbers follow a simple law, thus:

$$\begin{aligned} 2x&=896+2 \therefore x=449; \\ 4y&=896+4 \therefore y=225; \\ 8z&=896+8 \therefore z=113; \\ 16v&=896+16 \therefore v=57; \\ 32u&=896+32 \therefore u=29; \\ 64t&=896+64 \therefore t=15; \\ 128s&=896+128 \therefore s=8. \end{aligned}$$

CHAPTER X.

Art. 164, Page 118.

Ex. 1. What number is that whose third part exceeds its fourth part by 16?

Ex. 2. The sum of two numbers is 8 and their difference 2. What are those numbers?

Ex. 3. What fraction is that to the numerator of which if 4 be added the value is one half, but if 7 be added to the denominator its value is one fifth?

Ex. 4. Find two numbers whose difference is 6, such that five times the less may exceed four times the greater by 12.

Art. 180, Page 126.

Ex. 4. $x>12$.

Ex. 7. $x>49$, and $x<51$.

Ex. 8. $x>110$, and $x<126$.

Ex. 9. $x>14$, and $x<16$.

CHAPTER XI.

Art. 185, Page 128.

Ex. 2. $324x^4y^2z^6$.

Ex. 3. $343a^3b^6x^6$.

Ex. 4. $-512x^3y^6z^9$.

Ex. 5. $256a^4b^8c^{12}$.

Ex. 6. $625a^{12}b^8x^4$.

Ex. 7. $32a^5b^{15}x^{10}$.

Ex. 8. $-243a^5b^{10}x^{20}$.

Ex. 9. $729a^6b^{12}x^{18}$.

Ex. 10. $64a^{12}b^{18}x^{24}$.

Ex. 11. $128a^{14}x^{21}y^7$.

Ex. 12. $a^mb^{2m}x^{3m}$.

Art. 186, Page 129.

Ex. 3. $-\frac{216a^3x^6y^9}{125m^3n^3}$.

Ex. 4. $-\frac{a^3x^6}{729m^3y^9}$.

Ex. 5. $\frac{256a^4x^8y^4}{81b^4m^4}$.

Ex. 6. $\frac{625a^4x^{12}z^4}{16b^4m^4n^8}$.

Ex. 7. $-\frac{243a^5b^{10}x^{15}}{32m^5y^{10}}$.

Ex. 8. $-\frac{1024a^{10}x^{10}}{243b^5m^{15}}$.

Ex. 9. $\frac{729a^{12}b^6x^6y^{12}}{64m^6n^{12}}$.

Art. 187, Page 130.

Ex. 5. $-216a^3b^{-15}x^{-6}$.

Ex. 6. $-\frac{x^9}{27a^6z^6}$.

Ex. 7. $\frac{a^9x^{12}}{64b^3}$.

Ex. 8. $\frac{81b^8x^4}{a^{4n}}$.

Ex. 9. $\frac{a^{12}x^{16}}{256b^8}$.

Ex. 10. $-\frac{32a^5c^{10}y^5}{b^{15}x^{20}}$.

Art. 188, Page 131.

Ex. 2. $a^2+m^2+n^2+2am-2an-2mn.$

Ex. 3. $8a^6+36a^5+42a^4-9a^3-21a^2+9a-1.$

Ex. 4. $a^3+\frac{3a^2}{2}+\frac{3a}{4}+\frac{1}{8}.$

Ex. 5. $a^3+8b^3+27x^3+6a^2b+9a^2x+12ab^2+27ax^2+36b^2x$
$+54bx^2+36abx.$

Ex. 6. $a^4-4a^3b+6a^2b^2-4ab^3+b^4.$

Ex. 7. $16a^4-96a^3b+216a^2b^2-216ab^3+81b^4.$

Ex. 8. $a^{12}+4a^9b^3+6a^6b^6+4a^3b^9+b^{12}.$

Ex. 9. $a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5.$

Ex. 11. $\frac{8a^3+36a^2b+54ab^2+27b^3}{m^3-3m^2n+3mn^2-n^3}.$

Ex. 12. $\frac{a^6-3a^4b+3a^2b^2-b^3}{a^3-3a^2b^2+3ab^4-b^6}.$

Art. 189, Page 132.

Ex. 1. $a^2+b^2+c^2+d^2+x^2+2ab+2ac+2ad+2ax+2bc+2bd$
$+2bx+2cd+2cx+2dx.$

Ex. 2. $a^2+b^2+c^2-2ab+2ac-2bc.$

Ex. 3. $1+4x+10x^2+12x^3+9x^4.$

Ex. 4. $1-2x+3x^2-4x^3+3x^4-2x^5+x^6.$

Ex. 5. $a^2+4b^2+9a^2b^2+m^2-4ab+6a^2b-2am-12ab^2+4bm$
$-6abm.$

Ex. 6. $1-6x+15x^2-20x^3+15x^4-6x^5+x^6.$

Ex. 7. $a^2+4b^2+9c^2+16d^2-4ab+6ac-8ad-12bc+16bd-24cd.$

CHAPTER XII.

Art. 196, Page 135.

Ex. 3. $\pm 15a^mb^4x^3.$

Ex. 6. $-7a^2b^3x^4.$

Ex. 8. $\pm 4ab^3x^4.$

Ex. 11. $\pm\frac{5a^2b^2x^3}{8am^2y^2}.$

Ex. 13. $-\frac{5ab^3x^4}{6c^2z^3}.$

Ex. 14. $\pm\frac{4ax^3}{3b^2z^4}$.

Ex. 16. $-8a^{-1}b^{-2}x$.

Ex. 17. $\pm 4a^{-1}b^{-2}x$.

Ex. 20. $(a+b)(x+y)^2$.

ART. 198, PAGE 137.

Ex. 2. $a+b+c$.

Ex. 3. $3x^3-2x^2+x-1$.

Ex. 5. $a^3-3a^2b+3ab^2-b^3$.

Ex. 6. $a^2-2ab-2b^2$.

ART. 199, PAGE 138.

Ex. 2. $3a-4b$. Ex. 3. $3a^2-5ab$. Ex. 4. Impossible.

ART. 203, PAGE 141.

Ex. 2. 917.

Ex. 3. 1069.

Ex. 4. 5678.

Ex. 5. 8531.

Ex. 6. 59,319.

ART. 204, PAGE 142.

Ex. 2. $\frac{137}{413}$.

Ex. 3. $\frac{1}{8219}$.

Ex. 4. 3.143.

Ex. 5. 7.656.

Ex. 6. 0.747.

Ex. 7. 0.1865.

Ex. 10. 3.01662.

Ex. 11. 2.09165.

Ex. 12. 0.27735.

Ex. 13. 0.54233.

ART. 207, PAGE 145.

Ex. 2. x^2+2x-4.

Ex. 3. x^2+2x+2.

Ex. 4. b^2+b-1.

Ex. 5. $2x^2-3x+1$.

Ex. 6. $2x^2+4ax-3a^2$.

Ex. 7. $2x^2-3ax+4a^2$.

ART. 210, PAGE 149.

Ex. 4. 2345. Ex. 5. 5678. Ex. 6. 9123.

ART. 211, PAGE 149.

Ex. 3. 2.37.

Ex. 4. 12.37.

Ex. 5. 0.1234.

Ex. 10. 2.22398.

Ex. 11. 0.69336.

Ex. 12. 0.90856.

CHAPTER XIII.

Art. 215, Page 152.

Ex. 3. $7b\sqrt{6a}$.

Ex. 4. $28c\sqrt{5abc}$.

Ex. 5. $7ax^3y\sqrt{2}$.

Ex. 6. $3abc\sqrt{5bx}$.

Ex. 7. $12ab^2x^5\sqrt{6bx}$.

Ex. 11. $2ax^2\sqrt[6]{3ab}$.

Ex. 12. $27b\sqrt[3]{3a^2x}$.

Ex. 13. $a\sqrt{a-x}$.

Art. 218, Page 154.

Ex. 2. $\sqrt{6ab}$.

Ex. 3. $\sqrt[m]{a}$.

Ex. 4. $\sqrt[4]{5a^2bc^3}$.

Ex. 5. $\sqrt[3]{\frac{5a}{8b}}$.

Art. 221, Page 157.

Ex. 6. $\sqrt[6]{9}$, $\sqrt[6]{125}$, $\sqrt[6]{7}$.

Ex. 7. $\sqrt[6]{8a^3b^3}$, $\sqrt[6]{9a^2b^4}$, $\sqrt[6]{5ab^3}$.

Ex. 8. $(a+b)^{\frac{10}{20}}$, $(a-b)^{\frac{5}{20}}$, $(a^2-b^2)^{\frac{4}{20}}$.

Art. 222, Page 158.

Ex. 1. $3\sqrt{3}+4\sqrt{3}+5\sqrt{3}=12\sqrt{3}$.

Ex. 2. $28\sqrt{3}+15\sqrt{3}+8\sqrt{3}=51\sqrt{3}$.

Ex. 3. $6\sqrt{2}+8\sqrt{2}+9\sqrt{2}=23\sqrt{2}$.

Ex. 4. $6\sqrt{5}+9\sqrt{5}+8\sqrt{5}=23\sqrt{5}$.

Ex. 5. $\frac{3}{5}\sqrt{10}+\frac{1}{5}\sqrt{10}+\frac{1}{5}\sqrt{10}=\sqrt{10}$.

Ex. 6. $5\sqrt[3]{4}+3\sqrt[3]{4}+4\sqrt[3]{4}=12\sqrt[3]{4}$.

Ex. 7. $2\sqrt[3]{5}+3\sqrt[3]{5}+4\sqrt[3]{5}=9\sqrt[3]{5}$.

Ex. 8. $\frac{2}{3}\sqrt{15}+2\sqrt{15}+\sqrt{15}+\frac{1}{5}\sqrt{15}=3\frac{13}{15}\sqrt{15}$.

Ex. 9. $3c\sqrt{5c}+4c\sqrt{5c}+a\sqrt{5c}=(a+7c)\sqrt{5c}$.

Ex. 10. $3a^2b\sqrt{2ab}+5ab\sqrt{2ab}=(3a^2b+5ab)\sqrt{2ab}$.

Ex. 11. $\frac{a^2}{b}\sqrt{\frac{c}{b}}+\frac{ac}{d}\sqrt{\frac{c}{b}}+\frac{ad}{m}\sqrt{\frac{c}{b}}=\left(\frac{a^2}{b}+\frac{ac}{d}+\frac{ad}{m}\right)\sqrt{\frac{c}{b}}$.

Ex. 12. $2a\sqrt{ab}+5b\sqrt{ab}+5b\sqrt{ab}=(2a+10b)\sqrt{ab}$.

ART. 223, PAGE 159.

Ex. 1. $8\sqrt{7}-4\sqrt{7}=4\sqrt{7}$.
Ex. 2. $10\sqrt{5}-9\sqrt{5}=\sqrt{5}$.
Ex. 3. $10\sqrt{2}-3\sqrt{2}=7\sqrt{2}$.
Ex. 4. $4a^2\sqrt{5x}-2ax\sqrt{5x}=(4a^2-2ax)\sqrt{5x}$.
Ex. 5. $12a\sqrt{2}-9a\sqrt{2}=3a\sqrt{2}$.
Ex. 6. $4\sqrt[3]{3}-2\sqrt[3]{3}=2\sqrt[3]{3}$.
Ex. 7. $8\sqrt[3]{5}-6\sqrt[3]{5}=2\sqrt[3]{5}$.
Ex. 8. $3a\sqrt[3]{\frac{a^2x}{2b}}-\sqrt[3]{\frac{a^2x}{2b}}=(3a-1)\sqrt[3]{\frac{a^2x}{2b}}$.

ART. 224, PAGE 160.

Ex. 1. $6\sqrt{48}=24\sqrt{3}$.
Ex. 2. $15\sqrt{40}=30\sqrt{10}$.
Ex. 3. $\sqrt[2]{2}\times\sqrt[3]{3}=\sqrt[6]{8}\times\sqrt[6]{9}=\sqrt[6]{72}$.
Ex. 4. $35\sqrt{16}=140$.
Ex. 5. $cd\sqrt{a^2}=acd$.
Ex. 6. $35\sqrt[3]{72}=70\sqrt[3]{9}$.
Ex. 7. $\frac{1}{30}\sqrt[3]{102}$.
Ex. 8. $\frac{5}{2}\sqrt[3]{360}=5\sqrt[3]{45}$.
Ex. 9. $\frac{7}{2}\sqrt[3]{120}=7\sqrt[3]{15}$.

ART. 225, PAGE 161.

Ex. 3. $12xy$.
Ex. 4. $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}-\frac{1}{12}=1$. *Ans. a.*
Ex. 5. $\frac{1}{3}-\frac{3}{4}+\frac{4}{3}+\frac{1}{12}=1$. *Ans. a.*

ART. 226, PAGE 161.

Ex. 5. $\sqrt{1\frac{4}{5}}=\sqrt{\frac{9}{5}}=3\sqrt{\frac{1}{5}}$; $2\sqrt{9\frac{4}{5}}=2\sqrt{\frac{49}{5}}=14\sqrt{\frac{1}{5}}$; $2\sqrt{5}\times17\sqrt{\frac{1}{5}}=34$.

ART. 227, PAGE 162.

Ex. 1. $4\sqrt{18}=12\sqrt{2}$.
Ex. 2. $2\sqrt[3]{256}=8\sqrt[3]{4}$.

Ex. 3. $2\sqrt[3]{27}=6.$

Ex. 4. $2\sqrt[3]{4}.$

Ex. 5. $2\sqrt{2a^2}=2a\sqrt{2}.$

Ex. 6. $2\left(\frac{a^2b}{c}\right)^{\frac{1}{m}}.$

Ex. 7. $4\sqrt[6]{144}\div2\sqrt[6]{27}=2\sqrt[6]{\frac{144}{27}}=2\sqrt[6]{\frac{16}{3}}.$

Ex. 8. $\sqrt[5]{64}\div\sqrt[5]{32}=\sqrt[5]{2}.$

Ex. 9. $\sqrt[15]{a^{10}b^5c^5}\div\sqrt[15]{a^3b^6c^9}=\sqrt[15]{\frac{a^7}{bc^4}}.$

Art. 228, Page 163.

Ex. 2. $a^{\frac{5}{12}}.$

Ex. 3. $4\sqrt[6]{a^3b^3}\div2\sqrt[6]{a^2b^2}=2\sqrt[6]{ab}.$

Ex. 6. $4^{\frac{1}{2}}\div4^{\frac{3}{4}}=4^{-\frac{1}{4}}=\frac{1}{\sqrt[4]{4}}=\frac{1}{\sqrt{2}}.$

Art. 230, Page 164.

Ex. 2. $\frac{8}{27}\sqrt{27}=\frac{8}{9}\sqrt{3}.$

Ex. 3. $9\sqrt[3]{9}.$

Ex. 4. $17^3\sqrt{21^3}=17^3\times21\sqrt{21}=103{,}173\sqrt{21}.$

Ex. 5. $\frac{1}{6^4}\sqrt{6^4}=\frac{6^2}{6^4}=\frac{1}{6^2}=\frac{1}{36}.$

Ex. 6. $16\sqrt[6]{3^4a^{16}b^4}=16\sqrt[3]{3^2a^8b^2}=16a^2\sqrt[3]{9a^2b^2}.$

Ex. 7. $a^4b^4\sqrt{a^4b^4}=a^6b^6.$

Ex. 8. $(a+b)^{\frac{6}{3}}=(a+b)^2.$

Ex. 9. $\left(\frac{4}{5}\right)^7\times\left(\frac{5}{8}\right)^6=\frac{4}{5}\times\frac{1}{2^6}=\frac{1}{5.2^4}=\frac{1}{80}.$

Ex. 10. $(4ab^2)^{\frac{n}{3}}\times(2a^2b)^{\frac{n}{3}}=(8a^3b^3)^{\frac{n}{3}}=(2ab)^n.$

Art. 231, Page 165.

Ex. 3. $10^{\frac{3}{2}}=\sqrt{1000}=10\sqrt{10}.$

Ex. 4. $\frac{2}{3}a^{\frac{4}{3}}.$

Ex. 5. $\frac{2}{3}a^{\frac{1}{6}}$.
Ex. 6. $\frac{4}{5}a^2$.
Ex. 7. Cube root of $\left(\frac{a}{3}\right)^{\frac{3}{2}}$ is $\left(\frac{a}{3}\right)^{\frac{1}{2}}$.
Ex. 8. $3\sqrt[3]{5}=\sqrt[3]{135}$.
Ex. 9. $\frac{4}{9}\sqrt[3]{\frac{4}{9}}=\sqrt[3]{\frac{4^4}{9^4}}$. Its fourth root is $\sqrt[3]{\frac{4}{9}}$, which equals $\frac{1}{3}\sqrt[3]{12}$.

ART. 234, PAGE 167.

Ex. 3. $6\sqrt{-1}$.
Ex. 4. $16-\sqrt{-3}$.
Ex. 6. $1-2\sqrt{-1}-1=-2\sqrt{-1}$.
Ex. 9. $a^2b\sqrt{-b}\times a^2b^2\sqrt{-b}=a^4b^4\sqrt{-1}\times\sqrt{-1}=-a^4b^4$.
Ex. 10. $b-a$.
Ex. 11. $\sqrt{-119}=\sqrt{7}\sqrt{17}\sqrt{-1}$; $\sqrt{-133}=\sqrt{7}\sqrt{19}\sqrt{-1}$.
We have $-17\sqrt{7}-\sqrt{7}\sqrt{17}\sqrt{19}+\sqrt{7}\sqrt{17}\sqrt{19}+19\sqrt{7}=2\sqrt{7}$.

ART. 235, PAGE 167.

Ex. 4. $-\frac{a\sqrt{-1}}{b}$.
Ex. 5. $-\sqrt{-a}$.
Ex. 6. $2+\sqrt{2}+\sqrt{3}$.
Ex. 7. $-2\sqrt{-4}+\sqrt{5}=-4\sqrt{-1}+\sqrt{5}$.

ART. 237, PAGE 168.

Ex. 1. Multiplier, $\sqrt{5}-\sqrt{3}$; product, $5-3=2$.
Ex. 2. Multiplier, $\sqrt{3}+\sqrt{x}$; product, $3-x$.
Ex. 3. First multiplier, $\sqrt{3}+\sqrt[4]{x}$; product, $3-\sqrt{x}$; second multiplier, $3+\sqrt{x}$; product, $9-x$.

ART. 238, PAGE 169.

Ex. 1. First multiplier, $\sqrt{5}+\sqrt{3}+\sqrt{2}$; product, $6+2\sqrt{15}$; second multiplier, $6-2\sqrt{15}$; product, $36-60=-24$.
Ex. 2. First multiplier, $1+\sqrt{2}-\sqrt{3}$; product, $2\sqrt{2}$; second multiplier, $\sqrt{2}$; product, 4.

ART. 239, PAGE 170.

Ex. 3. $\frac{(3+\sqrt{2})\sqrt{2}}{9-2}=\frac{3\sqrt{2}+2}{7}$.

Ex. 4. $\frac{a^2-2a\sqrt{b}+b}{a^2-b}$.

Ex. 5. $\frac{4(\sqrt{3}-\sqrt{2}+1)}{(\sqrt{3}+\sqrt{2}+1)\times(\sqrt{3}-\sqrt{2}+1)}=\frac{4\sqrt{3}-4\sqrt{2}+4}{2+2\sqrt{3}}$

$$=\frac{-16-8\sqrt{2}+8\sqrt{6}}{-8}=2+\sqrt{2}-\sqrt{6}.$$

Ex. 6. $\frac{a\sqrt{b}}{b}$.

Ex. 7. $\frac{ab^{\frac{2}{3}}}{b}$.

Ex. 9. Multiply both numerator and denominator by the numerator, and we have

$$\frac{(1+a)^2+2(1+a)\sqrt{1-a^2}+1-a^2}{2a(1+a)},$$

which equals $\frac{(1+a)+2\sqrt{1-a^2}+1-a}{2a}$,

which equals $\frac{2+2\sqrt{1-a^2}}{2a}$.

ART. 240, PAGE 170.

Ex. 2. $\frac{7\sqrt{5}}{\sqrt{11}+\sqrt{3}}=\frac{7\sqrt{55}-7\sqrt{15}}{8}=\frac{\sqrt{2695}-\sqrt{735}}{8}=3.1003.$

Ex. 3. $\frac{\sqrt{6}}{\sqrt{7}+\sqrt{3}}=\frac{\sqrt{42}-\sqrt{18}}{4}=0.5595.$

Ex. 4. $\frac{\sqrt{3}}{3\sqrt{5}-3\sqrt{2}}=\frac{3\sqrt{15}+3\sqrt{6}}{45-18}=\frac{\sqrt{15}+\sqrt{6}}{9}=0.7025.$

Ex. 5. $\frac{9+2\sqrt{10}}{9-2\sqrt{10}}=\frac{121+36\sqrt{10}}{41}=\frac{121+\sqrt{12,960}}{41}=5.7278.$

ART. 244, PAGE 173.

Ex. 3. $x=\frac{11+\sqrt{121-120}}{2}=6;$

$y=\frac{11-\sqrt{121-120}}{2}=5.$

Ex. 4. $x=\frac{2+\sqrt{4-3}}{2}=\frac{3}{2};$

$y=\frac{2-\sqrt{4-3}}{2}=\frac{1}{2}.$

Ex. 5. $x=\frac{7+\sqrt{49-40}}{2}=5;$

$y=\frac{7-\sqrt{49-40}}{2}=2.$

Ex. 6. $x=\frac{18+\sqrt{324-320}}{2}=10;$

$y=\frac{18-\sqrt{324-320}}{2}=8.$

ART. 245, PAGE 173.

Ex. 8. $x=\frac{-\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{3}{4}}}{2}=\frac{1}{4};$

$y=\frac{-\frac{1}{2}-\sqrt{\frac{1}{4}+\frac{3}{4}}}{2}=-\frac{3}{4}.$

Therefore $\sqrt{x}=\frac{1}{2}$, and $\sqrt{y}=\frac{1}{2}\sqrt{-3}.$

Ex. 9. $x=\frac{\sqrt{4}}{2}=1;$

$y=\frac{-\sqrt{4}}{2}=-1.$

Ex. 10. $\sqrt{6+2\sqrt{5}}=\sqrt{5}+1;$

$\sqrt{6-2\sqrt{5}}=\sqrt{5}-1.$

Hence $\sqrt{6+2\sqrt{5}}-\sqrt{6-2\sqrt{5}}=2.$

Ex. 11. $$\sqrt{4+3\sqrt{-20}}=3+\sqrt{-5};$$
$$\sqrt{4-3\sqrt{-20}}=3-\sqrt{-5}.$$

Hence $$\sqrt{4+3\sqrt{-20}}+\sqrt{4-3\sqrt{-20}}=6.$$

Ex. 12. $$x=\frac{-3+\sqrt{9+16}}{2}=1;$$
$$y=\frac{-3-\sqrt{9+16}}{2}=-4;$$
$$\sqrt{x}=1;\ \sqrt{y}=2\sqrt{-1}.$$

Ex. 13. $$x=\frac{\sqrt{64}}{2}=4;$$
$$y=-\frac{\sqrt{64}}{2}=-4.$$

Hence $$\sqrt{x}=2;\ \sqrt{y}=2\sqrt{-1}.$$

Art. 246, Page 174.

Ex. 2. Clearing of fractions, and transposing, we have
$$\sqrt{ax+x^2}=a-x.$$

Squaring, $$ax+x^2=a^2-2ax+x^2.$$
Hence $$3ax=a^2;$$
$$x=\frac{a}{3}.$$

Ex. 3. $$17-5\sqrt{x}=-33.$$
Transposing, $$\sqrt{x}=10;$$
$$x=100.$$

Ex. 4. $$4x^2-7x-6=81-36x+4x^2.$$
Hence $$x=3.$$

Ex. 6. $$36+x=324+36\sqrt{x}+x.$$
Hence $$\sqrt{x}=-8;$$
$$x=64.$$

Ex. 7. $$x+4ab=4b^2+4b\sqrt{x}+x.$$
Hence $$\sqrt{x}=a-b;$$
$$x=(a-b)^2.$$

Ex. 8. Transposing a, and squaring, we have

$$x^2-2ax+a^2=a^2+x\sqrt{b^2+x^2-a^2}.$$

Hence $$x-2a=\sqrt{b^2+x^2-a^2}.$$

Squaring, $$x^2-4ax+4a^2=b^2+x^2-a^2.$$

Hence $$4ax=5a^2-b^2;$$

$$x=\frac{5a^2-b^2}{4a}.$$

ART. 248, PAGE 175.

Ex. 9. $$\sqrt{x^2-3x}=6-x.$$

Squaring, $$x^2-3x=36-12x+x^2.$$

Hence $$x=4.$$

Ex. 10. $$\sqrt{9x+13}+\sqrt{9x}=13\sqrt{9x+13}-13\sqrt{9x}.$$

Transposing, $$7\sqrt{9x}=6\sqrt{9x+13}.$$

Squaring, $$441x=324x+468.$$

Hence $$x=4.$$

Ex. 11. $$ax+b=cx+d.$$

Hence $$x=\frac{d-b}{a-c}.$$

Ex. 12. $$50\sqrt[10]{x+24}-9=21+35\sqrt[10]{x+24}.$$

Transposing, $$15\sqrt[10]{x+24}=30.$$

Hence $$x+24=2^{10}=1024;$$

$$x=1000.$$

Ex. 13. By Art. 84, $$\frac{3x-1}{\sqrt{3x}+1}=\sqrt{3x}-1.$$

Hence $$\sqrt{3x}-1=1+\frac{\sqrt{3x}-1}{2}.$$

Clearing of fractions, $$\sqrt{3x}=3.$$

Hence $$x=3.$$

Ex. 14. Clearing of fractions,

$$x+4m\sqrt{x}+n\sqrt{x}+4mn=x+2m\sqrt{x}+3n\sqrt{x}+6mn.$$

Uniting terms, $$2m\sqrt{x}-2n\sqrt{x}=2mn.$$

Hence $$\sqrt{x}=\frac{mn}{m-n}.$$

Ex. 15. Performing the multiplication indicated,

$$3x-6\sqrt{x}+75\sqrt{x}-150=5\sqrt{x}+15+3x+9\sqrt{x}.$$

Reducing, $\sqrt{x}=3.$

Hence $x=9.$

Ex. 16. Squaring, $2x-3n=9n+2x-6\sqrt{2nx}.$

Reducing, $\sqrt{2nx}=2n.$

Squaring, $2nx=4n^2.$

Hence $x=2n.$

Ex. 17. Clearing of fractions,

$$3x-4\sqrt{x}+120\sqrt{x}-160=15\sqrt{x}+3x+30+6\sqrt{x}.$$

Reducing, $\sqrt{x}=2.$

Squaring, $x=4.$

Ex. 18. Given, $\dfrac{\sqrt{6x}-2}{\sqrt{6x}+2}=\dfrac{4\sqrt{6x}-9}{4\sqrt{6x}+6}.$

Clearing of fractions,

$$24x-8\sqrt{6x}+6\sqrt{6x}-12=24x-9\sqrt{6x}+8\sqrt{6x}-18.$$

Uniting terms, $\sqrt{6x}=6.$

Squaring, $6x=36.$

Hence $x=6.$

Ex. 19. By Art. 84, $\dfrac{5x-9}{\sqrt{5x}+3}=\sqrt{5x}-3.$

Hence $\sqrt{5x}-3-1=\dfrac{\sqrt{5x}-3}{2}.$

Clearing of fractions, $\sqrt{5x}=5.$

Squaring, $5x=25.$

Hence $x=5.$

Ex. 20. Squaring, $4a+x=4b+4x+x-4\sqrt{bx+x^2}.$

Transposing, $\sqrt{bx+x^2}=x+b-a.$

Squaring, $bx+x^2=x^2+b^2+a^2+2bx-2ax-2ab.$

Transposing, $2ax-bx=a^2-2ab+b^2=(a-b)^2.$

Hence $x=\dfrac{(a-b)^2}{2a-b}.$

CHAPTER XIV.

ART. 255, PAGE 179.

Ex. 3. Clearing of fractions,

$$4x^2+5=405.$$

Reducing, $x^2=100.$

Hence $x=\pm 10.$

Ex. 5. Clearing of fractions,

$$x\sqrt{a^2+x^2}+a^2+x^2=2a^2.$$

Transposing, $x\sqrt{a^2+x^2}=a^2-x^2.$

Squaring, $x^2a^2+x^4=a^4-2a^2x^2+x^4.$

Transposing, $3a^2x^2=a^4.$

Hence $x=\pm\frac{a}{\sqrt{3}}.$

Ex. 6. Transposing, $ax^2-bx^2=2c+d.$

Dividing, $x^2=\frac{2c+d}{a-b}.$

Hence $x=\pm\left(\frac{2c+d}{a-b}\right)^{\frac{1}{2}}.$

Ex. 7. Clearing of fractions,

$$8x^2-72+10x^2=7-24x^2+299.$$

Reducing, $42x^2=378.$

Dividing, $x^2=9.$

Hence $x=\pm 3.$

Ex. 8. Transposing, $x^2=4a^2-12ab+9b^2.$

Hence $x=\pm(2a-3b).$

Ex. 9. Clearing of fractions,

$$11x^2-x-25=3x^2-x+25.$$

Reducing, $8x^2=50.$

Hence $x=\pm\frac{5}{2}.$

Ex. 10. Clearing of fractions,

$$x\sqrt{x^2-17}+x^2-17=4.$$

Transposing, $x\sqrt{x^2-17}=21-x^2.$

Squaring, $x^4-17x^2=441-42x^2+x^4.$

Reducing, $25x^2=441.$

Hence $x=\pm\frac{21}{5}.$

Ex. 11. Clearing of fractions,

$$2x^2+2a^2-2a^2x^2-2a^4=2a^2x^2-2a^4+2x^2-2a^2.$$

Uniting terms, $4a^2x^2=4a^2.$

Dividing, $x^2=1.$

Hence $x=\pm 1.$

Ex. 12. By multiplication,

$$x^2+3x-7=x+2+\frac{18}{x}.$$

Transposing, $x^2+2x=9+\frac{18}{x}.$

Resolving into factors,

$$x(x+2)=\frac{9}{x}(x+2).$$

Clearing of fractions, $x^2=9.$

Hence $x=\pm 3.$

Prob. 2. Let mx denote the sum of the numbers.

Then will nx " the greater number,

and $mx-nx$ " the less number.

Hence $mx(mx-nx)=a.$

Dividing, $x^2=\frac{a}{m^2-mn};$

Hence $x=\pm\sqrt{\frac{a}{m(m-n)}};$

$$nx=\pm\sqrt{\frac{an^2}{m(m-n)}};$$

$$mx=\pm\sqrt{\frac{am^2}{m(m-n)}};$$

$$mx-nx=\pm\sqrt{\frac{a(m-n)^2}{m(m-n)}}=\pm\sqrt{\frac{a(m-n)}{m}}.$$

Prob. 3. By the conditions, $20-\frac{x^2}{3}=8.$

Hence $x^2=36;$

$$x=\pm 6.$$

Prob. 4. By the conditions, $a-\frac{x^2}{m}=b.$

Hence $x^2=ma-mb;$

$$x=\pm\sqrt{m(a-b)}.$$

$\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{4}$ are in the ratio of 6, 8, and 9.

Prob. 5. Let $6x$ denote the first number.

Then will $8x$ " the second number,

and $9x$ " the third number.

Hence $36x^2+64x^2+81x^2=724.$

Reducing, $x^2=4.$

Hence $x=\pm 2.$

The numbers are $\pm 12;\ \pm 16;\ \pm 18.$

Prob. 6. Let mx, nx, px denote the numbers.

By the conditions, $m^2x^2+n^2x^2+p^2x^2=a.$

Hence $$x=\pm\sqrt{\frac{a}{m^2+n^2+p^2}}.$$

Prob. 7. Let x denote the greater part.

Then will $49-x$ " the less part.

By the conditions, $\frac{x}{49-x}:\frac{49-x}{x}::\frac{4}{3}:\frac{3}{4}.$

Clearing of fractions,

$$x^2:(49-x)^2::16:9.$$

Extracting the square root,

$$x:49-x::4:3.$$

Multiplying, $3x=196-4x.$

Hence $x=28;$

$$49-x=21.$$

Prob. 8. By the conditions, $\frac{x}{a-x}:\frac{a-x}{x}::m:n.$

Clearing of fractions,

$$x^2:(a-x)^2::m:n.$$

Extracting the square root,

$$x:a-x::\sqrt{m}:\sqrt{n}.$$

Multiplying, $x\sqrt{n}=a\sqrt{m}-x\sqrt{m}.$

Hence $$x=\frac{a\sqrt{m}}{\sqrt{m}+\sqrt{n}};$$

$$a-x=\frac{a\sqrt{m}+a\sqrt{n}-a\sqrt{m}}{\sqrt{m}+\sqrt{n}}.$$

Prob. 9. Let x denote a side of the smaller square.

Then will $x+10$ " " of the larger square.

By the conditions, $x^2:(x+10)^2::9:25$.
Extracting the square root,

$$x:x+10::3:5.$$

Multiplying, $5x=3x+30$.

Hence $x=15$;

$$x+10=25.$$

Prob. 10. By the conditions, $(x+a)^2:x^2::m:n$.
Extracting the square root,

$$x+a:x::\sqrt{m}:\sqrt{n}.$$

Multiplying, $x\sqrt{n}+a\sqrt{n}=x\sqrt{m}$.

Hence $x=\dfrac{a\sqrt{n}}{\sqrt{m}-\sqrt{n}}$;

$$x+a=\frac{a\sqrt{m}-a\sqrt{n}+a\sqrt{n}}{\sqrt{m}-\sqrt{n}}=\frac{a\sqrt{m}}{\sqrt{m}-\sqrt{n}}.$$

Prob. 11. Let x denote the distance B traveled.
Then will $x+18$ " " A traveled;
$\dfrac{x+18}{28}$ " the miles B traveled per hour;
$\dfrac{x}{15\frac{3}{4}}$ " the miles A traveled per hour.

Then $x:x+18::\dfrac{x+18}{28}:\dfrac{x}{15\frac{3}{4}}$.

Multiplying, $x^2=\dfrac{15\frac{3}{4}}{28}(x+18)^2=\dfrac{9}{16}(x+18)^2$.

Extracting the square root,

$$x=\tfrac{3}{4}(x+18).$$

Hence $x=54$;

$$x+18=72.$$

Prob. 12. Let x denote the distance B traveled.
Then will $x+a$ " " A traveled.

Then $x:x+a::\dfrac{x+a}{m}:\dfrac{x}{n}$.

Multiplying, $mx^2=n(x+a)^2$.

Extracting the square root,

$$x\sqrt{m}=x\sqrt{n}+a\sqrt{n}.$$

Hence $$x=\frac{a\sqrt{n}}{\sqrt{m}-\sqrt{n}};$$

$$x+a=\frac{a\sqrt{m}}{\sqrt{m}-\sqrt{n}}.$$

Prob. 13. $$256(x-1)^4=81x^4.$$

Extracting the fourth root,

$$4(x-1)=3x.$$

Hence $$x=4.$$

That is, one fourth part was drawn each time. The first time he draws $\frac{256}{4}=64$ gallons, and 192 gallons remain. The second time he draws $\frac{192}{4}=48$ gallons, and 144 remain; and so on.

Prob. 14. The first remainder $=a-\frac{a}{n}=\frac{na-a}{n}=\frac{a(n-1)}{n};$

the second remainder $=\frac{a(n-1)}{n}-\frac{a(n-1)}{n^2}=\frac{a(n-1)^2}{n^2};$

the third remainder $=\frac{a(n-1)^2}{n^2}-\frac{a(n-1)^2}{n^3}=\frac{a(n-1)^3}{n^3};$

the fourth remainder $=\frac{a(n-1)^4}{n^4}=b.$

Extracting the fourth root, $(n-1)a^{\frac{1}{4}}=nb^{\frac{1}{4}}.$

Hence $$n=\frac{a^{\frac{1}{4}}}{a^{\frac{1}{4}}-b^{\frac{1}{4}}}.$$

Prob. 15. Let x denote the number of days engaged.

A worked $x-4$ days, and his daily wages were $\frac{75}{x-4};$

B " $x-7$ days, " " " " $\frac{48}{x-7}.$

$75\left(\frac{x-7}{x-4}\right)=$ A's wages if he had played 7 days;

$48\left(\frac{x-4}{x-7}\right)=$ B's wages " " only 4 days.

Hence $$75\left(\frac{x-7}{x-4}\right)=48\left(\frac{x-4}{x-7}\right).$$

Clearing of fractions,

$$25(x-7)^2=16(x-4)^2.$$

Extracting the square root,

$$5(x-7)=4(x-4).$$

Reducing, $x=19.$

Prob. 16. A worked $x-a$ days, and his daily wages were $\frac{m}{x-a}$;

B " $x-b$ days, " " " " $\frac{n}{x-b}$.

Hence $$m\left(\frac{x-b}{x-a}\right)=n\left(\frac{x-a}{x-b}\right).$$

Clearing of fractions,

$$m(x-b)^2=n(x-a)^2.$$

Extracting the square root,

$$(x-b)\sqrt{m}=(x-a)\sqrt{n}.$$

Expanding, $$x\sqrt{m}-b\sqrt{m}=x\sqrt{n}-a\sqrt{n}.$$

Hence $$x=\frac{b\sqrt{m}-a\sqrt{n}}{\sqrt{m}-\sqrt{n}}.$$

Art. 259, Page 184.

Ex. 2. $$2x^2+8x=90.$$

Completing the square,

$$x^2+4x+4=45+4.$$

Extracting the square root,

$$x+2=\pm 7.$$

Hence $$x=+5 \text{ or } -9.$$

Art. 260, Page 185.

Ex. 6. Multiply by 28,

$$196x^2-84x+9=4480+9.$$

Extracting the square root,

$$14x-3=\pm 67.$$

Transposing, $$14x=+70 \text{ or } -64.$$

Hence $$x=+5 \text{ or } -\frac{32}{7}.$$

ART. 261, PAGE 186.

Ex. 9. Multiply by 12, $144x^2-3x=252$.

Completing the square,

$$144x^2-3x+\frac{1}{64}=\frac{16,128}{64}+\frac{1}{64}.$$

Extracting the square root,

$$12x-\frac{1}{8}=\pm\frac{127}{8}.$$

Transposing $\quad 12x=+16 \text{ or } -\frac{63}{4}.$

Reducing, $\quad x=+\frac{4}{3} \text{ or } -\frac{21}{16}.$

Ex. 10. Clearing of fractions,

$$3x^2-2x=133.$$

Completing the square,

$$9x^2-6x+1=399+1.$$

Extracting square root,

$$3x-1=\pm20.$$

Reducing, $\quad x=7 \text{ or } -\frac{19}{3}.$

Ex. 11. Completing the square,

$$x^2-x+\tfrac{1}{4}=\tfrac{841}{4}.$$

Extracting square root,

$$x-\tfrac{1}{2}=\pm\tfrac{29}{2}.$$

Transposing, $\quad x=15 \text{ or } -14.$

Ex. 12. Transposing, $\quad 3x^2+2x=85.$

Completing the square,

$$9x^2+6x+1=255+1.$$

Extracting square root,

$$3x+1=\pm16.$$

Reducing, $\quad x=5 \text{ or } -\frac{17}{3}.$

Ex. 13. Completing the square,

$$9x^2-6x+1=\tfrac{45}{4}+1.$$

Extracting square root,

$$3x-1=\pm\tfrac{7}{2}.$$

Reducing, $\quad x=\frac{3}{2} \text{ or } -\frac{5}{6}.$

Ex. 14. Multiply by $2x-1$, and we have

$$6x^2-3x-6x^2+40-\frac{6x^2-23x+10}{9-2x}=4x-2.$$

Transposing, $42-7x=\frac{6x^2-23x+10}{9-2x}$.

Clearing of fractions,

$$378-147x+14x^2=6x^2-23x+10.$$

Transposing, $8x^2-124x=-368$.

Dividing, $x^2-\frac{31x}{2}=-46$.

Completing the square,

$$x^2-\frac{31x}{2}+\frac{961}{16}=\frac{961}{16}-46=\frac{225}{16}.$$

Extracting square root,

$$x-\tfrac{31}{4}=\pm\tfrac{15}{4}.$$

Transposing, $x=\frac{23}{2}$ or 4.

Ex. 15. Clearing of fractions,

$$30x-72+6x=4x^2.$$

Transposing, $4x^2-36x=-72$.

Completing the square,

$$x^2-9x+\tfrac{81}{4}=\tfrac{81}{4}-\tfrac{72}{4}=\tfrac{9}{4}.$$

Extracting square root,

$$x=\tfrac{9}{2}\pm\tfrac{3}{2}=6 \text{ or } 3.$$

Ex. 16. Reducing, $\frac{10}{x}-\frac{10}{x+1}=\frac{3}{x+2}$.

Clearing of fractions,

$$10x^2+30x+20-10x^2-20x=3x^2+3x.$$

Reducing, $3x^2-7x=20$.

Completing the square,

$$36x^2-84x+49=240+49.$$

Extracting square root,

$$6x-7=\pm 17.$$

Reducing, $x=4$ or $-\frac{5}{3}$.

Ex. 17. Transposing,

$$x^2-x(\sqrt{3}+1)=-\tfrac{1}{2}\sqrt{3}.$$

Completing the square,

$$x^2-x(\sqrt{3}+1)+\left(\frac{\sqrt{3}+1}{2}\right)^2=\frac{3+2\sqrt{3}+1-2\sqrt{3}}{4}=1.$$

Extracting square root,

$$x=\frac{\sqrt{3}+1}{2}\pm 1=\frac{\sqrt{3}+3}{2}\text{ or }\frac{\sqrt{3}-1}{2}.$$

Ex. 18. Clearing of fractions,

$$3x^2-15x+12-3x^2+15x-18=-2x^2+12x-16.$$

Transposing, $2x^2-12x=-10.$

Completing the square,

$$x^2-6x+9=4.$$

Extracting square root,

$$x=3\pm 2=5\text{ or }1.$$

Ex. 19. Clearing of fractions,

$$2x^2+2a^2+4ax+2x^2=5ax+5x^2.$$

Transposing, $x^2+ax=2a^2.$

Completing the square,

$$x^2+ax+\frac{a^2}{4}=\frac{9a^2}{4}.$$

Extracting square root,

$$x=-\frac{a}{2}\pm\frac{3a}{2}=a\text{ or }-2a.$$

Ex. 20. Completing the square,

$$x^2-(a+b)x+\left(\frac{a+b}{2}\right)^2=\frac{a^2+2ab+b^2-4ab}{4}.$$

Extracting square root,

$$x=\frac{a+b}{2}\pm\frac{a-b}{2}=a\text{ or }b.$$

Ex. 21. Since the product of the two factors is zero, at least one of the factors must be zero.

If $3x-25=0$, then $x=8\frac{1}{3}$;

if $7x+29=0$, then $x=-\frac{29}{7}$.

These values may be found by the usual method.

Expanding, we have

$$21x^2-88x=725.$$

Completing the square,

$$x^2-\frac{88x}{21}+\left(\frac{44}{21}\right)^2=\frac{1936}{441}+\frac{15,225}{441}.$$

Extracting square root,

$$x=\frac{44}{21}\pm\frac{131}{21}=8\tfrac{1}{3} \text{ or } -4\tfrac{1}{7}.$$

Ex. 22. Clearing of fractions,

$$27x^2-36x+12+12x^2-60x+75=60x^2-190x+100.$$

Reducing, $21x^2-94x=-13.$

Completing the square,

$$x^2-\frac{94x}{21}+\left(\frac{47}{21}\right)^2=\frac{2209}{441}-\frac{273}{441}.$$

Extracting square root,

$$x=\frac{47}{21}\pm\frac{44}{21}=\frac{13}{3} \text{ or } \frac{1}{7}.$$

Ex. 23. Performing the multiplication,

$$x^2-3x+2+x^2-6x+8=12x-30.$$

Transposing, $2x^2-21x=-40.$

Completing the square,

$$x^2-\frac{21x}{2}+\left(\frac{21}{4}\right)^2=\frac{441}{16}-\frac{320}{16}.$$

Extracting square root,

$$x=\frac{21}{4}\pm\frac{11}{4}=8 \text{ or } \frac{5}{2}.$$

Ex. 24. Clearing of fractions,

$$170(x^2+3x+2)-170(x^2+2x)=51x^2+51x.$$

Reducing, $170(x+2)=51x^2+51x.$

Transposing, $51x^2-119x=340.$

Reducing, $x^2-\frac{7x}{3}=\frac{20}{3}.$

Completing the square,

$$x^2-\frac{7x}{3}+\left(\frac{7}{6}\right)^2=\frac{49}{36}+\frac{240}{36}.$$

Extracting square root,

$$x=\frac{7}{6}\pm\frac{17}{6}=4 \text{ or} -\frac{5}{3}.$$

Ex. 25. Multiplying,

$$a^3-x^3+a^3+x^3=\frac{ab(a^2-x^2)}{3a-4b+x}.$$

Reducing, $$2a^2=\frac{b(a^2-x^2)}{3a-4b+x}.$$

Clearing of fractions,

$$6a^3-8a^2b+2a^2x=a^2b-bx^2.$$

Transposing, $$bx^2+2a^2x=9a^2b-6a^3.$$

Dividing, $$x^2+\frac{2a^2x}{b}=\frac{9a^2b-6a^3}{b}.$$

Completing the square,

$$x^2+\frac{2a^2x}{b}+\frac{a^4}{b^2}=\frac{a^2(a^2-6ab+9b^2)}{b^2}.$$

Extracting square root,

$$x=-\frac{a^2}{b}\pm\frac{a^2-3ab}{b}=-3a \text{ or } 3a-\frac{2a^2}{b}.$$

ART. 262, PAGE 187.

Ex. 1. Completing the square,

$$y^2-13y+\left(\frac{13}{2}\right)^2=\frac{169}{4}-\frac{144}{4}.$$

Extracting square root,

$$y=\frac{13}{2}\pm\frac{5}{2}=9 \text{ or } 4.$$

Ex. 2. Completing the square,

$$y^2-35y+\left(\frac{35}{2}\right)^2=\frac{1225}{4}-\frac{864}{4}.$$

Extracting square root,

$$y=\frac{35}{2}\pm\frac{19}{2}=27 \text{ or } 8.$$

Ex. 3. Completing the square,

$$y^2+4y+4=25.$$

Extracting square root,

$$y=-2\pm5=3 \text{ or } -7.$$

Ex. 4. Completing the square,

$$y^2-26y+169=49.$$

Extracting square root,

$$y=13\pm7=20 \text{ or } 6.$$

Completing the square,

$$x^2+x+\tfrac{1}{4}=20\tfrac{1}{4}.$$

Extracting square root,

$$x=-\tfrac{1}{2}\pm\tfrac{9}{2}=4 \text{ or } -5.$$

Completing the square,

$$x^2+x+\tfrac{1}{4}=6\tfrac{1}{4}.$$

Extracting square root,

$$x=-\tfrac{1}{2}\pm\tfrac{5}{2}=2 \text{ or } -3.$$

Ex. 5. By substitution, $y^2+y=6.$

Completing the square,

$$y^2+y+\tfrac{1}{4}=6\tfrac{1}{4}.$$

Extracting square root,

$$y=-\tfrac{1}{2}\pm\tfrac{5}{2}=2 \text{ or } -3.$$

By substitution, $\sqrt[4]{x+12}=2 \text{ or } -3.$

Involving, $x+12=16 \text{ or } 81.$

Ex. 6. Completing the square,

$$y^2+y+\tfrac{1}{4}=\tfrac{49}{4}.$$

Extracting square root,

$$y=-\tfrac{1}{2}\pm\tfrac{7}{2}=3 \text{ or } -4.$$

By substitution, $\sqrt{2x^2+1}=3 \text{ or } -4.$

By involution, $2x^2+1=9 \text{ or } 16.$

Reducing, $x^2=4 \text{ or } 7\tfrac{1}{2}.$

Ex. 7. Put $x^2-6x=y.$

Completing the square,

$$y^2+8y+16=9009+16.$$

Extracting square root,

$$y=-4\pm95=91 \text{ or } -99.$$

By substitution, $x^2-6x=91 \text{ or } -99.$

Completing the square,

$$x^2-6x+9=100.$$

Extracting square root,

$$x=3\pm10=13 \text{ or } -7.$$

Also, $x^2-6x+9=9-99=-90.$

Extracting the square root,

$$x=3\pm\sqrt{-90}.$$

Ex. 8. Completing the square,

$$(x^2-x)^2-(x^2-x)+\tfrac{1}{4}=132\tfrac{1}{4}.$$

Extracting the square root,

$$x^2-x=\tfrac{1}{2}\pm\tfrac{23}{2}=12 \text{ or } -11.$$

Completing the square,

$$x^2-x+\tfrac{1}{4}=\tfrac{49}{4}.$$

Extracting the square root,

$$x=\tfrac{1}{2}\pm\tfrac{7}{2}=4 \text{ or } -3.$$

Also, $x^2-x+\tfrac{1}{4}=-11+\tfrac{1}{4}.$

Extracting the square root,

$$x=\tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{-43}.$$

Ex. 9. Completing the square,

$$x^4+4x^2+4=16.$$

Extracting the square root,

$$x^2=-2\pm4=2 \text{ or } -6.$$

Hence $x=\pm\sqrt{2} \text{ or } \pm\sqrt{-6}.$

Ex. 10. Completing the square,

$$x^6-8x^3+16=513+16.$$

Extracting the square root,

$$x^3=4\pm23=27 \text{ or } -19.$$

Hence $x=3 \text{ or } -\sqrt[3]{19}.$

Ex. 11. Completing the square,

$$x^{\frac{6}{5}}+x^{\frac{3}{5}}+\tfrac{1}{4}=756\tfrac{1}{4}.$$

Extracting the square root,

$$x^{\frac{3}{5}}=-\tfrac{1}{2}\pm\tfrac{55}{2}=27 \text{ or } -28.$$

Extracting the cube root,

$$x^{\frac{1}{5}}=3 \text{ or } -\sqrt[3]{28}.$$

Involving, $x=3^5 \text{ or } -\sqrt[3]{28^5}.$

Ex. 12. Multiplying, $x^6-\dfrac{x^3}{2}=-\dfrac{1}{16}.$

Completing the square,

$$x^6-\frac{x^3}{2}+\frac{1}{16}=0.$$

Extracting the square root,

$$x^3=\tfrac{1}{4}.$$

Extracting the cube root,

$$x=\sqrt[3]{\tfrac{1}{4}}=\tfrac{1}{2}\sqrt[3]{2}.$$

Ex. 13. Completing the square,

$$x^{\frac{2}{3}}+\frac{3x^{\frac{1}{3}}}{2}+\frac{9}{16}=1+\frac{9}{16}.$$

Extracting the square root,

$$x^{\frac{1}{3}}=-\tfrac{3}{4}\pm\tfrac{5}{4}=\tfrac{1}{2} \text{ or } -2.$$

Involving, $x=\tfrac{1}{8}$ or -8.

Ex. 14. Completing the square,

$$x-\frac{2\sqrt{x}}{3}+\tfrac{1}{9}=44\tfrac{1}{3}+\tfrac{1}{9}.$$

Extracting the square root,

$$\sqrt{x}=\tfrac{1}{3}\pm\tfrac{20}{3}=7 \text{ or } -\tfrac{19}{3}.$$

Involving, $x=49$ or $\tfrac{361}{9}$.

Ex. 15. Completing the square,

$$\sqrt{10+x}-\sqrt[4]{10+x}+\tfrac{1}{4}=2\tfrac{1}{4}.$$

Extracting the square root,

$$\sqrt[4]{10+x}=\tfrac{1}{2}\pm\tfrac{3}{2}=2 \text{ or } -1.$$

Involving, $10+x=16$ or 1.

Hence $x=6$ or -9.

Ex. 16. Completing the square,

$$x^6+20x^3+100=169.$$

Extracting the square root,

$$x^3=-10\pm13=3 \text{ or } -23.$$

Extracting the cube root,

$$x=\sqrt[3]{3} \text{ or } -\sqrt[3]{23}.$$

Ex. 17. Completing the square,

$$x^{2n}-\frac{2x^n}{3}+\frac{1}{9}=\frac{8}{3}+\frac{1}{9}.$$

Extracting the square root,

$$x^n=\tfrac{1}{3}\pm\tfrac{5}{3}=2 \text{ or } -\tfrac{4}{3}.$$

Ex. 18. Put $\sqrt{1+x-x^2}=y.$

Completing the square,

$$y^2-\frac{y}{2}+\frac{1}{16}=\frac{1}{16}-\frac{1}{18}=\frac{1}{144}.$$

Extracting the square root,

$$y=\tfrac{1}{4}\pm\tfrac{1}{12}=\tfrac{1}{3} \text{ or } \tfrac{1}{6}.$$

Squaring, $1+x-x^2=\tfrac{1}{9} \text{ or } \tfrac{1}{36}.$

Completing the square,

$$x^2-x+\tfrac{1}{4}=\tfrac{8}{9}+\tfrac{1}{4}=\tfrac{41}{36}.$$

Extracting the square root,

$$x=\tfrac{1}{2}\pm\tfrac{1}{6}\sqrt{41}.$$

Also, $x^2-x+\tfrac{1}{4}=\tfrac{35}{36}+\tfrac{1}{4}=\tfrac{11}{9}.$

Extracting the square root,

$$x=\tfrac{1}{2}\pm\tfrac{1}{3}\sqrt{11}.$$

Ex. 19. Completing the square,

$$\sqrt{x}+\sqrt[4]{x}+\tfrac{1}{4}=\tfrac{81}{4}.$$

Extracting the square root,

$$\sqrt[4]{x}=-\tfrac{1}{2}\pm\tfrac{9}{2}=4 \text{ or } -5.$$

Involving, $x=256 \text{ or } 625.$

Ex. 20. Completing the square,

$$(x^2-2x)^2+3(x^2-2x)+\tfrac{9}{4}=18+\tfrac{9}{4}.$$

Extracting the square root,

$$x^2-2x=-\tfrac{3}{2}\pm\tfrac{9}{2}=3 \text{ or } -6.$$

Completing the square,

$$x^2-2x+1=4.$$

Extracting the square root,

$$x=1\pm2=3 \text{ or } -1.$$

Also, $x^2-2x+1=-6+1.$

Extracting the square root,

$$x=1\pm\sqrt{-5}.$$

Ex. 21. Put $x^2+5x+28=y^2.$

By substitution, $y^2-5y=24.$

Completing the square,

$$y=\frac{5}{2}\pm\frac{11}{2}=8 \text{ or } -3.$$

Squaring, $x^2+5x+28=64$ or 9.

Completing the square,

$$x^2+5x+\frac{25}{4}=36+\frac{25}{4}.$$

Extracting the square root,

$$x=-\frac{5}{2}\pm\frac{13}{2}=4 \text{ or } -9.$$

Also, $x^2+5x+\frac{25}{4}=-19+\frac{25}{4}=-\frac{51}{4}$.

Extracting the square root,

$$x=-\frac{5}{2}\pm\frac{1}{2}\sqrt{-51}.$$

Ex. 22. Put $x^2-2x+2=y^2$.

By substitution, $y^2-2y+1=0$.

Extracting the square root,

$$y=1.$$

By substitution, $x^2-2x+2=1$.

Transposing, $x^2-2x+1=0$.

Extracting the square root,

$$x=1.$$

Ex. 23. By substitution, $y^4-y^2=20{,}592$.

Completing the square, $y^4-y^2+\frac{1}{4}=\frac{82{,}369}{4}$.

Extracting the square root,

$$y^2=\frac{1}{2}\pm\frac{287}{2}=144 \text{ or } -143.$$

Extracting the square root,

$$y=\pm 12.$$

By substitution, $x+\sqrt{x}=12$.

Completing the square,

$$x+\sqrt{x}+\frac{1}{4}=\frac{49}{4}.$$

Extracting the square root,

$$\sqrt{x}=-\frac{1}{2}\pm\frac{7}{2}=3 \text{ or } -4.$$

Squaring, $x=9$ or 16.

Ex. 24. Transposing, $25+x+\sqrt{25+x}=182$.

Completing the square,

$$y^2+y+\frac{1}{4}=\frac{729}{4}.$$

Extracting the square root,

$$y=-\tfrac{1}{2}\pm\tfrac{27}{2}=13 \text{ or } -14.$$

By substitution, $\sqrt{25+x}=13$ or -14.

Squaring, $25+x=169$ or 196.

Hence $x=144$ or 171.

Ex. 25. Squaring, we have

$$x-1=x^2-2x+1.$$

Completing the square,

$$x^2-3x+\tfrac{9}{4}=-2+\tfrac{9}{4}=\tfrac{1}{4}.$$

Extracting the square root,

$$x=\tfrac{3}{2}\pm\tfrac{1}{2}=2 \text{ or } 1.$$

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Prob. 2. Let x denote the breadth.

Then will $x+60$ " the length.

By the conditions, $x^2+60x=5500$.

Completing the square,

$$x=-30\pm80=50 \text{ or } -110.$$

Hence $x+60=110$ or -50.

The negative value satisfies the equation, but not the conditions of the problem.

Prob. 3. Let x denote the less number.

Then will $x+2a$ " the greater number.

By the conditions, $x^2+2ax+a^2=a^2+b$.

Extracting the square root,

$$x=-a\pm\sqrt{a^2+b}.$$

Hence $x+2a=a\pm\sqrt{a^2+b}$.

Prob. 4. Completing the square,

$$x^2-60x+900=36.$$

Extracting the square root,

$$x=30\pm6=36 \text{ or } 24.$$

Prob. 5. Let x denote the number of coins of one kind.

Then will $52-x$ " " of coins of the other kind.

By the conditions, $52x-x^2=100$.

Completing the square,

$$x^2-52x+26^2=676-100.$$

Extracting the square root,

$$x=26\pm24=50 \text{ or } 2;$$

that is, there were 2 coins worth 50 cents each, and 50 coins worth 2 cents each.

Prob. 6. Let x denote one of the numbers.

Then will $2a-x$ " the other number.

By the conditions, $2ax-x^2=b$.

Completing the square,

$$x=a\pm\sqrt{a^2-b}.$$

Hence $2a-x=a\mp\sqrt{a^2-b}$.

Prob. 8. Since the difference of the squares is 5, the sum of the squares must be $\frac{65}{5}$ or 13.

Hence if x denote the less number, $13-x^2$ will denote the square of the greater.

By the conditions, $13-2x^2=5$.

Transposing, $x^2=4=$the square of the less.

Hence $13-x^2=9=$the square of the greater.

The two numbers are therefore 2 and 3.

Prob. 9. Let x^2 denote the square of the less.

Then will $m-x^2$ " " of the greater.

By the conditions, $m-2x^2=a$.

Hence $x^2=\frac{m-a}{2}$.

Extracting the square root,

$$x=\sqrt{\frac{m-a}{2}}=\text{the less number.}$$

Also, $m-x^2=\frac{m+a}{2}$.

Hence $\sqrt{\frac{m+a}{2}}=$the greater number.

Prob. 10. Let x denote the length of one trench.

Then will $26-x$ " " of the other trench.

By the conditions, $(26-x)^2+x^2=356$.

Reducing, $2x^2-52x=-320.$

Completing the square,

$$x^2-26x+169=9.$$

Extracting the square root,

$$x=13\pm3=16 \text{ or } 10.$$

Prob. 11. Denote the two numbers by $x+a$ and $x-a$.

By the conditions, $2x^2+2a^2=2b.$

Reducing, $x=\sqrt{b-a^2}.$

Also, $x+a=a+\sqrt{b-a^2}.$

Prob. 12. Clearing of fractions,

$$80x+320=80x+x^2+4x.$$

Completing the square,

$$x^2+4x+4=324.$$

Extracting the square root,

$$x=-2\pm18=16 \text{ or } -20.$$

Prob. 13. Let x denote the number of articles.

Then will $\frac{a}{x}$ " the price of each.

By the conditions, $\frac{a}{x}=\frac{a}{x+2b}+c.$

Clearing of fractions,

$$ax+2ab=ax+cx^2+2bcx.$$

Completing the square,

$$x^2+2bx+b^2=\frac{2ab}{c}+b^2.$$

Extracting the square root,

$$x=-b\pm\sqrt{\frac{2ab+b^2c}{c}}.$$

Prob. 14. Let x denote the first number.

Then will $\frac{15}{x}$ " the second number,

and $\frac{21}{x}$ " the third number.

By the conditions, $\frac{225}{x^2}+\frac{441}{x^2}=74.$

Clearing of fractions, $74x^2=666$.
Reducing, $x^2=9$.
Extracting the square root,
$x=\pm 3$, the first number.

Hence $\frac{15}{\pm 3}=\pm 5$, the second number.

Also, $\frac{21}{\pm 3}=\pm 7$, the third number.

Prob. 15. Denote the three numbers by x, $\frac{a}{x}$, and $\frac{b}{x}$.

By the conditions, $\frac{a^2}{x^2}+\frac{b^2}{x^2}=c$.

Reducing, $x=\pm\sqrt{\frac{a^2+b^2}{c}}$.

Hence $\frac{a}{x}=a\sqrt{\frac{c}{a^2+b^2}}$.

Also, $\frac{b}{x}=b\sqrt{\frac{c}{a^2+b^2}}$.

Prob. 16. Let $8+x$ and $8-x$ denote the two numbers.
By the conditions, $1024+48x^2=1072$.
Reducing, $x^2=1$.
Extracting the square root,
$x=\pm 1$.
Hence $8\pm x=7$ or 9.

Prob. 17. Let $a+x$ and $a-x$ denote the two numbers.
By the conditions, $2a^3+6ax^2=2b$.

Reducing, $x=\sqrt{\frac{b-a^3}{3a}}$.

Hence $a\pm x=a+\sqrt{\frac{b-a^3}{3a}}$ or $a-\sqrt{\frac{b-a^3}{3a}}$.

Prob. 18. Let x denote the distance of the needle from the weakest magnet. Then will $20-x$ denote its distance from the strongest magnet.

By the conditions, $\frac{4}{x^2}=\frac{9}{(20-x)^2}$.

Extracting the square root,

$$2(20-x)=\pm 3x.$$

Hence $$40-2x=\pm 3x;$$

$$x=8 \text{ or } -40.$$

Prob. 19. By the conditions, $\frac{m}{x^2}=\frac{n}{(a-x)^2}$.

Extracting the square root,

$$(a-x)\sqrt{m}=\pm x\sqrt{n}.$$

Reducing, $$x=\frac{a\sqrt{m}}{\sqrt{m}\pm\sqrt{n}}.$$

Also, $$a-x=\frac{\pm a\sqrt{n}}{\sqrt{m}\pm\sqrt{n}}.$$

Prob. 20. Let x denote the distance from C to D. Then will $\frac{x}{20}$ denote the number of miles B traveled per hour, and also the number of hours he traveled before he met A. Hence $\left(\frac{x}{20}\right)^2$ denotes the distance B traveled; and $\frac{6x}{20}+45$ denotes the distance A traveled.

By the conditions, $\frac{6x}{20}+45+\frac{x^2}{400}=x.$

Reducing, $$x^2-280x=-18,000.$$

Completing the square,

$$x^2-280x+140^2=1600.$$

Extracting the square root,

$$x=140\pm 40=180 \text{ or } 100.$$

Prob. 21. Let x denote the distance from C to D.

Then will $\frac{x}{n}$ " the number of miles B traveled per hour.

By the conditions, $\frac{ax}{n}+b+\frac{x^2}{n^2}=x.$

Completing the square,

$$x^2+(an-n^2)x+\left(\frac{an-n^2}{2}\right)^2=\left(\frac{an-n^2}{2}\right)^2-bn^2.$$

Extracting the square root,

$$x=-\frac{an-n^2}{2}\pm\sqrt{\left(\frac{an-n^2}{2}\right)^2-bn^2},$$

or $$x=n\left\{\frac{n-a}{2}\pm\sqrt{\left(\frac{n-a}{2}\right)^2-b}\right\}.$$

Prob. 22. Let x denote the cost of the horse.

The loss in selling was $x\times\frac{x}{100}$.

By the conditions, $\frac{x^2}{100}=x-24$.

Completing the square,

$$x^2-100x+2500=100.$$

Extracting the square root,

$$x=50\pm10=60 \text{ or } 40.$$

Prob. 23. Let x denote the number of melons less than 18.

Then $18+x$ denotes the price of each;

$18-x$ " the number of melons.

By the conditions, $324-x^2=315$.

Reducing, $x^2=9$.

Extracting the square root,

$$x=\pm3.$$

Hence $18\pm3=21$ or 15.

Prob. 24. Let x denote the length of the produced part.

By the conditions, $\frac{a}{2}\left(x+\frac{a}{2}\right)=x^2$.

Reducing, $4x^2-2ax=a^2$.

Completing the square,

$$x^2-\frac{ax}{2}+\frac{a^2}{16}=\frac{5a^2}{16}.$$

Extracting the square root,

$$x=\frac{a}{4}\pm\frac{a}{4}\sqrt{5}.$$

ART. 268, PAGE 197.

Ex. 1. Reducing, $3y^2+7y=26.$

Completing the square,

$$y^2+\frac{7y}{3}+\left(\frac{7}{6}\right)^2=\frac{49}{36}+\frac{312}{36}.$$

Extracting the square root,

$$y=-\tfrac{7}{6}\pm\tfrac{19}{6}=2 \text{ or } -\tfrac{13}{3}.$$

Hence $x=7-2y=3$ or $\tfrac{47}{3}.$

Ex. 2. From Eq. (2), $x=\dfrac{1+3y}{2}.$

Substituting in (1),

$$\frac{1+6y+9y^2}{2}+\frac{y+3y^2}{2}-5y^2=20.$$

Reducing, $2y^2+7y=39.$

Completing the square,

$$y^2+\frac{7y}{2}+\left(\frac{7}{4}\right)^2=\frac{49}{16}+\frac{39}{2}.$$

Extracting the square root,

$$y=-\tfrac{7}{4}\pm\tfrac{19}{4}=3 \text{ or } -\tfrac{13}{2}.$$

Hence $x=5$ or $-\tfrac{37}{4}.$

Ex. 3. From Eq. (2), $y=x+2.$

Substituting in (1), $10x+x+2=3x^2+6x.$

Reducing, $3x^2-5x=2.$

Completing the square,

$$x^2-\frac{5x}{3}+\left(\frac{5}{6}\right)^2=\frac{25}{36}+\frac{2}{3}.$$

Extracting the square root,

$$x=\tfrac{5}{6}\pm\tfrac{7}{6}=2 \text{ or } -\tfrac{1}{3}.$$

Hence $y=4$ or $\tfrac{5}{3}.$

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Ex. 4. Clearing of fractions,

$$12v-24=v^2+v.$$

Completing the square,

$$v^2-11v+(\tfrac{11}{2})^2=\tfrac{121}{4}-\tfrac{96}{4}.$$

Extracting the square root,

$$v=\tfrac{11}{2}\pm\tfrac{5}{2}=3 \text{ or } 8.$$

Hence $y^2=1$ or $\frac{1}{8}$.

Extracting the square root,

$$y=\pm 1 \text{ or } \pm\sqrt{\tfrac{1}{8}}.$$

Also, $x=vy=\pm 3$ or $\pm\dfrac{8}{\sqrt{6}}$.

Ex. 5. By substitution, $\dfrac{77}{v^2+v}=\dfrac{12}{v-1}$.

Clearing of fractions,

$$12v^2-65v=-77.$$

Completing the square,

$$v^2-\frac{65v}{12}+\left(\frac{65}{24}\right)^2=\frac{4225}{576}-\frac{77}{12}.$$

Extracting the square root,

$$v=\tfrac{65}{24}\pm\tfrac{23}{24}=\tfrac{7}{4} \text{ or } \tfrac{11}{3}.$$

Hence $y^2=16$ or $\frac{9}{2}$.

Extracting the square root,

$$y=\pm 4 \text{ or } \pm\frac{3}{\sqrt{2}}.$$

Also, $x=vy=\pm 7$ or $\pm\dfrac{11}{\sqrt{2}}$.

Ex. 6. By substitution,

$$\frac{20}{2v^2+3v+1}=\frac{41}{5v^2+4}.$$

Clearing of fractions,

$$18v^2-123v=-39.$$

Completing the square,

$$v^2-\frac{41v}{6}+\left(\frac{41}{12}\right)^2=\frac{1681}{144}-\frac{312}{144}.$$

Extracting the square root,

$$v=\tfrac{41}{12}\pm\tfrac{37}{12}=\tfrac{1}{3} \text{ or } \tfrac{13}{2}$$

Hence $y^2=9$ or $\frac{4}{21}$.

Extracting the square root,

$$y=\pm 3 \text{ or } \pm\frac{2}{\sqrt{21}}.$$

Also, $x=vy=\pm1$ or $\pm\frac{13}{\sqrt{21}}$.

Ex. 8. Assume $x=4+v$ and $y=4-v$.

By substitution,

$$x^5+y^5=2048+1280v^2+40v^4=3368.$$

Reducing, $v^4+32v^2=33$.

Completing the square,

$$v^4+32v^2+16^2=256+33.$$

Extracting the square root,

$$v^2=-16\pm17=1 \text{ or } -33.$$

Extracting the square root,

$$v=\pm1.$$

Hence $x=4\pm1=5$ or 3.

Also, $y=4\mp1=3$ or 5.

Ex. 9. Assume $x=z+v$ and $y=z-v$.

By substitution, $2z^3+6zv^2=341$, (1)

and $2z^3-2zv^2=330$. (2)

Subtract (2) from (1), $8zv^2=11$. (3)

Substitute (3) in (1), $2z^3+\frac{33}{4}=341$.

Clearing of fractions, $8z^3=1331$.

Hence $z=\frac{11}{2}$.

From (3), $44v^2=11$.

Hence $v=\pm\frac{1}{2}$;

$$x=z+v=\tfrac{11}{2}\pm\tfrac{1}{2}=6 \text{ or } 5;$$

$$y=z-v=\tfrac{11}{2}\mp\tfrac{1}{2}=5 \text{ or } 6.$$

Ex. 10. We find $x=10-y$.

By substitution, $10y-y^2-y^2=8$.

Completing the square,

$$y^2-5y+(\tfrac{5}{2})^2=\tfrac{25}{4}-4.$$

Extracting the square root,

$$y=\tfrac{5}{2}\pm\tfrac{3}{2}=4 \text{ or } 1.$$

Hence $x=10-y=6$ or 9.

Also, $x=-12-y$.

By substitution, $-12y-y^2-y^2=8$.

Completing the square,

$$y^2+6y+9=9-4.$$

Extracting the square root,

$$y=-3\pm\sqrt{5}.$$

Hence $$x=-12-y=-9\mp\sqrt{5}.$$

Ex. 11. We have $$x=6-y.$$

Hence $$xy=6y-y^2=8.$$

Completing the square,

$$y^2-6y+9=1.$$

Extracting the square root,

$$y=3\pm1=4 \text{ or } 2.$$

Hence $$x=6-y=2 \text{ or } 4.$$

Also, $$xy=6y-y^2=-12.$$

Completing the square,

$$y^2-6y+9=21.$$

Extracting the square root,

$$y=3\pm\sqrt{21}.$$

Hence $$x=6-y=3\mp\sqrt{21}.$$

Ex. 12. We find $$\frac{x}{y}=\frac{5}{3}.$$

Hence $$x=\frac{5y}{3}=y+2;$$

$$y=3 \text{ and } x=5.$$

Also, $$\frac{x}{y}=-\frac{17}{3}.$$

Hence $$x=-\frac{17y}{3}=y+2.$$

Therefore $$y=-\tfrac{3}{10} \text{ and } x=\tfrac{17}{10}.$$

Ex. 13. Squaring Eq. (2),

$$x^2+2xy+y^2=4.$$

Add this to Eq. (1), $$2x^2=5.$$

Hence $$x=\pm\sqrt{\tfrac{5}{2}}.$$

Also, $$y=2-x=2\mp\sqrt{\tfrac{5}{2}}.$$

Ex. 14. Clearing of fractions, we have

$$bx+ay=ab; \quad (3)$$

$$bx+ay=4xy=ab. \quad (4)$$

Hence $x=\frac{ab}{4y}.$

By substitution in (3), $\frac{ab^2}{4y}+ay=ab.$

Reducing, $y^2-by+\frac{b^2}{4}=0.$

Extracting the square root,

$$y=\frac{b}{2}.$$

Hence $x=\frac{ab}{2b}=\frac{a}{2}.$

Ex. 15. Put $x=z+v$ and $y=z-v.$

From Eq. (1), $3z^2+v^2=84.$ (3)

From Eq. (2), $2z+\sqrt{z^2-v^2}=14.$ (4)

Squaring (4), $z^2-v^2=196-56z+4z^2.$

Transposing, $3z^2+v^2=56z-196=84.$

Hence $z=5.$

From Eq. (3), $v^2=84-3z^2=9.$

Hence $v=\pm 3;$

$$x=5\pm 3=8 \text{ or } 2;$$

$$y=5\mp 3=2 \text{ or } 8.$$

Ex. 16. From Eq. (2), $xy=180.$ (3)

From Eq. (1), $y+x=\frac{xy}{5}=36.$ (4)

Hence $x=36-y.$

Substituting in (3), $36y-y^2=180.$

Completing the square,

$$y^2-36y+18^2=324-180.$$

Extracting the square root,

$$y=18\pm 12=30 \text{ or } 6.$$

Also, $x=36-y=6$ or $30.$

Ex. 17. Put $x=z+v$ and $y=z-v.$

Substitute in (2),

$$2z(z^2-v^2)=30 \quad \therefore z^2-v^2=\frac{15}{z}.$$

Substitute in (1), $2(z^2+v^2)(z^2-v^2)^2=468.$

By substitution, $2(z^2+v^2)\times\frac{225}{z^2}=468.$

Hence $225z^2+225v^2=234z^2.$

Reducing, $225v^2=9z^2.$

Extracting the square root,

$$15v=3z.$$

Hence $5v=z.$

Therefore $x=6v,\ y=4v,$ and $x=\frac{2y}{3}.$

Substitute in Eq. (2), $\left(\frac{2y}{3}+y\right)\frac{2y^2}{3}=30.$

Reducing, $10y^3=270.$

Extracting the cube root,

$$y=3 \text{ and } x=2.$$

Ex. 17. *Otherwise.* Squaring Eq. (2),

$$x^4y^2+2x^3y^3+x^2y^4=900. \quad (3)$$

Subtracting Eq. (1), $2x^3y^3=432.$ (4)

Hence $xy=6.$ (5)

Dividing (2) by (5), $x+y=5.$

Squaring, $x^2+2xy+y^2=25.$

Subtracting 4 times (5),

$$x^2-2xy+y^2=1.$$

Extracting the square root,

$$x-y=\pm 1.$$

Hence $x=3$ or 2;

$y=2$ or 3.

Ex. 18. Add twice (1) to (2),

$$x^2+2xy+y^2=b+2a.$$

Extracting the square root,

$$x+y=\pm\sqrt{b+2a}.$$

Subtract twice (1) from (2), and we obtain

$$x-y=\pm\sqrt{b-2a}.$$

Hence $2x=\pm\sqrt{b+2a}\pm\sqrt{b-2a};$

$2y=\pm\sqrt{b+2a}\mp\sqrt{b-2a}.$

Ex. 19. By substitution, $v^3+z^3=72;$ (3)

$v+z=6$ or $v=6-z.$ (4)

Substitute (4) in (3),

$$216-108z+18z^2=72.$$

Reducing, $z^2-6z=-8.$

Completing the square,

$$z^2-6z+9=1.$$

Extracting the square root,

$$z=3\pm1=4 \text{ or } 2.$$

Also, $v=6-z=2$ or 4;

$$x=2^3 \text{ or } 4^3;$$

$$y=4^3 \text{ or } 2^3.$$

Ex. 20. From Eq. (1), $(x+y)xy=20.$ (3)

From Eq. (2), $x+y=\frac{5xy}{4}.$ (4)

Substitute (4) in (3), $\frac{5}{4}x^2y^2=20.$ (5)

Extracting the square root,

$$xy=\pm4. \quad (6)$$

Combining (4) and (6), we find

$$x^2+2xy+y^2=25.$$

Also, $x^2-2xy+y^2=9.$

Hence $x+y=\pm5$;

$$x-y=\pm3.$$

Therefore $x=4$ or 1;

$$y=1 \text{ or } 4.$$

Ex. 21. From Eq. (2), $x^2+y^2=\frac{x^2y^2}{2}=8.$ (3)

Hence $xy=4.$ (4)

Add twice (4) to (1), $x^2+2xy+y^2=16.$

Extracting the square root,

$$x+y=\pm4.$$

Subtract twice (4) from (1),

$$x^2-2xy+y^2=0.$$

Extracting the square root,

$$x-y=0.$$

Hence $x=\pm2=y.$

Ex. 22. Divide (1) by (2),

$$x^4+x^3y+x^2y^2+xy^3+y^4=1031. \quad (3)$$

aise (2) to 4th power,

$$x^4-4x^3y+6x^2y^2-4xy^3+y^4=81. \quad (4)$$

ubtract (4) from (3), and divide by 5,

$$x^3y+xy^3-x^2y^2=190. \quad (5)$$

rom Eq. (2), $x^2+y^2=9+2xy$. (6)

Iultiply (6) by xy,

$$x^3y+xy^3=9xy+2x^2y^2. \quad (7)$$

ubstitute (7) in (5), $9xy+x^2y^2=190$.

Completing the square,

$$x^2y^2+9xy+\left(\frac{9}{2}\right)^2=\frac{81}{4}+\frac{760}{4}.$$

Extracting the square root,

$$xy=-\tfrac{9}{2}\pm\tfrac{29}{2}=10 \text{ or } -19.$$

From Eq. (2), $x^2-2xy+y^2=9$.

Hence $x^2+2xy+y^2=49$.

Extracting the square root,

$$x+y=\pm 7.$$

Therefore $2x=10$ or -4;

$$y=x-3=2 \text{ or } -5.$$

.23. Divide (1) by (2),

$$x^2-xy+y^2=19. \quad (3)$$

Add (3) to (2), $x^2+y^2=34$. (4)

Hence $xy=15$. (5)

Add (5) to (2), $x^2+2xy+y^2=64$.

Subtract 3 times (5) from (2),

$$x^2-2xy+y^2=4.$$

Extracting the square root,

$$x+y=\pm 8;$$

$$x-y=\pm 2.$$

Therefore $x=\pm 5$ or ± 3;

$$y=\pm 3 \text{ or } \pm 5.$$

x. 24. By substitution, $zv=80$. (3)

Also, $z+v=18 \quad \therefore z=18-v$. (4)

Substitute (4) in (3), $18v-v^2=80$.

Completing the square,

$$v^2-18v+81=1.$$

Extracting the square root,

$$v=9\pm1=10 \text{ or } 8.$$

Hence $z=8$ or 10.

Therefore $x=z-7=1$ or 3;

$$y=v-6=4 \text{ or } 2.$$

Prob. 1. Denote the two parts by x and $100-x$.

By the conditions, $\sqrt{x}+\sqrt{100-x}=14.$

Squaring, $x+100-x+2\sqrt{100x-x^2}=196.$

Reducing, $\sqrt{100x-x^2}=48.$

Squaring, $100x-x^2=2304.$

Completing the square,

$$x^2-100x+2500=196.$$

Extracting the square root,

$$x=50\pm14=64 \text{ or } 36.$$

Prob. 2. By the conditions,

$$\sqrt{x}+\sqrt{a-x}=b.$$

Squaring, $x+a-x+2\sqrt{ax-x^2}=b^2.$

Reducing, $2\sqrt{ax-x^2}=b^2-a.$

Squaring, $4ax-4x^2=b^4-2ab^2+a^2.$

Completing the square,

$$4x^2-4ax+a^2=2ab^2-b^4.$$

Extracting the square root,

$$2x-a=\pm\sqrt{2ab^2-b^4}.$$

Hence $x=\frac{a}{2}\pm\frac{b}{2}\sqrt{2a-b^2}.$

Prob. 3. Let $4+x$ and $4-x$ denote the two numbers.

By the conditions,

$$512+192x^2+2x^4=706.$$

Completing the square,

$$x^4+96x^2+48^2=2304+97.$$

Extracting the square root,

$$x^2+48=49.$$

Extracting the square root,

$$x=\pm1;$$

$4\pm1=5$ or 3, the two numbers.

Prob. 4. Let $a+x$ and $a-x$ denote the two numbers.

By the conditions, $2a^4+12a^2x^2+2x^4=2b$.

Completing the square,

$$x^4+6a^2x^2+9a^4=9a^4-a^4+b.$$

Extracting the square root,

$$x^2=-3a^2\pm\sqrt{8a^4+b}.$$

Hence $$a\pm x=a\pm\sqrt{-3a^2\pm\sqrt{8a^4+b}}.$$

Prob. 5. Let $3+x$ and $3-x$ denote the two numbers.

By the conditions, $486+540x^2+30x^4=1056$.

Completing the square,

$$x^4+18x^2+81=81+19.$$

Extracting the square root,

$$x^2=-9\pm10=1 \text{ or } -19.$$

Extracting the square root,

$$x=\pm1 \text{ or } \pm\sqrt{-19}.$$

Hence $3\pm x=4$ or 2, the required numbers.

Prob. 6. Let $a+x$ and $a-x$ denote the two numbers.

By the conditions, $2a^5+20a^3x^2+10ax^4=b$.

Completing the square,

$$x^4+2a^2x^2+a^4=\frac{b-2a^5}{10a}+a^4.$$

Extracting the square root,

$$x^2=\pm\sqrt{\frac{b}{10a}+\frac{4a^4}{5}}-a^2.$$

Hence $$a\pm x=a\pm\sqrt{\sqrt{\frac{b}{10a}+\frac{4a^4}{5}}-a^2}.$$

Prob. 7. Let x denote the greater number.

Then will $\frac{120}{x}$ denote the less number.

By the conditions, $(x+8)\left(\frac{120}{x}+5\right)=300$.

Completing the square,

$$x^2-28x+196=196-192.$$

Extracting the square root,

$$x=14\pm2=16 \text{ or } 12.$$

Hence $$\frac{120}{x}=7\tfrac{1}{2} \text{ or } 10.$$

Prob. 8. By the conditions, $(x+b)\left(\frac{a}{x}+c\right)=d.$

Expanding, $$x^2+\frac{ab}{c}=\frac{d-a-bc}{c}.x.$$

Completing the square,
$$x^2-mx+\frac{m^2}{4}=\frac{m^2}{4}-\frac{ab}{c}.$$

Extracting the square root,
$$x=\frac{m}{2}\pm\sqrt{\frac{m^2}{4}-\frac{ab}{c}}.$$

Also, $$\frac{a}{x}=\frac{a}{\frac{m}{2}\pm\sqrt{\frac{m^2}{4}-\frac{ab}{c}}}.$$

Prob. 9. By the conditions, $xy=x^2-y^2;$ (1)

$$xy=x+y.$$ (2)

Divide (1) by (2), $1=x-y$ or $x=y+1.$ (3)

Substitute (3) in (2), $y^2+y=2y+1.$

Completing the square,
$$y^2-y+\tfrac{1}{4}=1+\tfrac{1}{4}.$$

Extracting the square root,
$$y=\tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{5}=1.618 \text{ or } -0.618.$$

Also, $$x=\tfrac{3}{2}\pm\tfrac{1}{2}\sqrt{5}=2.618 \text{ or } +0.382.$$

Prob. 10. Let x and $100-x$ denote the two parts.

By the conditions,
$$100x-x^2=(100-x)^2-x^2=10{,}000-200x.$$

Reducing, $$x^2-300x=-10{,}000.$$

Completing the square,
$$x^2-300x+150^2=22{,}500-10{,}000.$$

Extracting the square root,
$$x=150\pm111.803=38.197 \text{ or } 261.803.$$

Also, $$100-x=61.803 \text{ or } -161.803.$$

Prob. 11. Let x and $a-x$ denote the two parts.

By the conditions, $ax-x^2=a^2-2ax.$

Completing the square,

$$x^2-3ax+\frac{9a^2}{4}=\frac{5a^2}{4}.$$

Extracting the square root,

$$x=\frac{3a}{2}\pm\frac{a}{2}\sqrt{5}.$$

Hence $$a-x=-\frac{a}{2}\mp\frac{a}{2}\sqrt{5}.$$

Prob. 12. By the conditions, $x+y=a;$ (1)

$$\frac{1}{x}+\frac{1}{y}=b. \quad (2)$$

Clearing of fractions, $x+y=bxy=a.$ (3)

Hence $$x=\frac{a}{by}. \quad (4)$$

Substituting (4) in (1), $\frac{a}{by}+y=a.$

Reducing, $$y^2-ay=-\frac{a}{b}.$$

Completing the square,

$$y=\frac{a}{2}\pm\sqrt{\frac{a^2}{4}-\frac{a}{b}}.$$

Art. 275, Page 204.

Ex. 2. $x^2+11x=-28.$

Ex. 3. $x^2+4x=45.$

Ex. 4. $x^2-5x=66.$

Ex. 5. $x^2+x=2.$

Ex. 6. $x^2-\frac{x}{6}=\frac{1}{6}.$

Ex. 7. $x^2+\frac{x}{12}=\frac{1}{12}.$

Ex. 8. $x^2-2x=4.$

Ex. 9. $x^2-2x=-6.$

Art. 276, Page 205.

Ex. 1. $(x+2)(x+4).$

Ex. 2. $(x-3)(x+9).$

Ex. 3. $(x-6)(x+4).$

Ex. 4. Completing the square,

$$x^2+73x+(\tfrac{73}{2})^2=\tfrac{5329}{4}-780.$$

Extracting the square root,

$$x=-\tfrac{73}{2}\pm\tfrac{47}{2}=-13 \text{ or } -60.$$

Ex. 5. Completing the square,

$$x^2-88x+44^2=1936-1612.$$

Extracting the square root,

$$x=44\pm18=62 \text{ or } 26.$$

Ex. 6. We have $\qquad 2x^2+x=6.$

Completing the square,

$$x^2+\frac{x}{2}+\frac{1}{16}=\frac{1}{16}+3.$$

Extracting the square root,

$$x=-\tfrac{1}{4}\pm\tfrac{7}{4}=-2 \text{ or } \tfrac{3}{2}.$$

Ex. 7. We have $\qquad 3x^2-10x=25.$

Completing the square,

$$x^2-\frac{10x}{3}+\left(\frac{5}{3}\right)^2=\frac{25}{9}+\frac{75}{9}.$$

Extracting the square root,

$$x=\tfrac{5}{3}\pm\tfrac{10}{3}=5 \text{ or } -\tfrac{5}{3}.$$

CHAPTER XV.

Art. 300, Page 217.

Ex. 1. 25.

Ex. 2. $12a^4$.

Ex. 3. $6ab^7$.

Ex. 4. $15x^4y^2$.

Ex. 5. $a-b$.

Ex. 6. Product of extremes $=9a^2-16b^2$; product of means $=9a^2-10ab-16b^2$. *No* proportion.

Ex. 7. Product of extremes $=225a^4-244a^2b^2+64b^4$; product of means $=225a^4-385a^2b^2+64b^4$. *No* proportion.

Ex. 8. Product of extremes $=a^4+ab^3-a^3b-b^4$; product of means $=a^4-b^4$. *No* proportion.

Ex. 9. Product of extremes $=a^6-b^6$; product of means $=a^6-b^6$. *Proportion.*

Art. 319, Page 225.

Ex. 5. Divide (1) by $x-y$,

$$x^2+xy+y^2:xy::7:2. \qquad (3)$$

By composition, $x^2+2xy+y^2:xy::9:2.$ (4)

Substitute Eq. (2) in (4),

$$36:xy::9:2. \qquad (5)$$

Hence $xy=8.$ (6)

Substitute Eq. (2) in (6),

$$6y-y^2=8.$$

Completing the square,

$$y=3\pm1=4 \text{ or } 2.$$

Hence $x=6-y=2$ or 4.

Ex. 6. Substitute (1) in (2),

$$\sqrt{y}:\sqrt{a-x}::5:2. \qquad (3)$$

Squaring, $y:a-x::25:4.$ (4)

From Eq. (2), $\sqrt{y-x}:\sqrt{a-x}::3:2.$ (5)

Squaring, $y-x:a-x::9:4.$ (6)

By division, $y-a:a-x::5:4.$ (7)

Comparing (4) and (7),

$$y:y-a::5:1.$$

Hence $y=\frac{5a}{4}.$

Substitute in (4) $\frac{5a}{4}:a-x::25:4.$

Reducing, $a:a-x::5:1.$

Hence $x=\frac{4a}{5}.$

Ex. 7. By composition and division,

$$2x:2\sqrt{x}::5\sqrt{x}+6:\sqrt{x}+6.$$

Reducing, $\sqrt{x}:1::5\sqrt{x}+6:\sqrt{x}+6.$

Hence $x+\sqrt{x}=6.$

Completing the square,

$$x+\sqrt{x}+\tfrac{1}{4}=6\tfrac{1}{4}.$$

Extracting the square root,

$$\sqrt{x}=-\tfrac{1}{2}\pm\tfrac{5}{2}=2 \text{ or } -3.$$

Squaring, $x=4$ or 9.

Ex. 8. By the conditions,

$$x+1:x+5::x+5:x+13.$$

Hence $x^2+14x+13=x^2+10x+25.$

Therefore $x=3.$

Ex. 9. By the conditions,

$$x+a:x+b::x+b:x+c.$$

Hence $x^2+ax+cx+ac=x^2+2bx+b^2.$

Therefore $x=\dfrac{b^2-ac}{a-2b+c}.$

Ex. 10. By the conditions,

$$2x+4:3x+4::5:7.$$

Hence $14x+28=15x+20.$

Therefore $x=8.$

The numbers are 16 and 24.

Ex. 11. By the conditions,

$$mx+a:nx+a::p:q.$$

Hence $mqx+aq=npx+ap.$

Therefore $x=\dfrac{ap-aq}{mq-np};$

$$mx=\frac{qm(p-q)}{mq-np};\ nx=\frac{an(p-q)}{mq-np}.$$

Ex. 12. By the conditions,

$$x-y:x+y::2:3.$$

Whence $x:y::5:1$, or $x=5y.$

Also, $x+y:xy::3:5.$

By substitution, $6y:5y^2::3:5.$

Whence $y=2$ and $x=10$

Ex. 13. By the conditions,

$$x-y:x+y::m:n.$$

Hence $x:y::n+m:n-m;$

$$x=y\cdot\frac{n+m}{n-m}.$$

Also, $$x+y : xy :: n : p.$$

Hence $$y\cdot\frac{2n}{n-m} : y^2\cdot\frac{n+m}{n-m} :: n : p.$$

Therefore $$y=\frac{2p}{n+m} \text{ and } x=\frac{2p}{n-m}.$$

Ex. 14. By the conditions,

$$x+y : 42 :: x-y : 6.$$

Hence $$x : y :: 48 : 36 :: 4 : 3;$$

$$y=\frac{3x}{4}.$$

Also, $$x : y :: x+y : 42.$$

Hence $$42x=xy+y^2=\frac{3x^2}{4}+\frac{9x^2}{16}.$$

Therefore $$x=32;$$

$$y=24.$$

Ex. 15. By the conditions,

$$x+y : x-y :: a : b.$$

Therefore $$x : y :: a+b : a-b.$$

Hence $$x=y\,\frac{a+b}{a-b}.$$

Also, $$x : y :: x+y : a.$$

By substitution, $$y\cdot\frac{a+b}{a-b} : y :: y\,\frac{2a}{a-b} : a.$$

Therefore $$a+b : 1 :: 2y : 1.$$

Hence $$y=\frac{a+b}{2},\quad x=\frac{(a+b)^2}{2(a-b)}.$$

Ex. 16. Let $3x$ and $2x$ denote the two numbers.

By the conditions,

$$81x^4-16x^4 : 27x^3+8x^3 :: 26 : 7.$$

Reducing, $$65x : 35 :: 26 : 7.$$

Hence $$5x : 5 :: 2 : 1;$$

$$x=2.$$

Ex. 17. Let mx and nx denote the two numbers.

By the conditions,

$$m^4x^4-n^4x^4 : m^3x^3+n^3x^3 :: p : q.$$

Reducing, $(m^4-n^4)qx=(m^3+n^3)p.$

Hence $$x=\frac{p(m^3+n^3)}{q(m^4-n^4)}.$$

Ex. 18. Let x denote the diameter of a circle, and y its area. Then, since y varies as x^2, $y=mx^2$. The sum of the areas of the circles whose diameters are 6 inches and 8 inches is $m(6^2+8^2)$; that is, $m(36+64)$; that is, $m10^2$; that is, the area of a circle whose diameter is 10 inches.

Ex. 19. Let x denote the radius of a sphere, and y its volume. Then, since y varies as x^3, $y=mx^3$. The sum of the volumes of three spheres whose radii are 3, 4, and 5 inches is $m(3^3+4^3+5^3)$; that is, $m\times216$; that is, $m6^3$; that is, the volume of a sphere whose radius is 6 inches.

Ex. 20. The volumes of the three globes may be denoted by mr^3, mr'^3, and mr''^3. Let R denote the radius of the single sphere which is formed. Then its volume is $m\text{R}^3$.

Hence $$m\text{R}^3=m(r^3+r'^3+r''^3);$$
$$\text{R}^3=r^3+r'^3+r''^3.$$

Hence $$\text{R}=\sqrt[3]{r^3+r'^3+r''^3}.$$

CHAPTER XVI.

ART. 326, PAGE 229.

Formula 4. By Formula (3),
$$2s=an+ln.$$

Substituting from (1),
$$2s=an+an+n^2d-nd.$$

Hence $$d=\frac{2s-2an}{n^2-n}.$$

Formula 5. By Formula (4),
$$2an=2s-n^2d+nd.$$

Hence $$a=\frac{2s-n^2d+nd}{2n}.$$

Also, by Formula (4),
$$l=\frac{2s}{n}-a=\frac{2s}{n}-l+(n-1)d.$$

Hence $$2l=\frac{2s}{n}+(n-1)d.$$

Formula 9. By Formula (8),

$$n=\frac{2s}{a+l}=\frac{l-a+d}{d}.$$

Hence $$2ds=al-a^2+ad+l^2-al+ld.$$

Transposing, $$l^2+ld=2ds+a^2-ad.$$

Completing the square,

$$l^2+ld+\frac{d^2}{4}=2ds+a^2-ad+\frac{d^2}{4}.$$

Hence $$l=-\frac{d}{2}\pm\sqrt{2ds+\left(a-\frac{d}{2}\right)^2}.$$

Also, $$l=a+(n-1)d=\frac{2s}{n}-a.$$

Hence $$an+n^2d-nd=2s-an.$$

Reducing, $$n^2+\frac{2a-d}{d}.n=\frac{2s}{d}.$$

Completing the square,

$$n^2+\frac{2a-d}{d}.n+\left(\frac{2a-d}{2d}\right)^2=\frac{(2a-d)^2+8ds}{4d^2}.$$

Extracting the square root,

$$n=\frac{d-2a}{2d}\pm\frac{\sqrt{(2a-d)^2+8ds}}{2d}.$$

Formula 10. By Formula (8),

$$a^2-ad=l^2+ld-2ds.$$

Completing the square,

$$a^2-ad+\frac{d^2}{4}=l^2+ld+\frac{d^2}{4}-2ds.$$

Extracting the square root,

$$a=\frac{d}{2}\pm\sqrt{\left(l+\frac{d}{2}\right)^2-2ds}.$$

Also, $$a=l-(n-1)d=\frac{2s}{n}-l.$$

Clearing of fractions,

$$ln-n^2d+nd=2s-ln.$$

Transposing, $n^2d-2ln-dn=-2s.$

Completing the square,

$$n^2-\frac{2l+d}{d}.n+\left(\frac{2l+d}{2d}\right)^2=\left(\frac{2l+d}{2d}\right)^2-\frac{2s}{d}.$$

Extracting the square root,

$$n=\frac{2l+d}{2d}\pm\frac{\sqrt{(2l+d)^2-8ds}}{2d}.$$

Ex. 2. 13.

Ex. 3. $7\frac{3}{4}$.

Ex. 4. $-\frac{1}{2}$.

Ex. 6. 1700.

Ex. 7. $61\frac{1}{2}$.

Ex. 8. -2.

Ex. 9. From Formula (9),

$$n=\frac{3-4\pm\sqrt{1+24\times 442}}{6}=17.$$

Ex. 10. From Formula (10),

$$a=1\pm\sqrt{20^2-4\times 99}=1\pm 2=+3 \text{ or } -1.$$

Ex. 11. From Formula (8),

$$d=\frac{92^2-5^2}{2910-97}=3.$$

Ex. 13. $l=1+2(n-1)=2n-1.$

Ex. 14. $s=\frac{n}{2}\{2+2(n-1)\}=n^2.$

Ex. 15. $s=\frac{n}{2}\{2+n-1\}=\frac{n(n+1)}{2}.$

Ex. 16. $s=\frac{n}{2}\{4+2(n-1)\}=n(n+1).$

Ex. 17. $d=\frac{49}{7}=7.$ The numbers are 8, 15, 22, 29, 36, and 43.

Ex. 18. $d=\frac{2\frac{2}{3}}{8}=\frac{1}{3}.$ The numbers are $\frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}, 2, \frac{7}{3}$, and $\frac{8}{3}$.

Ex. 19.

$$l=16+19\times 32=624;$$

$$s=\frac{624+16}{2}\times 20=6400.$$

Ex. 20. $s=2n(n+1)=200\times 101=20,200.$

Art. 327, Page 231.

Prob. 1. Denote the digits by $x-y$, x, and $x+y$.

By the conditions,

$$100(x-y)+10x+x+y+198=100(x+y)+10x+x-y.$$

Uniting terms, $198y=198.$

Hence $y=1.$

Also, $$\frac{100(x-y)+10x+x+y}{3x}=26.$$

Hence $33x=99;$

$x=3;$

$x-1=2$ and $x+1=4.$

Prob. 2. By the conditions,

$$(x-y)^2+x^2+(x+y)^2=1232; \quad (1)$$
$$x^2-(x^2-y^2)=16. \quad (2)$$

Hence $y=\pm4.$

From Eq. (1), $3x^2+2y^2=1232.$

Hence $3x^2=1200;$

$x=\pm20.$

Prob. 3. By the conditions,

$$(x-y)^2+x^2+(x+y)^2=a; \quad (1)$$
$$x^2-(x^2-y^2)=b. \quad (2)$$

From Eq. (2), $y=\sqrt{b}.$

From Eq. (1), $3x^2+2y^2=a.$

By substitution, $x^2=\frac{a-2b}{3}.$

Hence $x=\sqrt{\frac{a-2b}{3}}.$

Prob. 4. By the conditions,

$$(x-3y)+(x-y)+(x+y)+(x+3y)=28.$$

Hence $x=7.$

Also, $(x^2-9y^2)(x^2-y^2)=585.$

By substitution, $9y^4-490y^2+2401=585.$

Completing the square,

$$y^4-\frac{490y^2}{9}+\left(\frac{245}{9}\right)^2=\frac{60,025}{81}-\frac{1816}{9}.$$

Extracting the square root,

$$y^2=\tfrac{245}{9}\pm\tfrac{209}{9}=4.$$

Hence $$y=\pm 2;$$

$$x-3y=1;\ x-y=5;\ x+y=9;\ x+3y=13.$$

Prob. 5. Let x denote the number of days A travels.
Then will $x-5$ " " " B travels.

A travels $\frac{x(x+1)}{2}$ miles.

By the conditions, $\frac{x(x+1)}{2}=(x-5)12.$

Reducing, $x^2+x=24x-120.$

Completing the square,

$$x^2-23x+(\tfrac{23}{2})^2=\tfrac{529}{4}-120.$$

Extracting the square root,

$$x=\tfrac{23}{2}\pm\tfrac{7}{2}=15 \text{ or } 8.$$

Prob. 6. By the conditions,

$$\frac{x(x+1)}{2}=(x-a)b.$$

Reducing, $x^2+x=2bx-2ab.$

Completing the square,

$$x^2-(2b-1)x+\left(\frac{2b-1}{2}\right)^2=\left(\frac{2b-1}{2}\right)^2-\frac{8ab}{4}.$$

Extracting the square root,

$$x=\frac{2b-1}{2}\pm\frac{\sqrt{(2b-1)^2-8ab}}{2}.$$

Prob. 7. Let x denote the number of days the second person travels.

The distance he travels is $\frac{x}{2}(24+x-1)=\frac{x}{2}\times(x+23).$

The distance the first person travels is $(x+3)^2$.

By the conditions, $(x+3)^2=\frac{x^2+23x}{2}.$

Expanding, $2x^2+12x+18=x^2+23x.$

Completing the square,

$$x^2-11x+(\tfrac{11}{2})^2=(\tfrac{11}{2})^2-18.$$

Extracting the square root,

$$x=\tfrac{11}{2}\pm\tfrac{7}{2}=9 \text{ or } 2.$$

Prob. 8. Let x denote the required number of days.

The number of miles A traveled is $\frac{x}{2}(1+x)$.

The number of miles B traveled is $\frac{x}{2}(42-2x)$.

By the conditions, $\frac{x}{2}(43-x)=165$.

Expanding, $x^2-43x=-330$.

Completing the square,

$$x^2-43x+(\tfrac{43}{2})^2=\tfrac{1849}{4}-330.$$

Extracting the square root,

$$x=\tfrac{43}{2}\pm\tfrac{23}{2}=10 \text{ or } 33.$$

Art. 335, Page 236.

Formula 2. Substitute in Formula (1),

$$s=\frac{lr-\dfrac{l}{r^{n-1}}}{r-1}.$$

Multiply numerator and denominator by r^{n-1}, and we have

$$s=\frac{lr^n-l}{r^n-r^{n-1}}.$$

Formula 3. From Formula (2),

$$lr^n-l=s(r^n-r^{n-1})=(r-1)sr^{n-1}.$$

Hence $$l=\frac{(r-1)sr^{n-1}}{r^n-1}.$$

Formula 4. From Formula (1),

$$sr-s=lr-a.$$

Substitute the value of r,

$$s\left(\frac{l}{a}\right)^{\frac{1}{n-1}}-s=l\left(\frac{l}{a}\right)^{\frac{1}{n-1}}-a.$$

Multiply by $a^{\frac{1}{n-1}}$,

$$sl^{\frac{1}{n-1}}-sa^{\frac{1}{n-1}}=l^{\frac{n}{n-1}}-a^{\frac{n}{n-1}}.$$

Hence $$s=\frac{l^{\frac{n}{n-1}}-a^{\frac{n}{n-1}}}{l^{\frac{1}{n-1}}-a^{\frac{1}{n-1}}}.$$

Formula 5. From Formula (1),

$$sr - lr = s - a.$$

Hence $$r = \frac{s-a}{s-l}.$$

From Formula (1), $$\frac{l}{a} = r^{n-1} = \left(\frac{s-a}{s-l}\right)^{n-1}.$$

Hence $$l(s-l)^{n-1} = a(s-a)^{n-1}.$$

Formula 6. From Formula (2),

$$sr^n - sr^{n-1} = lr^n - l.$$

Hence $$sr^n - lr^n - sr^{n-1} = -l.$$

Formula 7. From Formula (1),

$$r^{n-1} = \frac{l}{a}.$$

Hence $$(n-1)\log. r = \log. l - \log. a.$$

Therefore $$n-1 = \frac{\log. l - \log. a}{\log. r}$$

Formula 8. From Formula (5),

$$\left(\frac{s-a}{s-l}\right)^{n-1} = \frac{l}{a}.$$

Hence $$(n-1)\{\log.(s-a) - \log.(s-l)\} = \log. l - \log. a.$$

Therefore $$n-1 = \frac{\log. l - \log. a}{\log.(s-a) - \log.(s-l)}.$$

Formula 9. From Formula (5),

$$ar^n = a + rs - s.$$

Hence $$r^n = \frac{a+rs-s}{a}.$$

Therefore $$n.\log. r = \log.(a+rs-s) - \log. a.$$

Hence $$n = \frac{\log.(a+rs-s) - \log. a}{\log. r}.$$

Formula 10. By Formula (2),

$$r^{n-1} = \frac{l}{a} = \frac{l}{lr - rs + s}.$$

Hence $$(n-1)\log. r = \log. l - \log.(lr - rs + s).$$

Therefore $$n-1 = \frac{\log. l - \log.(lr-rs+s)}{\log. r}.$$

Ex. 2. $l=2.3^9=2\times 19{,}683=39{,}366.$

Ex. 4. $s=\dfrac{3^{12}-1}{2}=\dfrac{531{,}441-1}{2}=265{,}720.$

Ex. 5. $r=\dfrac{s-a}{s-l}=\dfrac{1023-1}{1023-512}=2.$

Ex. 6. $a=\dfrac{l}{r^n-1}=\dfrac{2048}{2^{11}}=\dfrac{2048}{2048}=1.$

Ex. 7. $s=\dfrac{ar^n-a}{r-1}=\dfrac{6(\frac{3}{4})^6-6}{-\frac{1}{4}}=\dfrac{10{,}101}{512}=19\frac{373}{512}.$

Ex. 8. $s=\dfrac{8(\frac{1}{2})^{15}-8}{-\frac{1}{2}}=\dfrac{32{,}767}{2048}=15\frac{2047}{2048}.$

Ex. 9. $$r=\left(\frac{162}{2}\right)^{\frac{1}{4}}=81^{\frac{1}{4}}=3.$$

6, 18, and 54 are the required numbers.

Ex. 10. $$r=\left(\frac{256}{4}\right)^{\frac{1}{3}}=4.$$

16 and 64 are the required numbers.

Ex. 11. $$r=\left(\frac{b}{a}\right)^{\frac{1}{4}}.$$

The numbers are $a\left(\frac{b}{a}\right)^{\frac{1}{4}},\ a\left(\frac{b}{a}\right)^{\frac{1}{2}},\ a\left(\frac{b}{a}\right)^{\frac{3}{4}}$;

or $(a^3b)^{\frac{1}{4}},\ (ab)^{\frac{1}{2}},\ (ab^3)^{\frac{1}{4}}.$

Ex. 13. $s=\dfrac{1}{1-\frac{1}{3}}=\frac{3}{2}.$

Ex. 14. $s=\dfrac{1}{1-\frac{1}{4}}=\frac{4}{3}.$

Ex. 15. $r=1-\dfrac{a}{s}=1-\dfrac{1}{\frac{5}{4}}=\frac{1}{5}.$

Ex. 16. $a=s(1-r)=\frac{2}{3}\cdot\frac{9}{10}=\frac{3}{5}.$

Ex. 17. $a=\dfrac{n}{n-1}\left(1-\dfrac{1}{n}\right)=\dfrac{n}{n-1}\times\dfrac{n-1}{n}=1.$

Ex. 18. $s=\dfrac{a}{1-r}=\dfrac{3}{\frac{1}{3}}=9.$

Ex. 19. $s=\frac{4}{3}\div\frac{1}{4}=\frac{16}{3}$.

Ex. 20. $s=2^{32}-1=16^8-1=42,949,672.96-1$.

Prob. 3. By the conditions,

$$x+xy+xy^2=210; \quad (1)$$

$$xy^2-x=90. \quad (2)$$

Subtract (2) from (1),

$$2x+xy=120. \quad (3)$$

Hence $$x=\frac{120}{y+2}=\frac{90}{y^2-1}.$$

Therefore $$\frac{4}{y+2}=\frac{3}{y^2-1}.$$

Clearing of fractions,

$$4y^2-4=3y+6.$$

Completing the square,

$$y^2-\frac{3y}{4}+\left(\frac{3}{8}\right)^2=\frac{9}{64}+\frac{10}{4}.$$

Extracting the square root,

$$y=\frac{3}{8}\pm\frac{13}{8}=2 \text{ or } -\frac{5}{4}.$$

Also $$x=\frac{120}{y+2}=30 \text{ or } 160.$$

The numbers are 30, 60, and 120;

or +160, −200, +250.

Prob. 4. Let x denote the second number, and y the ratio.

By the conditions, $$\frac{x}{y}+x+xy=42; \quad (1)$$

$$\frac{x}{y}+xy=34. \quad (2)$$

Subtract (2) from (1), $x=8$.

Substitute in (2), $$\frac{8}{y}+8y=34.$$

Clearing of fractions,

$$8y^2-34y=-8.$$

Completing the square,

$$y^2-\frac{17y}{4}+\left(\frac{17}{8}\right)^2=\frac{289}{64}-1.$$

Extracting the square root,

$$y=\tfrac{17}{8}\pm\tfrac{15}{8}=4 \text{ or } \tfrac{1}{4}.$$

Prob. 5. By the conditions,

$$\frac{x^3}{y^3}+x^3+x^3y^3=584; \qquad (1)$$

$$x^3=64. \qquad (2)$$

Hence $x=4.$

Substitute in (1), $\dfrac{1}{y^3}+1+y^3=\dfrac{584}{64}=\dfrac{73}{8}.$

Reducing, $y^6-\dfrac{65y^3}{8}=-1.$

Completing the square,

$$y^6-\frac{65y^3}{8}+\left(\frac{65}{16}\right)^2=\frac{4225}{256}-1.$$

Extracting the square root,

$$y^3=\tfrac{65}{16}\pm\tfrac{63}{16}=8 \text{ or } \tfrac{1}{8}.$$

Extracting the cube root, $y=2$ or $\tfrac{1}{2}$.

Prob. 6. Let x, xy, xy^2, xy^3 denote the four numbers.

By the conditions, $xy^3-xy^2=36; \qquad (1)$

$$xy-x=4. \qquad (2)$$

Divide (1) by (2), $y^2=9.$

Hence $y=\pm 3.$

From Eq. (2), $x=\dfrac{4}{y-1}=2 \text{ or } -1.$

The numbers are 2, 6, 18, and 54;

or $-1, +3, -9, +27.$

Prob. 7. By the conditions,

$$x+xy^2=a; \qquad (1)$$

$$xy+xy^3=b. \qquad (2)$$

Divide (2) by (1), $y=\dfrac{b}{a}.$

Substitute in (1), $x+\dfrac{b^2x}{a^2}=a.$

Reducing, $a^2x+b^2x=a^3.$

Hence $x=\dfrac{a^3}{a^2+b^2}.$

Prob. 8. By the conditions,

$$x+xy^3:xy+xy^2::7:3.$$

Hence $3+3y^3=7y+7y^2.$

Divide by $1+y$, $3-3y+3y^2=7y.$

Completing the square,

$$y^2-\frac{10y}{3}+\left(\frac{5}{3}\right)^2=\frac{25}{9}-\frac{9}{9}.$$

Extracting the square root,

$$y=\tfrac{5}{3}\pm\tfrac{4}{3}=3 \text{ or } \tfrac{1}{3}.$$

Also, $xy^3-xy=24.$

By substitution, $27x-3x=24.$

Hence $x=1;$

or $\frac{x}{27}-\frac{x}{3}=24.$

Clearing of fractions, $x-9x=648.$

Hence $x=-81.$

The numbers are 1, 3, 9, and 27;

or $-81, -27, -9, -3.$

Prob. 9. By the conditions,

$$x+xy+xy^2+xy^3=700; \quad (1)$$

$$xy^3-x=\tfrac{37}{12}(xy^2-xy). \quad (2)$$

Divide (2) by x, $y^3-1=\tfrac{37}{12}(y^2-y).$ (3)

Divide (3) by $y-1$,

$$12(y^2+y+1)=37y.$$

Completing the square,

$$y^2-\frac{25y}{12}+\left(\frac{25}{24}\right)^2=\frac{625}{576}-1.$$

Hence $y=\tfrac{25}{24}\pm\tfrac{7}{24}=\tfrac{4}{3}$ or $\tfrac{3}{4}.$

Substitute in (1),

$$x+\frac{4x}{3}+\frac{16x}{9}+\frac{64x}{27}=700.$$

Clearing of fractions,

$$27x+36x+48x+64x=27\times 700.$$

Reducing, $175x=27\times 700.$

Hence $x=108.$

Prob. 10. By the conditions,

$$x+xy+xy^2+xy^3+xy^4+xy^5=1365; \qquad (1)$$

$$xy^2+xy^3=80. \qquad (2)$$

Substitute (2) in (1),

$$\frac{80}{y^2}+80+80y^2=1365.$$

Reducing, $$\frac{16}{y^2}+16y^2=257.$$

Completing the square,

$$y^4-\frac{257y^2}{16}+\left(\frac{257}{32}\right)^2=\frac{66,049}{1024}-\frac{1024}{1024}.$$

Extracting the square root,

$$y^2=\tfrac{257}{32}\pm\tfrac{255}{32}=16 \text{ or } \tfrac{1}{16}.$$

Extracting the square root,

$$y=\pm 4 \text{ or } \pm\tfrac{1}{4}.$$

Substitute in (2), $16x+64x=80.$

Hence $x=1;$

or $16x-64x=80.$

Hence $x=-\tfrac{5}{3}.$

The numbers are 1, 4, 16, 64, 256, and 1024;

or $-\tfrac{5}{3}, +\tfrac{20}{3}, -\tfrac{80}{3}, +\tfrac{320}{3}, -\tfrac{1280}{3}, +\tfrac{5120}{3}.$

CHAPTER XVII.

Art. 337, Page 241.

Ex. 4. $$\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{5+\cfrac{1}{6}}}}}.$$

Ex. 5. $$\cfrac{1}{3+\cfrac{1}{22+\cfrac{1}{1+\cfrac{2}{9}}}}.$$

Ex. 6. $\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{5+\cfrac{1}{6}}}}$.

ART. 338, PAGE 242.

Ex. 2. $\frac{47}{162}$.

Ex. 3. $\frac{41}{94}$.

Ex. 4. $\frac{445}{612}$.

ART. 341, PAGE 243.

Ex. 2. $\frac{1}{3}, \frac{1}{4}, \frac{3}{11}, \frac{16}{59}$.

Ex. 3. $\frac{1}{3}, \frac{2}{7}, \frac{7}{24}, \frac{16}{55}$.

ART. 342, PAGE 244.

Ex. 4. The number is $3+\cfrac{1}{7+\cfrac{1}{15+\cfrac{1}{1}}}$.

Ex. 5. The fraction is $\cfrac{1}{4+\cfrac{1}{7+\cfrac{1}{1+\cfrac{1}{4}}}}$.

Ex. 6. The fraction is $\cfrac{1}{3+\cfrac{1}{6+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2}}}}}$.

Ex. 7. The number is $1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{29}}}}$.

Values, $\frac{3}{2}, \frac{8}{5}, \frac{235}{147}$.

Ex. 8. The fraction is $\cfrac{1}{12+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1.}}}}}}$

Values, $\frac{1}{12}, \frac{2}{25}, \frac{3}{37}, \frac{8}{99}, \frac{11}{136}, \frac{19}{235}$.

Ex. 9. The number is $3+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{3.}}}}$

Values, $\frac{10}{3}, \frac{13}{4}, \frac{23}{7}, \frac{82}{25}$.

Ex. 10. The number is $2+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{7.}}}$

Values, $\frac{9}{4}, \frac{11}{5}, \frac{86}{39}$.

Ex. 11. The fraction is $\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{5.}}}}$

Values, $\frac{1}{4}, \frac{1}{5}, \frac{2}{9}, \frac{11}{50}$.

Art. 347, Page 247.

Ex. 2. $26.25.24.23=358{,}800$.

Ex. 3. $12.11.10.9.8.7=665{,}280$.

Ex. 5. $1.2.3.4=24$.

Ex. 6. $1.2.3.4.5.6=720$.

Ex. 7. $1.2.3.4.5.6.7.8.9.10.11.12=479{,}001{,}600$.

Ex. 9. $\frac{8.7.6.5}{1.2.3.4}=70$.

Ex. 10. $\frac{10.9.8.7.6.5}{1.2.3.4.5.6}=210$.

Ex. 11. The first arm can be placed in n distinct positions; so also can the second: thus with these two arms we can form n^2 signals. The third arm is also capable of n distinct positions, and each position may be combined with any pair of positions of the first and second arms: thus we have $n^2 \times n$, that is n^3 signals. In like manner, with 4 arms we can make n^4 signals, and with m arms we can make n^m signals.

Note.—The telegraph here referred to is that which was in common use in Europe before the invention of the electro-magnetic telegraph. It consisted of an upright post with two movable arms, each of which could be exhibited in various positions. If each arm can assume six different positions, then two arms can be made to furnish 36 different combinations, and each combination may be employed to denote a letter of the alphabet or a numeral; or it may denote a word or sentence in a telegraphic dictionary.

Ex. 12. $1.2.3.4.5.6.7.8.9 = 362,880$.

CHAPTER XVIII.

ART. 359, PAGE 253.

Ex. 3. $a^7 - 7a^6x + 21a^5x^2 - 35a^4x^3 + 35a^3x^4 - 21a^2x^5 + 7ax^6 - x^7$.

Ex. 4. $+105a^{13}b^2$.

Ex. 5. $+1225a^2x^{48}$.

Ex. 6. $+252a^5x^5$.

Ex. 9.

Coefficients,	1	4	6	4	1
Powers of $2x$,	$16x^4$	$8x^3$	$4x^2$	$2x$	1
Powers of $5a^2$,	1	$5a^2$	$25a^4$	$125a^6$	$625a^8$.

Ex. 10.

Coefficients,	1	4	6	4	1
Powers of x^3,	x^{12}	x^9	x^6	x^3	1
Powers of $4y^2$,	1	$4y^2$	$16y^4$	$64y^6$	$256y^8$

Result, $x^{12} + 16x^9y^2 + 96x^6y^4 + 256x^3y^6 + 256y^8$.

Ex. 13.

Coefficients,	1	5	10	10	5	1
Powers of $5a^2$,	$3125a^{10}$	$625a^8$	$125a^6$	$25a^4$	$5a^2$	1
Powers of $-4x^2y$,	1	$-4x^2y$	$+16x^4y^2$	$-64x^6y^3$	$+256x^8y^4$	$-1024x^{10}y^5$

Result,

$3125a^{10} - 12,500a^8x^2y + 20,000a^6x^4y^2 - 16,000a^4x^6y^3 + 6400a^2x^8y^4 - 1024x^{10}y^5$.

Ex. 14.

Coefficients,							
	1	6	15	20	15	6	1
Powers of a^2x,							
	$a^{12}x^6$	$a^{10}x^5$	a^8x^4	a^6x^3	a^4x^2	a^2x	1
Powers of by^2,							
	1	by^2	b^2y^4	b^3y^6	b^4y^8	b^5y^{10}	b^6y^{12}

Result, $a^{12}x^6+6a^{10}x^5by^2+15a^8x^4b^2y^4+20a^6x^3b^3y^6+15a^4x^2b^4y^8+6a^2xb^5y^{10}+b^6y^{12}$.

Ex. 15.

Coefficients,	1	5	10	10	5	1
Powers of ax,	a^5x^5	$-a^4x^4$	$+a^3x^3$	$-a^2x^2$	$+ax$	-1

Result, $a^5x^5-5a^4x^4+10a^3x^3-10a^2x^2+5ax-1$.

Ex. 16. $+\frac{12.11.10.9}{1.2.3.4}a^{16}b^8=+495a^{16}b^8$.

Ex. 17. $\frac{9.8.7.6}{1.2.3.4}.3^5x^{\frac{5}{2}}4^4y^2=7,838,208x^{\frac{5}{2}}y^2$.

Ex. 18.

Coefficients,							
	1	6	15	20	15	6	1
Powers of 5,							
	5^6	5^5	5^4	5^3	5^2	5	1
Powers of $-\frac{x}{6}$,							
	1	$-\frac{x}{6}$	$+\frac{x^2}{6^2}$	$-\frac{x^3}{6^3}$	$+\frac{x^4}{6^4}$	$-\frac{x^5}{6^5}$	$+\frac{x^6}{6^6}$

Result, $5^6-\frac{6.5^5x}{6}+\frac{15.5^4x^2}{6^2}-\frac{20.5^3x^3}{6^3}+\frac{15.5^2x^4}{6^4}-\frac{6.5x^5}{6^5}+\frac{x^6}{6^6}$;

or $15,625-3125x+\frac{3125x^2}{12}-\frac{625x^3}{54}+\frac{125x^4}{432}-\frac{5x^5}{1296}+\frac{x^6}{46,656}$.

ART. 361, PAGE 255.

Ex. 2. $(x+y)^5=x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$;

$(x+a+b)^5=x^5+5(a+b)x^4+10(a^2+2ab+b^2)x^3+10(a^3+3a^2b+3ab^2+b^3)x^2+5(a^4+4a^3b+6a^2b^2+4ab^3+b^4)x+a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$.

Ex. 4. $1+10x+55x^2+200x^3+530x^4+1052x^5+1590x^6+1800x^7+1485x^8+810x^9+243x^{10}$.

ART. 363, PAGE 257.

Ex. 2. 87. Ex. 3. 1.58490.

ART. 364, PAGE 258.

Ex. 1. The square root of this polynomial is $a^2-2ab+b^2$, whose square root is $a-b$.

Ex. 2. The square root of this polynomial is $a^3+3a^2b+3ab^2+b^3$, whose cube root is $a+b$.

Ex. 3. The square root of this polynomial is

$$16x^4+32x^3y+24x^2y^2+8xy^3+y^4.$$

The square root of the latter polynomial is $4x^2+4xy+y^2$, whose square root is $2x+y$, which is therefore the eighth root of the given polynomial.

CHAPTER XIX.

ART. 369, PAGE 260.

Ex. 3. Given	1	16	81	256	625	1296	
1st diff.,	15	65	175	369	671		
2d diff.,	50	110	194	302			
3d diff.,	60	84	108				
4th diff.,	24	24					
Ex. 4. Given	1	32	243	1024	3125	7776	16,807
1st diff.,	31	211	781	2101	4651	9031	
2d diff.,	180	570	1320	2550	4380		
3d diff.,	390	750	1230	1830			
4th diff.,	360	480	600				
5th diff.,	120	120					
Ex. 5. Given	1	3	6	10	15	21	
1st diff.,	2	3	4	5	6		
2d diff.,	1	1	1	1			

ART. 370, PAGE 261.

Ex. 3. $a=1$, $D'=4$, $D''=5$, $D'''=2$.

$$T_{13}=1+12.4+\frac{12.11}{2}.5+\frac{12.11.10}{2.3}.2=819.$$

Ex. 4. $a=1$, $D'=3$, $D''=2$.

$$T_{15}=1+14.3+\frac{14.13}{2}.2=225.$$

Ex. 5. $a=1$, $D'=7$, $D''=12$, $D'''=6$.

$$T_{20}=1+19.7+\frac{19.18}{2}.12+\frac{19.18.17}{2.3}.6=8000.$$

Ex. 6. $a=1$, $D'=2$, $D''=1$.

$$T_n=1+2(n-1)+\frac{(n-1)(n-2)}{2}=\frac{n^2+n}{2}.$$

Ex. 7. $a=1$, $D'=3$, $D''=3$, $D'''=1$.

$$T_n=1+3(n-1)+\frac{3(n-1)(n-2)}{2}+\frac{(n-1)(n-2)(n-3)}{2.3}=\frac{n(n+1)(n+2)}{6}.$$

Ex. 8. $a=1$, $D'=4$, $D''=6$, $D'''=4$, $D''''=1$.

$$T_n=1+4(n-1)+\frac{6(n-1)(n-2)}{2}+\frac{4(n-1)(n-2)(n-3)}{2.3}$$
$$+\frac{(n-1)(n-2)(n-3)(n-4)}{2.3.4}=\frac{n(n+1)(n+2)(n+3)}{24}.$$

Art. 371, Page 263.

Ex. 2. $a=1$, $D'=3$, $D''=3$, $D'''=1$.

$$S=20+\frac{20.19}{2}.3+\frac{20.19.18}{2.3}.3+\frac{20.19.18.17}{2.3.4}=8855.$$

Ex. 3. $a=1$, $D'=1$.

$$S=n+\frac{n(n-1)}{2}=\frac{n^2+n}{2}.$$

Ex. 4. $a=1$, $D'=3$, $D''=2$.

$$S=n+\frac{3n(n-1)}{2}+\frac{2n(n-1)(n-2)}{2.3}=\frac{2n^3+3n^2+n}{6}=\frac{n(n+1)(2n+1)}{6}.$$

Ex. 5. $a=1$, $D'=7$, $D''=12$, $D'''=6$.

$$S=n+\frac{7n(n-1)}{2}+\frac{12n(n-1)(n-2)}{2.3}+\frac{6n(n-1)(n-2)(n-3)}{2.3.4}$$
$$=\frac{n^4+2n^3+n^2}{4}=\frac{(n^2+n)^2}{4}.$$

Ex. 6. $a=1$, $D'=2$, $D''=1$.

$$S=n+\frac{2n(n-1)}{2}+\frac{n(n-1)(n-2)}{2.3}=\frac{n(n^2+3n+2)}{6}.$$

Ex. 7. Each term of this series is double the corresponding term of the preceding series.

Ex. 8. $a=1$, $D'=3$, $D''=3$, $D'''=1$.

$$S=n+\frac{3n(n-1)}{2}+\frac{3n(n-1)(n-2)}{2.3}+\frac{n(n-1)(n-2)(n-3)}{2.3.4}$$
$$=\frac{n(n^3+6n^2+11n+6)}{24}.$$

ART. 372, PAGE 264.

Ex. 2. $T=a+\frac{D'}{2}-\frac{D''}{8}$.

$$\begin{array}{r}3.914868\\+\ 10815\\+\ \ \ 29\\\hline 3.925712\end{array}$$, *Ans.*

Ex. 3. $T=a+\frac{3D'}{4}-\frac{3D''}{32}$.

$$\begin{array}{r}3.914868\\+\ 16222\\+\ \ \ 22\\\hline 3.931112\end{array}$$, *Ans.*

Ex. 4. $T=a+\frac{3D'}{5}-\frac{3D''}{25}$.

$$\begin{array}{r}3.914868\\+\ 12978\\+\ \ \ 28\\\hline 3.927874\end{array}$$, *Ans.*

Ex. 5. $T=a+\frac{33D'}{100}-\frac{D''}{9}$.

$$\begin{array}{r}3.914868\\+\ 7137\\+\ \ \ 26\\\hline 3.922031\end{array}$$, *Ans.*

Ex. 6. $D'=+0.090538$, $D''=-0.001448$, $D'''=+0.000067$.

$T=a+\frac{3D'}{10}-\frac{21D''}{200}+\frac{D'''}{17}$.

$$\begin{array}{r}5.477226\\+\ 27162\\+\ \ 152\\+\ \ \ \ 4\\\hline 5.504544\end{array}$$, *Ans.*

Ex. 7. $T=a+\frac{4D'}{10}-\frac{12D''}{100}+\frac{D'''}{16}$.

$$\begin{array}{r}5.477226\\+\ 36215\\+\ \ 174\\+\ \ \ \ 4\\\hline 5.513619\end{array}$$, *Ans.*

Ex. 8. $T=a+\frac{D'}{2}-\frac{D''}{8}+\frac{D'''}{16}$.

$$\begin{array}{r} 5.477226 \\ +\ 45269 \\ +\ 181 \\ +\ 4 \\ \hline 5.522680 \end{array}$$, *Ans.*

Ex. 9. $T=a+\frac{6D'}{10}-\frac{12D''}{100}+\frac{D'''}{18}$.

$$\begin{array}{r} 5.477226 \\ +\ 54323 \\ +\ 174 \\ +\ 4 \\ \hline 5.531727 \end{array}$$, *Ans.*

Ex. 10. $T=a+\frac{8D'}{10}-\frac{8D''}{100}+\frac{D'''}{31}$.

$$\begin{array}{r} 5.477226 \\ +\ 72431 \\ +\ 116 \\ +\ 2 \\ \hline 5.549775 \end{array}$$, *Ans.*

Art. 373, Page 265.

Ex. 2.

$$\begin{array}{l} 1+x)1 \quad (1-x+x^2-x^3+x^4+\text{ etc.} \\ \quad 1+x \\ \hline \quad -x \\ \quad -x-x^2 \\ \hline \quad\quad +x^2 \\ \quad\quad +x^2+x^3 \\ \hline \quad\quad\quad -x^3 \\ \quad\quad\quad -x^3-x^4 \\ \hline \quad\quad\quad\quad +x^4 \end{array}$$

Ex. 3.

$$\begin{array}{l} a+x)a \quad (1-\frac{x}{a}+\frac{x^2}{a^2}-\frac{x^3}{a^3}+\text{ etc.} \\ \quad a+x \\ \hline \quad -x \\ \quad -x-\frac{x^2}{a} \\ \hline \quad\quad +\frac{x^2}{a} \\ \quad\quad \frac{x^2}{a}+\frac{x^3}{a^2} \\ \hline \quad\quad\quad -\frac{x^3}{a^2}. \end{array}$$

Ex. 4.

$$a-x)a\quad\left(1+\frac{x}{a}+\frac{x^2}{a^2}+\frac{x^3}{a^3}+\text{ etc.}\right.$$
$$\underline{a-x}$$
$$+x$$
$$\underline{x-\frac{x^2}{a}}$$
$$\frac{x^2}{a}$$
$$\underline{\frac{x^2}{a}-\frac{x^3}{a^2}}$$
$$\frac{x^3}{a^2}.$$

Ex. 5.

$$1-x)1+x\;(1+2x+2x^2+2x^3+\text{ etc.}$$
$$\underline{1-x}$$
$$+2x$$
$$\underline{2x-2x^2}$$
$$2x^2$$
$$\underline{2x^2-2x^3}$$
$$2x^3.$$

Ex. 6.

$$a-x)a+x\;\left(1+\frac{2x}{a}+\frac{2x^2}{a^2}+\frac{2x^3}{a^3}+\text{ etc.}\right.$$
$$\underline{a-x}$$
$$2x$$
$$\underline{2x-\frac{2x^2}{a}}$$
$$\frac{2x^2}{a}$$
$$\underline{\frac{2x^2}{a}-\frac{2x^3}{a^2}}$$
$$\frac{2x^3}{a^2}.$$

Ex. 7.

$$1-x+x^2)1\quad(1+x-x^3-x^4+x^6+\text{ etc.}$$
$$\underline{1-x+x^2}$$
$$x-x^2$$
$$\underline{x-x^2+x^3}$$
$$-x^3$$
$$\underline{-x^3+x^4-x^5}$$
$$-x^4+x^5$$
$$\underline{-x^4+x^5-x^6}$$
$$x^6.$$

Ex. 8.

$$\begin{array}{l} 1-x+x^2)\,1-x\,(1-x^2-x^3+x^5+\text{ etc.} \\ \quad 1-x+x^2 \\ \quad\quad -x^2 \\ \quad\quad -x^2+x^3-x^4 \\ \quad\quad\quad -x^3+x^4 \\ \quad\quad\quad -x^3+x^4-x^5 \\ \quad\quad\quad\quad x^5. \end{array}$$

Ex. 9.

$$\begin{array}{l} 1-x-x^2)\,1+x\,(1+2x+3x^2+5x^3+\text{ etc.} \\ \quad 1-x-x^2 \\ \quad\quad 2x+x^2 \\ \quad\quad 2x-2x^2-2x^3 \\ \quad\quad\quad 3x^2+2x^3 \\ \quad\quad\quad 3x^2-3x^3-3x^4 \\ \quad\quad\quad\quad 5x^3+3x^4. \end{array}$$

Art. 374, Page 267.

Ex. 2.

$$a^2+x\left(a+\frac{x}{2a}-\frac{x^2}{8a^3}+\frac{x^3}{16a^5}-\frac{5x^4}{128a^7}+\text{ etc.}\right.$$

$$a^2$$

$$2a+\frac{x}{2a}\Big)\quad x$$

$$x+\frac{x^2}{4a^2}$$

$$2a+\frac{x}{a}-\frac{x^2}{8a^3}\Big)\quad -\frac{x^2}{4a^2}$$

$$-\frac{x^2}{4a^2}-\frac{x^3}{8a^4}+\frac{x^4}{64a^6}$$

$$2a+\frac{x}{a}-\frac{x^2}{4a^3}+\frac{x^3}{16a^5}\Big)\quad \frac{x^3}{8a^4}-\frac{x^4}{64a^6}$$

$$\frac{x^3}{8a^4}+\frac{x^4}{16a^6}-\frac{x^5}{64a^8}+\frac{x^6}{256a^{10}}$$

$$-\frac{5x^4}{64a^6}+\frac{x^5}{64a^8}-\frac{x^6}{256a^{10}}.$$

Ex. 3.

$$\begin{array}{rl}
 & a^4+x\left(a^2+\frac{x}{2a^2}-\frac{x^2}{8a^6}+\frac{x^3}{16a^{10}}-\text{ etc.}\right. \\
 & a^4 \\
2a^2+\frac{x}{2a^2}\Big) & \overline{\quad x \quad} \\
 & x+\frac{x^2}{4a^4} \\
2a^2+\frac{x}{a^2}-\frac{x^2}{8a^6}\Big) & \overline{\quad -\frac{x^2}{4a^4} \quad} \\
 & -\frac{x^2}{4a^4}-\frac{x^3}{8a^8}+\frac{x^4}{64a^{12}} \\
 & \overline{\qquad \frac{x^3}{8a^8}-\frac{x^4}{64a^{12}}.}
\end{array}$$

Ex. 4.

$$\begin{array}{rl}
 & a^4-x\left(a^2-\frac{x}{2a^2}-\frac{x^2}{8a^6}-\frac{x^3}{16a^{10}}-\text{ etc.}\right. \\
 & a^4 \\
2a^2-\frac{x}{2a^2}\Big) & \overline{\quad -x \quad} \\
 & -x+\frac{x^2}{4a^4} \\
2a^2-\frac{x}{a^2}-\frac{x^2}{8a^6}\Big) & \overline{\quad -\frac{x^2}{4a^4} \quad} \\
 & -\frac{x^2}{4a^4}+\frac{x^3}{8a^8}+\frac{x^4}{64a^{12}} \\
 & \overline{\qquad -\frac{x^3}{8a^8}-\frac{x^4}{64a^{12}}.}
\end{array}$$

Ex. 5.

$$\begin{array}{rl}
 & a^2+x^2\left(a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-\text{ etc.}\right. \\
 & a^2 \\
2a+\frac{x^2}{2a}\Big) & \overline{\quad x^2 \quad} \\
 & x^2+\frac{x^4}{4a^2} \\
2a+\frac{x^2}{a}-\frac{x^4}{8a^3}\Big) & \overline{\quad -\frac{x^4}{4a^2} \quad} \\
 & -\frac{x^4}{4a^2}-\frac{x^6}{8a^4}+\frac{x^8}{64a^6} \\
 & \overline{\qquad \frac{x^6}{8a^4}-\frac{x^8}{64a^6}.}
\end{array}$$

ART. 380, PAGE 270.

Ex. 3. Assume

$$\frac{1+2x}{1-x-x^2}=A+Bx+Cx^2+Dx^3+Ex^4+\text{ etc.}$$

Hence

$$1+2x=A+(B-A)x+(C-A-B)x^2+(D-B-C)x^3+\text{ etc.}$$

Therefore

$$A=1;\ B-A=2 \because B=3;\ C=A+B=4;\ D=B+C=7.$$

Ex. 4. Assume

$$\frac{1-x}{1-2x-3x^2}=A+Bx+Cx^2+Dx^3+\text{ etc.}$$

Hence

$$1-x=A+(B-2A)x+(C-3A-2B)x^2+(D-3B-2C)x^3+\text{ etc.}$$

Therefore

$$A=1;\ B-2A=-1 \because B=1;\ C=3A+2B=5;\ D=3B+2C=13.$$

Each coefficient is equal to twice the preceding one, plus three times the coefficient next preceding.

Ex. 5. Assume

$$\sqrt{1-x}=A+Bx+Cx^2+Dx^3+Ex^4+\text{ etc.}$$

Squaring,

$$1-x=A^2+2ABx+(B^2+2AC)x^2+(2BC+2AD)x^3$$
$$+(C^2+2BD+2AE)x^4+\text{ etc.}$$

Hence

$$A=1;\ 2AB=-1 \because B=-\tfrac{1}{2};\ B^2+2AC=0 \because C=-\tfrac{1}{8};$$
$$D=-\tfrac{1}{16};\ E=-\tfrac{5}{128},\text{ etc.}$$

Ex. 6. Assume $\frac{1-x}{1+x+x^2}=A+Bx+Cx^2+Dx^3+\text{ etc.}$

Hence

$$1-x=A+(A+B)x+(A+B+C)x^2+(B+C+D)x^3+\text{ etc.}$$

Therefore

$$A=1;\ A+B=-1 \because B=-2;\ A+B+C=0 \because C=+1;$$
$$B+C+D=0 \because D=+1,\text{ etc.}$$

Ex. 7. Assume

$$\sqrt{a^2-x^2}=A+Bx+Cx^2+Dx^3+Ex^4+\text{ etc.}$$

Squaring,

$$a^2-x^2=A^2+2ABx+(B^2+2AC)x^2+(2BC+2AD)x^3$$
$$+(C^2+2BD+2AE)x^4+\text{ etc.}$$

Therefore

$$A=a;\ 2AB=0 \therefore B=0;\ B^2+2AC=-1 \therefore C=-\frac{1}{2a};$$

$$2BC+2AD=0 \therefore D=0;\ C^2+2BD+2AE=0 \therefore E=-\frac{1}{8a^3}, \text{ etc.}$$

Hence $\sqrt{a^2-x^2}=a-\frac{x^2}{2a}-\frac{x^4}{8a^3}-\frac{x^6}{32a^5}-$ etc.

Art. 382, Page 272.

Ex. 2. Assume $\frac{5x+1}{x^2-1}=\frac{A}{x+1}+\frac{B}{x-1}.$

Hence $5x+1=Ax-A+Bx+B.$

Therefore $1=-A+B;$

$5=A+B.$

Hence $A=2$ and $B=3.$

Ex. 3. Assume $\frac{5x-19}{x^2-8x+15}=\frac{A}{x-5}+\frac{B}{x-3}.$

Hence $5x-19=Ax-3A+Bx-5B.$

Therefore $19=3A+5B;$

$5=A+B.$

Hence $A=3$ and $B=2.$

Ex. 4. Assume $\frac{3x^2-1}{x^3-x}=\frac{A}{x+1}+\frac{B}{x-1}+\frac{C}{x}.$

Therefore $3x^2-1=Ax^2-Ax+Bx^2+Bx+Cx^2-C.$

Hence $1=C;$

$0=-A+B;$

$3=A+B+C.$

Hence $A=1$, $B=1$, and $C=1.$

Ex. 5. Assume

$$\frac{2x^2-6x+6}{(x-1)(x-2)(x-3)}=\frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{x-3}.$$

Therefore

$$2x^2-6x+6=Ax^2-5Ax+6A+Bx^2-4Bx+3B+Cx^2-3Cx+2C.$$

Hence $6=6A+3B+2C;$

$6=5A+4B+3C;$

$2=A+B+C.$

Hence $A=1$, $B=-2$, and $C=3$.

Ex. 6. Assume

$$\frac{5x^2+2x-1}{(x+1)(x-1)(2x+1)}=\frac{A}{x+1}+\frac{B}{x-1}+\frac{C}{2x+1}.$$

Hence

$$5x^2+2x-1=2Ax^2-Ax-A+2Bx^2+3Bx+B+Cx^2-C.$$

Therefore

$$1=A-B+C;$$
$$2=-A+3B;$$
$$5=2A+2B+C.$$

Hence $A=1$, $B=1$, and $C=1$.

Ex. 7. Assume

$$\frac{13+21x+2x^2}{1-5x^2+4x^4}=\frac{A}{1+x}+\frac{B}{1-x}+\frac{C}{1+2x}+\frac{D}{1-2x}.$$

Therefore

$$13+21x+2x^2=A-Ax-4Ax^2+4Ax^3+B+Bx-4Bx^2-4Bx^3$$
$$+C-2Cx-Cx^2+2Cx^3+D+2Dx-Dx^2-2Dx^3.$$

Hence

$$13=A+B+C+D;$$
$$21=-A+B-2C+2D;$$
$$-2=4A+4B+C+D;$$
$$0=4A-4B+2C-2D.$$

Hence $A=1$, $B=-6$, $C=2$, and $D=16$.

ART. 383, PAGE 273.

Ex. 2.

$$y=\begin{array}{l|l|l|l} Ay+B & y^2+C & y^3+D & y^4+\text{ etc.} \\ -\frac{1}{2}A^2 & -AB & -AC & \\ & +\frac{1}{4}A^3 & -\frac{1}{2}B^2 & \\ & & +\frac{3}{4}A^2B & \\ & & -\frac{1}{8}A^4 & \end{array}$$

Hence

$$A=+1;\ B-\tfrac{1}{2}A^2=0\ \because B=+\tfrac{1}{2};\ C-AB+\tfrac{1}{4}A^3=0\ \because C=+\tfrac{1}{4};$$
$$D-AC-\tfrac{1}{2}B^2+\tfrac{3}{4}A^2B-\tfrac{1}{8}A^4=0\ \because D=+\tfrac{1}{8}.$$

Ex. 3.

$$y=Ay+\left(B-\frac{A^2}{2}\right)y^2+\left(C-AB+\frac{A^3}{3}\right)y^3$$
$$+\left(D-AC-\frac{B^2}{2}+A^2B-\frac{A^4}{4}\right)y^4$$
$$+\left(E-AD-BC+A^2C+AB^2-A^3B+\frac{A^5}{5}\right)y^5+\text{ etc.}$$

Hence

$$A=1;\ B=\tfrac{1}{2};\ C=AB-\frac{A^3}{3}=\tfrac{1}{6};\ D=AC+\frac{B^2}{2}-A^2B+\frac{A^4}{4}$$

$$=\tfrac{1}{24};\ E=AD+BC-A^2C-AB^2+A^3B-\frac{A^5}{5}=\tfrac{1}{120}, \text{etc.}$$

Ex. 4. Proceeding in the usual manner, we find $B=0$, $D=0$, $F=0$, etc. Hence it is more convenient to omit these terms, and assume $x=Ay+Cy^3+Ey^5+Gy^7+$ etc.,
and we find

$$y=Ay+(C+A^3)y^3+(E+3A^2C+A^5)y^5+(G+3A^2E+3AC^2$$
$$+5A^4C+A^7)y^7+(I+3A^2G+6ACE+C^3+5A^4E+10A^3C^2$$
$$+7A^6C+A^9)y^9+\text{etc.}$$

Hence

$$A=+1;\ C=-1;\ E=+2;\ G=-5;\ I=+14, \text{etc.}$$

Ex. 5. $y=Ay+(B+3A^3)y^2+(C+6AB+5A^5)y^3+(D+6AC$
$+3B^2+15A^2B+7A^4)y^4+(E+6AD+6BC+15A^2C+15AB^2$
$+28A^3B+9A^5)y^5+$ etc.
Hence $A=1;\ B=-3;\ C=+13;\ D=-67;\ E=+381$, etc.

ART. 384, PAGE 274.

Ex. 1.

$$s=\frac{A}{2}s+\left(\frac{B}{2}+\frac{A^2}{8}\right)s^2+\left(\frac{C}{2}+\frac{AB}{4}+\frac{A^3}{48}\right)s^3$$
$$+\left(\frac{D}{2}+\frac{AC}{4}+\frac{B^2}{8}+\frac{A^2B}{16}+\frac{A^4}{384}\right)s^4+\text{etc.}$$

Hence $A=2;\ B=-1;\ C=+\tfrac{2}{3};\ D=-\tfrac{1}{2}$, etc.

Value of x,

	+0.500000
	−0.062500
	+0.010417
	−0.001953
	+0.000390
	0.446354

Ex. 2. Assume $x=As+Cs^3+Es^5+Gs^7$, etc.,
and we have

$$s=2As+(2C+3A^3)s^3+(2E+9A^2C+4A^5)s^5$$
$$+(2G+9A^2E+9AC^2+20A^4C+5A^7)s^7+\text{etc.}$$

Hence

$A=\frac{1}{2}$; $C=-\frac{3}{16}$; $E=+\frac{19}{128}$; $G=-\frac{152}{1024}$, etc.

Value of x,

$$\begin{array}{r} +0.2500 \\ -0.0234 \\ +0.0046 \\ -0.0012 \\ \hline 0.2300 \end{array}$$

Ex. 3. Assume $x=As+Cs^3+Es^5+Gs^7+$ etc.,
and we have

$$s=As+\left(C-\frac{A^3}{3}\right)s^3+\left(E-A^2C+\frac{A^5}{5}\right)s^5$$
$$+\left(G-A^2E-AC^2+A^4C-\frac{A^7}{7}\right)s^7+\text{ etc.}$$

Hence $A=1$; $C=\frac{1}{3}$; $E=\frac{2}{15}$; $G=\frac{17}{315}$, etc.

Value of x,

$$\begin{array}{r} +0.33333 \\ +0.01235 \\ +0.00055 \\ +0.00002 \\ \hline 0.34625 \end{array}$$

Ex. 4. This series is similar to that in Ex. 3, Art. 383, except the signs of the terms.

Value of x,

$$\begin{array}{r} +0.2000000 \\ -0.0200000 \\ +0.0013333 \\ -0.0000667 \\ +0.0000027 \\ -0.0000001 \\ \hline 0.1812692 \end{array}$$

Art. 389, Page 279.

Ex. 4. $a^{-2}+2a^{-3}b+3a^{-4}b^2+4a^{-5}b^3+5a^{-6}b^4+$ etc.,

or $$\frac{1}{a^2}+\frac{2b}{a^3}+\frac{3b^2}{a^4}+\frac{4b^3}{a^5}+\frac{5b^4}{a^6}+\text{ etc.}$$

Ex. 2. **Art. 390, Page 280.**

$$(x-a)^{\frac{1}{2}}=x^{\frac{1}{2}}-\tfrac{1}{2}x^{-\frac{1}{2}}a-\frac{1}{2.4}x^{-\frac{3}{2}}a^2-\frac{1.3}{2.4.6}x^{-\frac{5}{2}}a^3-\frac{1.3.5}{2.4.6.8}x^{-\frac{7}{2}}a^4-\text{etc.}$$

$$=x^{\frac{1}{2}}\left\{1-\frac{a}{2x}-\frac{a^2}{2.4x^2}-\frac{3a^3}{2.4.6x^3}-\frac{3.5a^4}{2.4.6.8x^4}-\text{etc.}\right\}.$$

Ex. 6.

$$(a+x)^{\frac{2}{3}}=a^{\frac{2}{3}}\left\{1+\frac{2x}{3a}-\frac{2x^2}{3.6a^2}+\frac{2.4x^3}{3.6.9a^3}-\frac{2.4.7x^4}{3.6.9.12a^4}+\frac{2.4.7.10x^5}{3.6.9.12.15a^5}-\text{etc.}\right\}.$$

Art. 393, Page 283.

Ex. 3. $\sqrt[3]{9}=(8+1)^{\frac{1}{3}}$. *Computation.*

```
 2.0833333
-0.0034722
+0.0002411
-0.0000201
+0.0000019
-0.0000002
 2.0800838, Ans.
```

Ex. 4. $\sqrt[3]{31}=(27+4)^{\frac{1}{3}}$. *Computation.*

```
 3.1481481
-0.0073159
+0.0006021
-0.0000594
+0.0000064
-0.0000007
 3.1413806, Ans.
```

Ex. 5. $\sqrt[5]{30}=(32-2)^{\frac{1}{5}}$. *Computation.*

```
 2.0000000
-0.0250000
-0.0006250
-0.0000234
-0.0000010
 1.9743506, Ans.
```

CHAPTER XX.

Art. 411, Page 293.

Ex. 3. 355.5. Ex. 4. 2.220.

ART. 412, PAGE 294.

Ex. 3. 27.69.

Ex. 4. 317.5.

ART. 413, PAGE 295.

Ex. 3. 3.207.

Ex. 4. 0.4605.

ART. 414, PAGE 295.

Ex. 4. 0.6156.

ART. 415, PAGE 296.

Ex. 2. 36.70.

Ex. 3. 7380.

ART. 417, PAGE 296.

Ex. 2. $x=1.544$.

Ex. 3. $x=0.7093$.

Ex. 4. $x=0.8451$.

Ex. 5. $x=2.262$.

Ex. 6. $x=3.831$.

ART. 419, PAGE 298.

Ex. 5. $620.70.

Ex. 6. $270.70.

Ex. 7. 5 per cent.

Ex. 8. 11.89 years.

Ex. 9. 14.20 years.

ART. 422, PAGE 300.

Ex. 3. $8515.00.

Ex. 4. $5972.00.

Ex. 5.

$$\frac{Ar}{a}+1=(1+r)^n;$$

$$\frac{A}{a}=10 \because \frac{Ar}{a}+1=\tfrac{7}{5};$$

$$n=\frac{\log.\frac{7}{5}}{\log. 1.04}=8.59 \text{ years.}$$

ART. 423, PAGE 300.

Ex. 4. 36.98 years.

ART. 429, PAGE 304.

Computation of log. 2,	0.666666
	0.024691
	0.001646
	0.000131
	0.000012
	0.000001
log. 2=	0.693147
Computation of log. 3,	0.693147
	0.400000
	0.005333
	0.000128
	0.000004
log. 3=	1.098612
Computation of log. 5,	1.386294
	0.222222
	0.000915
	0.000007
log. 5=	1.609438
Computation of log. 7,	1.791759
	0.153846
	0.000303
	0.000002
log. 7=	1.945910

CHAPTER XXI.

ART. 435, PAGE 308.

Ex. 3. The first member is divisible by $x-2$, and gives

$$x^2-9x+18=0.$$

Ex. 4. The first member is divisible by $x-4$, and gives

$$x^2+5x-14=0.$$

Ex. 5. The first member is divisible by $x+1$, and gives

$$x^3-39x^2+249x+289=0.$$

Ex. 6. The first member is divisible by $x+5$, and gives

$$x^4+x^3-15x^2-37x-22=0.$$

Ex. 7. The first member is divisible by $x-3$, and gives

$$x^6+4x^5-2x^4-20x^3-11x^2+16x+12=0.$$

Art. 436, Page 309.

Ex. 1. Dividing by $x-1$, we have

$$x^2+4x-12=0,$$

which, being solved, gives

$$x=+2 \text{ or } -6.$$

Ex. 2. Dividing successively by $x-1$ and $x-3$, we have

$$x^2-6x+8=0,$$

which, being solved, gives $x=2$ or 4.

Ex. 3. Dividing successively by $x-3$ and $x-5$, we have

$$x^2-4x+1=0,$$

which, being solved, gives

$$x=2\pm\sqrt{3}.$$

Ex. 4. Dividing successively by $x-2$ and $x-3$, we have

$$4x^2+6x+1=0,$$

which gives $$x=\frac{-3\pm\sqrt{5}}{4}.$$

Ex. 5. Dividing successively by $x-2$ and $x+2$, we have

$$x^2-6x+4=0,$$

which gives $$x=3\pm\sqrt{5}.$$

Art. 437, Page 310.

$$\begin{array}{r}
-1+\sqrt{-3} \\
-1+\sqrt{-3} \\
\hline
1-\sqrt{-3} \\
-\sqrt{-3}-3 \\
\hline
-2-2\sqrt{-3} = \text{the square.} \\
-1+\sqrt{-3} \\
\hline
2+2\sqrt{-3} \\
-2\sqrt{-3}+6 \\
\hline
8 = \text{the third power.}
\end{array}$$

$$\begin{array}{r}
-1-\sqrt{-3} \\
-1-\sqrt{-3} \\
\hline
1+\sqrt{-3} \\
+\sqrt{-3}-3 \\
\hline
-2+2\sqrt{-3} = \text{the square.} \\
-1-\sqrt{-3} \\
\hline
2-2\sqrt{-3} \\
+2\sqrt{-3}+6 \\
\hline
8 = \text{the third power.}
\end{array}$$

Ex. 1. $x^4=81 \because x^2=\pm 9 \because x=\pm\sqrt{9}=+3$ or -3.

Also, $x=\pm\sqrt{-9}=\pm 3\sqrt{-1}$.

Ex. 2. $x^6=64 \because x^3=\pm 8 \because x=+2$ or -2.

Dividing x^6-64 by x^2-4, we obtain

$$x^4+4x^2+16=0,$$

which gives $x^2=-2\pm 2\sqrt{-3}$.

Hence $x=\pm\sqrt{-2\pm 2\sqrt{-3}}$.

ART. 438, PAGE 312.

Ex. 3. $x^5-4x^3+2x^2+3x-2=0$.
Ex. 5. $x^4-6x^3+18x^2-26x+21=0$.
Ex. 6. $x^4-6x^3+11x^2-10x+2=0$.

ART. 441, PAGE 314.

Ex. 2. Substitute $\frac{y}{6}$ for x.

Ex. 3. Substitute $\frac{y}{6}$ for x.

Ex. 4. Substitute $\frac{y}{6}$ for x, and we obtain

$$y^3+13y^2+6y-72=0.$$

Ex. 5. Substitute $\frac{y}{2}$ for x, and we obtain

$$y^4-18y^2+64y+33=0.$$

Ex. 6. Substitute $\frac{y}{3}$ for x, and we obtain

$$y^3-14y^2+63y-90=0.$$

ART. 442, PAGE 315.

Ex. 1. -1, -3, and $+2$.
Ex. 2. -1, -2, and -3.
Ex. 3. $+1$, -5, $-1-\sqrt{-1}$, and $-1+\sqrt{-1}$.

ART. 443. PAGE 316.

Ex. 1. One root is $1-\sqrt{-1}$. Dividing the given polynomial by x^2-2x+2, we obtain $x+2=0$. Hence -2 is a root of the equation.

Ex. 2. One root is $2-\sqrt{-1}$. Dividing the given polynomial by x^2-4x+5, we obtain $x+3=0$. Hence -3 is a root of the equation.

Ex. 3. One root is $1-2\sqrt{-1}$. Dividing the given polynomial by x^2-2x+5, we obtain $x+1=0$. Hence -1 is a root of the equation.

Ex. 4. One root of the equation is $2-\sqrt{3}$. Dividing the given polynomial by x^2-4x+1, we obtain $x^2-1=0$. Hence $+1$ and -1 are roots of the equation.

Ex. 5. Two roots of the equation are $-1-\sqrt{-1}$, and $1+\sqrt{-3}$. Dividing the given polynomial by x^2+2x+2, and this result by x^2-2x+4, we obtain $x^4-4=0$. Hence $x=\pm\sqrt{2}$ or $\pm\sqrt{-2}$.

ART. 444, PAGE 317.

Ex. 1. Substitute $y-1$ in place of x.

$$\begin{array}{rl} x^3= & y^3-3y^2+3y-1 \\ 3x^2= & \quad\; 3y^2-6y+3 \\ -4x= & \qquad\quad -4y+4 \\ +1= & \qquad\qquad\; +1 \\ \hline \textit{Ans.}\; & y^3 \qquad -7y+7=0 \end{array}$$

Ex. 2. Substitute $y+1$ in place of x.

$$\begin{array}{rl} x^3= & y^3+3y^2+3y+1 \\ -2x^2= & \quad -2y^2-4y-2 \\ 3x= & \qquad\qquad 3y+3 \\ -4= & \qquad\qquad\quad -4 \\ \hline \textit{Ans.}\; & y^3+y^2+2y-2=0 \end{array}$$

Ex. 3. Substitute $y-3$ in place of x.

$$\begin{array}{rl} x^4= & y^4-12y^3+54y^2-108y+81 \\ 9x^3= & \qquad 9y^3-81y^2+243y-243 \\ 12x^2= & \qquad\qquad 12y^2-72y+108 \\ -14x= & \qquad\qquad\qquad -14y+42 \\ \hline \textit{Ans.}\; & y^4-3y^3-15y^2+49y-12=0 \end{array}$$

Ex. 4. Substitute $y+2$ for x.

$$\begin{array}{rl}
5x^4= & 5y^4+40y^3+120y^2+160y+80 \\
-12x^3= & \quad -12y^3-\ 72y^2-144y-96 \\
3x^2= & \qquad\qquad 3y^2+\ 12y+12 \\
4x= & \qquad\qquad\qquad 4y+\ 8 \\
-5= & \qquad\qquad\qquad\quad -\ 5 \\
\hline
\textit{Ans.} & 5y^4+28y^3+\ 51y^2+\ 32y-\ 1=0
\end{array}$$

Ex. 5. Substitute $y-2$ for x.

$$\begin{array}{rl}
x^5= & y^5-10y^4+40y^3-\ 80y^2+\ 80y-\ 32 \\
10x^4= & \quad 10y^4-80y^3+240y^2-320y+160 \\
42x^3= & \qquad 42y^3-252y^2+504y-336 \\
86x^2= & \qquad\qquad 86y^2-344y+344 \\
70x= & \qquad\qquad\qquad 70y-140 \\
12= & \qquad\qquad\qquad\quad 12 \\
\hline
\textit{Ans.} & y^5 \qquad +\ 2y^3-\ \ 6y^2-\ 10y+\ \ 8=0
\end{array}$$

ART. 445, PAGE 318.

Ex. 1.

$$\begin{array}{rl}
x^3= & y^3+6y^2+12y+\ 8 \\
-6x^2= & \quad -6y^2-24y-24 \\
8x= & \qquad\qquad 8y+16 \\
-2= & \qquad\qquad\quad -\ 2 \\
\hline
\textit{Ans.} & y^3 \qquad -\ 4y-\ 2=0
\end{array}$$

Ex. 2.

$$\begin{array}{rl}
x^4= & y^4+16y^3+\ 96y^2+256y+\ 256 \\
-16x^3= & \quad -16y^3-192y^2-768y-1024 \\
-6x= & \qquad\qquad\qquad -6y-\ \ 24 \\
15= & \qquad\qquad\qquad\quad 15 \\
\hline
\textit{Ans.} & y^4 \qquad -\ 96y^2-518y-\ 777=0
\end{array}$$

Ex. 3. Put $x=y-3$.

$$\begin{array}{rl}
x^5= & y^5-15y^4+\ 90y^3-270y^2+\ 405y-\ 243 \\
15x^4= & \quad 15y^4-180y^3+810y^2-1620y+1215 \\
12x^3= & \qquad 12y^3-108y^2+\ 324y-\ 324 \\
-20x^2= & \qquad\qquad -\ 20y^2+\ 120y-\ 180 \\
14x= & \qquad\qquad\qquad 14y-\ \ 42 \\
-25= & \qquad\qquad\qquad\quad -\ \ 25 \\
\hline
\textit{Ans.} & y^5 \qquad -\ 78y^3+412y^2-\ 757y+\ 401=0
\end{array}$$

Ex. 4. Put $x=y+2$.

$$\begin{aligned} x^4 &= y^4+8y^3+24y^2+32y+16 \\ -8x^3 &= \quad -8y^3-48y^2-96y-64 \\ 5 &= \qquad\qquad\qquad\qquad\qquad 5 \\ \hline \textit{Ans. } y^4 & \qquad -24y^2-64y-43=0 \end{aligned}$$

Art. 446, Page 319.

Ex. 2. When $x=4$, the first member of the equation reduces to -6; and when $x=5$, it reduces to $+10$. Hence there must be a root between 4 and 5; that is, 4 is the first figure of one of the roots.

Ex. 3. When $x=-2$, the expression reduces to $-\ 4$.

$x=-1$,	"	"	$+\ 9$.
$x=\ \ 0$,	"	"	$+\ 8$.
$x=+1$,	"	"	$-\ 1$.
$x=+2$,	"	"	-12.
$x=+3$,	"	"	-19.
$x=+4$,	"	"	-16.
$x=+5$,	"	"	$+\ 3$.

Hence the initial figures of the roots are

$-1, 0$, and $+4$.

Art. 448, Page 321.

Ex. 1. There are three variations of sign, and therefore three positive roots.

Ex. 2. There is only one permanence, and therefore only one negative root.

Ex. 3. There are three variations of sign, and therefore three positive roots.

Art. 450, Page 322.

Ex. 2. 1st derivative, $4x^3-24x^2+28x+4$.

2d	"	$12x^2-48x+28$.
3d	"	$24x-48$.
4th	"	24.

Ex. 3. 1st derivative, $5x^4+12x^3+6x^2-6x-2.$
2d " $20x^3+36x^2+12x-6.$
3d " $60x^2+72x+12.$
4th " $120x+72.$
5th " $120.$

Ex. 4. $nx^{n-1}+A(n-1)x^{n-2}+B(n-2)x^{n-3}+\dots+T.$

ART. 451, PAGE 323.

Ex. 2. The first derivative is $3x^2-26x+55$. Find the greatest common divisor between this and the given polynomial.

$$\begin{array}{r|l} 3x^3-39x^2+165x-225 & 3x^2-26x+55 \\ 3x^3-26x^2+\ \ 55x & x,\ -13 \\ \hline -13x^2+110x-225 & \end{array}$$

Multiply by 3, $$\begin{array}{r} -39x^2+330x-675 \\ -39x^2+338x-715 \\ \hline -\ \ 8x+\ \ 40 \end{array}$$

$$\begin{array}{r|l} 3x^2-26x+55 & x-5 \\ 3x^2-15x & 3x-11 \\ \hline -11x+55 & \end{array}$$

The greatest common divisor is $x-5$.

Ex. 3. The first derivative is $3x^2-14x+16$.

$$\begin{array}{r|l} 3x^3-21x^2+48x-\ \ 36 & 3x^2-14x+16 \\ 3x^3-14x^2+16x & x,\ -7 \\ \hline -\ \ 7x^2+32x-\ \ 36 & \end{array}$$

Multiply by 3, $$\begin{array}{r} -21x^2+96x-108 \\ -21x^2+98x-112 \\ \hline -\ \ 2x+\ \ 4 \end{array}$$

$$\begin{array}{r|l} 3x^2-14x+16 & x-2 \\ 3x^2-6x & 3x-8 \\ \hline -8x\ +16 & \end{array}$$

The greatest common divisor is $x-2$.

Ex. 4. The first derivative is $4x^3-12x-8$.

$$\begin{array}{r|l} x^4-6x^2-8x-3 & x^3-3x-2 \\ x^4-3x^2-2x & x \\ \hline -3x^2-6x-3 & \end{array}$$

$$\begin{array}{r|l} x^3-3x-2 & x^2+2x+1 \\ x^3+2x^2+x & x-2 \\ \hline -2x^2-4x-2 & \end{array}$$

The greatest common divisor is x^2+2x+1 or $(x+1)^2$. Hence the equation has three roots equal to -1.

Ex. 5. The first derivative is $3x^2-6x-9$.

$$\begin{array}{r|l} x^3-3x^2-9x+27 & x^2-2x-3 \\ x^3-2x^2-3x & x-1 \\ \hline -\ x^2-6x+27 & \\ -\ x^2+2x+\ 3 & \\ \hline -8x+24 & \end{array}$$

$$\begin{array}{r|l} x^2-2x-3 & x-3 \\ x^2-3x & x+1 \\ \hline x-3 & \end{array}$$

The greatest common divisor is $x-3$. Hence the equation has two roots, each equal to $+3$.

Ex. 6. The first derivative is $3x^2+16x+20$.

$$\begin{array}{r|l} 3x^3+24x^2+60x+48 & 3x^2+16x+20 \\ 3x^3+16x^2+20x & x,\ +1 \\ \hline 8x^2+40x+48 & \end{array}$$

Multiply by $\frac{3}{8}$, $3x^2+15x+18$

$$\begin{array}{r} 3x^2+15x+18 \\ 3x^2+16x+20 \\ \hline -\quad x-\ 2 \end{array}$$

$$\begin{array}{r|l} 3x^2+16x+20 & x+2 \\ 3x^2+\ 6x & 3x+10 \\ \hline 10x+20 & \end{array}$$

The greatest common divisor is $x+2$. Hence the equation has two roots, each equal to -2.

ART. 458, PAGE 330.

Ex. 4.

$$X = x^3-7x+7,$$
$$X_1 = 3x^2-7,$$
$$R = 2x-3,$$
$$R_1 = +1.$$

When $x=+2$,	the signs are	$+ + + +$,	giving	no	variations.
$x=+1\frac{1}{2}$,	"	$- - 0 +$,	"	1	"
$x=+1$,	"	$+ - - +$,	"	2	"
$x=-3$,	"	$+ + - +$,	"	2	"
$x=-4$,	"	$- + - +$,	"	3	"

Ex. 5.

$$X=2x^4-20x+19,$$
$$X_1=2x^3-5,$$
$$R=15x-19,$$
$$R_1=+3157.$$

When $x=-\infty$,	the signs are	$+ - - +$,	giving	2	variations.
$x=0$,	"	$+ - - +$,	"	2	"
$x=+1$,	"	$+ - - +$,	"	2	"
$x=+1\frac{1}{2}$,	"	$- + + +$,	"	1	"
$x=+2$,	"	$+ + + +$,	"	0	"
$x=+\infty$,	"	$+ + + +$,	"	0	"

Hence this equation has but two real roots.

Ex. 6.

$$X=x^5+2x^4+3x^3+4x^2+5x-20,$$
$$X_1=5x^4+8x^3+9x^2+8x+5,$$
$$R=-7x^3-21x^2-42x+255,$$
$$R_1=-13x+14,$$
$$R_2=-388,147.$$

When $x=-\infty$,	the signs are	$- + + + -$,	giving	2	variations.
$x=0$,	"	$- + + + -$,	"	2	"
$x=+1$,	"	$- + + + -$,	"	2	"
$x=+2$,	"	$+ + + - -$,	"	1	"
$x=+\infty$,	"	$+ + - - -$,	"	1	"

Hence this equation has but one real root, and it is situated between 1 and 2.

Ex. 7.

$$X=x^3+3x^2+5x-178,$$
$$X_1=3x^2+6x+5,$$
$$R=-4x+539,$$
$$R_1=-884,579.$$

When $x=-\infty$,	the signs are	$- + + -$,	giving	2	variations.
$x=0$,	"	$- + + -$,	"	2	"
$x=+4$,	"	$- + + -$,	"	2	"
$x=+5$,	"	$+ + + -$,	"	1	"
$x=+\infty$,	"	$+ + - -$,	"	1	"

Hence this equation has but one real root, and it is situated between 4 and 5.

Ex. 8.
$$X = x^4 - 12x^2 + 12x - 3,$$
$$X_1 = x^3 - 6x + 3,$$
$$R = 2x^2 - 3x + 1,$$
$$R_1 = 17x - 9,$$
$$R_2 = +8.$$

When $x = -\infty$,	the signs are	$+ - + - +$,	giving	4	variations.
$x = -4$,	"	$+ - + - +$,	"	4	"
$x = -3$,	"	$- - + - +$,	"	3	"
$x = 0$,	"	$- + + - +$,	"	3	"
$x = +\frac{1}{2}$,	"	$+ + 0 - +$,	"	2	"
$x = +1$,	"	$- - 0 + +$,	"	1	"
$x = +2$,	"	$- - + + +$,	"	1	"
$x = +3$,	"	$+ + + + +$,	"	0	"
$x = +\infty$,	"	$+ + + + +$,	"	0	"

Hence this equation has four real roots.

Ex. 9.
$$X = x^4 - 8x^3 + 14x^2 + 4x - 8,$$
$$X_1 = x^3 - 6x^2 + 7x + 1,$$
$$R = 5x^2 - 17x + 6,$$
$$R_1 = 76x - 103,$$
$$R_2 = +45,375.$$

When $x = -\infty$,	the signs are	$+ - + - +$,	giving	4	variations.
$x = -1$,	"	$+ - + - +$,	"	4	"
$x = 0$,	"	$- + + - +$,	"	3	"
$x = +1$,	"	$+ + - - +$,	"	2	"
$x = +2$,	"	$+ - - + +$,	"	2	"
$x = +3$,	"	$- - 0 + +$,	"	1	"
$x = +5$,	"	$- + + + +$,	"	1	"
$x = +6$,	"	$+ + + + +$,	"	0	"
$x = +\infty$,	"	$+ + + + +$,	"	0	"

Hence this equation has four real roots.

ART. 459, PAGE 332.

Ex. 1. When $y = 2$, Eq. 2 reduces to $x + 2 - 5 = 0$ or $x = 3$.
When $y = 3$, " " $x + 3 - 5 = 0$ or $x = 2$.

Ex. 2. When $y=2$, Eq. 1 reduces to $x+8x-18=0$ or $x=2$.

When $y=\frac{1}{2}$, " " $x+\frac{x}{8}-18=0$ or $x=16$.

Ex. 3. When $y=1$, Eq. 2 reduces to $x^3-x^2+x-3=0$ or $x=3$.

Ex. 4. Since $x-2y=0$, when $y=1$, $x=2$; and when $y=0$, $x=0$.

Ex. 5. When $x=1$, Eq. 2 reduces to $y^2+y=0$. Whence

$$y=0 \text{ or } -1.$$

When $y=1$, Eq. 2 reduces to $x^2-5x+6=0$. Whence

$$x^2-5x+\frac{25}{4}=\tfrac{1}{4};$$

$$x=\tfrac{5}{2}\pm\tfrac{1}{2}=3 \text{ or } 2.$$

Ex. 6. When $y=2$, Eq. 2 reduces to $x^2+4x=0$. Whence

$$x=0 \text{ or } -4.$$

When $y=3$, Eq. 1 reduces to

$$x^3+6x^2+6x+5=0, \quad (1)$$

and Eq. 2 reduces to $x^2+6x+5=0$. (2)

Multiply (2) by x, $x^3+6x^2+5x=0$. (3)

Subtract (3) from (1), $x+5=0$.

Hence $x=-5$.

CHAPTER XXII.

ART. 464, PAGE 337.

Ex. 5.

$1-12+47-72+36$	1.
$1-11+36-36$	2.
$1-\ \ 9+18$	3.
$1-\ \ 6$	6.

Ex. 6.

$1+2-7-\ \ 8+12$	1.
$1+3-4-12$	2.
$1+5+6$	$-2.$
$1+3$	$-3.$

The four roots are 1, 2, -2, and -3.

Ex. 7.

$1+\ \ 0-55-\ \ 30+504$	$+3.$
$1+\ \ 3-46-168$	$+7.$
$1+10+24$	$-4.$
$1+\ \ 6$	$-6.$

The four roots are $+3$, $+7$, -4, and -6.

Ex. 8.

$$\begin{array}{l|r} 1+0-25+60-36 & 1. \\ 1+1-24+36 & 2. \\ 1+3-18 & 3. \\ 1+6 & -6. \end{array}$$

The four roots are 1, 2, 3, and -6.

Ex. 9.

$$\begin{array}{l|r} 1-1-1+19-42 & 2. \\ 1+1+1+21 & -3. \\ 1-2+7 & \end{array}$$

Supplying the letters to the last coefficients, we have

$$x^2-2x+7=0.$$

Hence $$x^2-2x+1=-6;$$

$$x=1\pm\sqrt{-6}.$$

The four roots are therefore $+2$, -3, and $1\pm\sqrt{-6}$.

Ex. 10.

$$\begin{array}{l|r} 1+5+\ 1-16-20-16 & +2. \\ 1+7+15+14+\ 8 & -2. \\ 1+5+\ 5+\ 4 & -4. \\ 1+1+\ 1 & \end{array}$$

Supplying the letters, we have

$$x^2+x+1=0.$$

Hence $$x^2+x+\tfrac{1}{4}=-\tfrac{3}{4};$$

$$x=-\tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{-3}.$$

The five roots therefore are $+2$, -2, -4, and $-\frac{1}{2}\pm\frac{1}{2}\sqrt{-3}$.

Art. 467, Page 343.

Ex. 2. Dividing the original equation by $x-3.21312$, we obtain

$$x^2+14.21312x-56.33154=0.$$

Hence $$x^2+14.21312x+50.50320=106.83474;$$

$$x=-7.10656\pm10.33609$$

$$=3.22953 \text{ or } -17.44265.$$

Ex. 4. Dividing the equation $x^3+1.5x^2-425=0$ by

$$x-7.050256,$$

we obtain $$x^2+8.550256x+60.281494=0,$$

where q is negative, and greater than $\frac{p^2}{4}$. See Art. 280.

Ex. 7.

1 − 15	+ 63	= + 50 (7.39543 −
− 8	+ 7	49
− 1	0 = 1st divisor.	1 = 1st dividend.
+ 6.3	1.89	0.567
6.6	3.87 = 2d divisor.	0.433 = 2d dividend.
6.99	4.4991	0.404919
7.08	5.1363 = 3d divisor.	28081 = 3d dividend.
7.175	5.172175	25860875
7.180	5.208075 = 4th div'r.	2220125 = 4th dividend.
7.1854	5.21094916	2084379664
7.1858	5.21382348 = 5th div'r.	135745336 = 5th div'd.

Dividing the original equation by $x - 7.39543$, we obtain

$$x^2 - 7.60457x + 6.76094 = 0.$$

Hence $\quad x^2 - 7.60457x + 14.45737 = 7.69643;$

$$x = 3.80228 \pm 2.77424$$

$$= 6.57652 \text{ or } 1.02804.$$

Ex. 8. Changing the signs of the alternate terms, we have

$$x^3 - 9x^2 + 24x = 17.$$

1 − 9	+ 24	= + 17 (4.53209 −
− 5	4	16
− 1	0 = 1st divisor.	1 = 1st dividend.
+ 3.5	1.75	0.875
4.0	3.75 = 2d divisor.	0.125 = 2d dividend.
4.53	3.8859	0.116577
4.56	4.0227 = 3d divisor.	8423 = 3d dividend.
4.592	4.031884	8063768
4.594	4.041072 = 4th divisor.	359232 = 4th dividend.
4.59609	4.0414856481	363733

Dividing the above equation by $x - 4.53209$, we obtain

$$x^2 - 4.46791x + 3.75103 = 0.$$

Hence $\quad x^2 - 4.46791x + 4.99055 = 1.23952;$

$$x = 2.23395 \pm 1.11334$$

$$= 3.34729 \text{ or } 1.12061.$$

Hence the roots of the original equation are

$$-1.12061, -3.34729, \text{ and } -4.53209.$$

Ex. 9.

1+0	+0	=48,228,544 (364
3	9	27
6	27=1st divisor.	21228=1st dividend.
96	3276	19656
102	3888=2d divisor.	1572544=2d dividend.
1084	393136	1572544

Ex. 10. Let x denote the less number; then $x+2$ will denote the greater.

By the conditions, $(x^2+2x)(2x+2)=100$;

$$x^3+3x^2+2x=50.$$

1+3	+2	=50 (2.77449+
5	12	24
7	26=1st divisor.	26=1st dividend.
9.7	32.79	22.953
10.4	40.07=2d divisor.	3.047=2d dividend.
11.17	40.8519	2.859633
11.24	41.6387=3d div'r.	0.187367=3d dividend.
11.314	41.683956	0.166735824
11.318	41.729228=4th div.	0.020631176=4th divid'd.
11.3224		

The two numbers are 2.77449 and 4.77449.

Ex. 11. Let $x+3$ and $x-3$ denote the two numbers. The difference of their cubes is $18x^2+54$.

Hence $$2x(18x^2+54)=5000;$$

$$x^3+3x=138\tfrac{8}{9}.$$

1+0	+3	138.888 (4.98571+
4	19	76
8	51=1st divisor.	62.888=1st dividend.
12.9	62.61	56.349
13.8	75.03=2d divisor.	6.539888=2d dividend.
14.78	76.2124	6.096992
14.86	77.4012=3d divisor.	0.442896=3d dividend.
14.945	77.475925	0.387379
14.950	77.550675=4th div'r.	55517=4th dividend.
14.9557	77.56114399	54292
		1225

The two numbers are 7.98571 and 1.98571.

Ex. 12. Let $x+2$ and $x-2$ denote the two numbers. The sum of their cubes is $2x^3+24x$, which equals 850. Hence

$$x^3+12x=425.$$

1+0	+12	=425 (6.9874
6	48	288
12	120=1st divisor.	137=1st dividend.
18.9	137.01	123.309
19.8	154.83=2d divisor.	13.691=2d dividend.
20.78	156.4924	12.519392
20.86	158.1612=3d divisor.	1.171608=3d dividend.
20.947	158.307829	1.108154803
20.954	158.454507=4th divisor.	63453197=4th div'd.
20.9614	158.46289156	63385156

The required numbers are 4.9874 and 8.9874.

Ex. 13. Let x denote the number of partners.

Then $10x$ denotes what each partner contributed;

$10x^2$ " the whole capital;

$x+6$ " the gain per cent.

The whole profit is $\dfrac{10x^2(x+6)}{100}=392.$

Hence $x^3+6x^2=3920;$

$$x=14.$$

Ex. 14. Let x denote the second digit.

Then $9-x$ will denote the first digit,

and $x+3$ " " the third digit.

The product of the three digits is $27x+6x^2-x^3$.

Hence $27x+6x^2-x^3+38(9-x)=336.$

Therefore $x^3-6x^2+11x-6=0.$

$$\begin{array}{l|l} 1-6+11-6 & 3. \\ 1-3+\ 2 & 2. \\ 1-1 & 1. \end{array}$$

The roots of this equation are 1, 2, and 3.

Ex. 15. Let x denote the number of merchants.

$25x$ denotes what each contributed;

$25x^2$ " " they together contributed;

$25x^2+4775$ denotes the entire stock.

They gain x per cent.; that is, $\frac{25x^3+4775x}{100}$.

$6x$ denotes what each received;

$6x^2$ " " they together received.

Hence $6x^2+126=\frac{25x^3+4775x}{100}=\frac{x^3+191x}{4}$.

Therefore $x^3-24x^2+191x=504$.

$1-24+191-504$	7.
$1-17+\ 72$	8.
$1-\ 9$	9.

The roots of this equation are 7, 8, and 9.

Art. 468, Page 347.

Ex. 1. This equation can be resolved into two quadratic factors, viz., $x^2-2x-2=0$, and $x^2-6x+4=0$.

The first of these equations gives

$$x=1\pm\sqrt{3}$$
$$=2.7320508, \text{ or } -0.7320508.$$

The second of these equations gives

$$x=3\pm\sqrt{5}$$
$$=5.2360680, \text{ or } 0.7639320.$$

Ex. 3.

1+0	−12	+12	=3 (2.858083.
2	− 8	− 4	−8
4	0	− 4=1st divisor.	11=1st dividend.
6	12	+11.232	8.9856
8.8	19.04	32.608=2d divisor.	2.0144=2d dividend.
9.6	26.72	34.388125	1.71940625
10.4	35.04	36.196500=3d div'r.	0.29499375=3d dividend.
11.25	35.6025	36.491110112	0.291928880896
11.30	36.1675	36.786450848=4th div'r.	3064869104=4th div.
11.35	36.7350	36.789411639	2943152931
11.408	36.826264	36.792372504=5th div'r.	121716173=5th div.
11.416	36.917592		
11.424	37.008984		
11.43208	37.0098985664		
11.43216	37.0108131392		

1+0	−12	+12	=3	(0.606018.
0.6	−11.64	+ 5.016	3.0096	
1.2	−10.92	− 1.536=1st divisor.	−0.0096=1st dividend.	
1.8	− 9.84	− 1.594953384		9569720304
2.406	− 9.825564	− 1.653819936=2d div'r.		30279696=2d div.
2.412	− 9.811092	− 1.653917901		16539179
2.418	− 9.796584	− 1.654015867=3d div'r.		13740517=3d div.
2.42401	− 9.7965597599			
2.42402	− 9.7965355197			
2.42403				
2.424048				
2.424056				

Divide $x^4-12x^2+12x-3$ by $x-2.858083$, and we obtain

$$x^3+2.858083x^2-3.831360x+1.049654=0.$$

Divide this equation by $x-0.606018$, and we obtain

$$x^2+3.464101x-1.732052=0;$$

from which we obtain

$$x=-1.732050\pm2.175328$$

$$=+0.443278, \text{ or } -3.907378.$$

Ex. 4. We may proceed with this equation in the usual way, or we may resolve it into the two quadratic factors

$$x^2-4x+2=0, \text{ and } x^2-12x+29=0.$$

The first equation gives us $x=2\pm\sqrt{2}$

$$=3.4142136, \text{ or } 0.5857864.$$

The second equation gives us $x=6\pm\sqrt{7}$

$$=8.6457513, \text{ or } 3.3542487.$$

Ex. 5. *First Root.*

1−20	+150	−520	+806	=407	(0.934684+
−19.1	+132.81	−400.471	+445.5761	401.01849	
−18.2	+116.43	−295.684	+179.4605=1st divisor.	5.98151=1st dividend.	
−17.3	+100.86	−204.910	+173.39027231	5.2017081693	
−16.4	+ 86.10	−202.340923	+167.39670005=2d divisor.	0.7798018307=2d div'd.	
−15.5	+ 85.6359	−199.785742	+166.609069331856	0.666436277327424	
−15.47	+ 85.1727	−197.244430	+165.822784633680=3d div.	0.113365553372576=3d d.	
−15.44	+ 84.7104	−196.907679536	+165.705073922674	0.099423044353604	
−15.41	+ 84.2490	−196.571174544	+165.587393446328=4th div.	0.013942509018972=4th d.	
−15.38	+ 84.187616	−196.234914960			
−15.35	+ 84.126248	−196.184518342584			
−15.346	+ 84.064896	−196.134127243536			
−15.342	+ 84.003560				
−15.338	+ 83.99436236				
−15.334	+ 83.98516508				
−15.330					
−15.3294					
−15.3288					

Second Root.

1−20	+150	−520	+806	=407 (3.308423+
−17	+ 99	−223	+137	411
−14	+ 57	− 52	− 19=1st divisor.	−4=1st dividend.
−11	+ 24	+ 20	− 13.1269	3.93807
− 8	0	+ 19.577	− 7.4995=2d divisor.	0.06193=2d dividend.
− 5	− 1.41	+ 18.758	− 7.359268187904	0.058874145503232
− 4.7	− 2.73	+ 17.570	− 7.219366347520=3d div.	0.003055854496768=3d div.
− 4.4	− 3.96	+ 17.528976512		
− 4.1	− 5.10	+ 17.487730048		
− 3.8	− 5.127936			
− 3.5	− 5.155808			
− 3.492				
− 3.484				

Third Root.

1−20	+150	−520	+806	=407 (3.824325
−17	+ 99	−223	+187	411
−14	+ 57	− 52	− 19=1st divisor.	−4=1st dividend.
−11	+ 24	+ 20	− 5.1504	−4.12032
− 8	0	+ 17.312	+ 4.8080=2d divisor.	0.12032=2d dividend.
− 5	− 3.36	+ 12.443	+ 4.92255216	0.0984510432
− 4.2	− 6.08	+ 5.920	+ 5.03324880=3d divisor.	0.0218689568=3d dividend.
− 3.4	− 8.16	+ 5.727608	+ 5.054460646656	0.0202178426
− 2.6	− 9.60	+ 5.534832	+ 5.075517562880=4th div'r.	0.0016511142=4th dividend.
− 1.8	− 9.6196	+ 5.341680		
− 1.0	− 9.6388	+ 5.302961664		
− 0.98	− 9.6576	+ 5.264229056		
− 0.96	− 9.6760			
− 0.94	− 9.679584			
− 0.92	− 9.683152			
− 0.90				
− 0.896				
− 0.892				

Fourth Root.

1−20	+150	−520	+806	=407 (4.879508
−16	+ 86	−176	+102	408
−12	+ 38	− 24	+ 6=1st divisor.	−1=1st dividend.
− 8	+ 6	0	+ 0.0096	+0.00768
− 4	− 10	− 7.488	− 11.1520=2d divisor.	−1.00768=2d dividend.
+ 0.8	− 9.36	− 13.952	− 12.48984399	0.8742890793
+ 1.6	− 8.08	− 18.880	− 13.84251195=3d divisor.	0.1333909207=3d dividend.
+ 2.4	− 6.16	− 19.112057	− 14.01834041	0.1261650637
+ 3.2	− 3.60	− 19.323828	− 14.19435942=4th divisor.	0.0072258570=4th dividend.
+ 4.07	− 3.3151	− 19.514970		
+ 4.14	− 3.0253	− 19.536495921		
+ 4.21	− 2.7306	− 19.557668034		
+ 4.28	− 2.4310			
+ 4.359	− 2.391769			
+ 4.368	− 2.352457			

Fifth Root.

1−20	+150	−520	+806	=407 (7.053058
−13	+ 59	−107	+ 57	399
− 6	+ 17	+.12	+141=1st divisor.	8=1st dividend.
+ 1	+ 24	+180	+150.20188125	7.5100940625
+ 8	+ 80	+184.037625	159.60753125=2d div'r.	0.4899059375=2d dividend.
+15.05	+ 80.7525	+188.113000	160.18495763	0.4805548729
+15.10	+ 81.5075	+192.226250	160.76313207=3d div'r.	0.0093510646=3d dividend.
+15.15	+ 82.2650	+192.475462277		
+15.20	+ 83.0250	+192.724811858		
+15.253	+ 83.070759			
+15.256	+ 83.116527			

Ex. 6.

1+0	+0	+0	18339659776 (368
3	9	27	81
6	27	108=1st divisor.	1023965=1st dividend.
9	54	144936	869616
126	6156	186624=2d divisor	1543499776=2d dividend.
132	6948	192937472	1543499776
138	7776		
1448	789184		

Ex. 7.

1+0	+0	+0	+0	26286674882643 (483
4	16	64	256	1024
8	48	256	1280=1st divisor.	160466748=1st dividend.
12	96	640	19050496	152403968
16	160	781312	26542080=2d divisor.	806278082643=2d dividend.
208	17664	936448	268759360881	806278082643
216	19392	1105920		
224	21184	1112853627		
232	23040			
2403	2311209			

Ex. 8. Let x denote the first digit; then $9-x$ will denote the second, $10-x$ the third, and $11-x$ the fourth digit.

Hence

$$990x-299x^2+30x^3-x^4+36(10x-x^2)=3024-300x.$$

Therefore $x^4-30x^3+335x^2-1650x+3024=0.$

$$\begin{array}{l|l} 1-30+335-1650+3024 & 6. \\ 1-24+191-\ 504 & 7. \\ 1-17+\ 72 & 8. \\ 1-\ 9 & 9. \end{array}$$

Hence the value of x is either 6, 7, 8, or 9.

ART. 469, PAGE 351.

Ex. 3. $$h=\frac{r^3-9r-10}{-3r^2+9}.$$

Suppose $x=3$, $\qquad h=0.5$ nearly.

Suppose $x=3.5$,

$$h=\frac{1.375}{-27.75}=-0.05 \text{ nearly.}$$

Suppose $x=3.45$,

$$h=\frac{0.013625}{-26.7075}=-0.0005 \text{ nearly.}$$

Suppose $x=3.4495$,

$$h=\frac{0.000273837}{-26.6971507}=-0.0000103$$

Hence $\qquad x=3.4494897.$

Ex. 4. $$h=\frac{r^3+9r^2+4r-80}{-3r^2-18r-4}.$$

Suppose $x=2$, $\qquad h=+0.5$ nearly.

Suppose $x=2.5$, $\qquad h=-0.028$ nearly.

Suppose $x=2.472$,

$$h=\frac{0.009086}{66.828352}=0.0001359.$$

Hence $\qquad x=2.4721359.$

Art. 470, Page 352.

Ex. 2. Assume $x=5$; the result is -10. Hence 5 is too small.

Assume $x=6$; the result is $+80$. Hence 6 is too large, and is about 8 times as much too large as 5 is too small.

Assume $x=5.1$; the result is -2.629. Hence 5.1 is too small.

Assume $x=5.2$; the result is $+5.088$.

Then $\qquad 7.717:0.1::2.629:0.034.$

Hence $\qquad x=5.134$ nearly.

Assume $x=5.134$; the result is -0.044342, a little too small.

Assume $x=5.135$; the result is $+0.032285$.

Then $\qquad 0.076627:0.001::0.044342:0.000579.$

Hence $\qquad x=5.134579$ nearly.

Ex. 3. Assume $x=10$; the result is -1050. Hence 10 is too small.

Assume $x=11$; the result is $+3453$. Hence 11 is too great, and is about three times as much too great as 10 is too small.

Assume $x=10.25$; the result is -45.8086. Hence 10.25 is too small.

Assume $x=10.26$; the result is -4.0352.

Then $41.7734 : 0.01 :: 4.0352 : 0.000965$.

Hence $x=10.260965$ nearly.

Ex. 4. Assume $x=1$; the result is -1. Hence 1 is a little too small.

Assume $x=1.1$; the result is $+0.83$. Hence 1.1 is a little too great.

Assume $x=1.05$; the result is -0.16945. Hence 1.05 is too small.

Assume $x=1.06$; the result is $+0.01689$. Hence 1.06 is a little too great.

Assume $x=1.059$; the result is -0.00206.

Then $0.01895 : 0.001 :: 0.00206 : 0.000109$.

Hence $x=1.059109$ nearly.

EXAMPLES FOR PRACTICE.

EQUATIONS OF THE FIRST DEGREE WITH ONE UNKNOWN QUANTITY, PAGE 355.

Ex. 1. Uniting terms, $11\frac{5}{8}+3x-\frac{7x}{3}=\frac{3x}{4}$.

Clearing of fractions,

$$279+72x-56x=18x.$$

Hence $$2x=279;$$

$$x=139\tfrac{1}{2}.$$

Ex. 2. $$2ax+19ab-10a^2=bx+7b^2.$$

Transposing,

$$(2a-b)x=10a^2-19ab+7b^2=(5a-7b)(2a-b).$$

Reducing, $$x=5a-7b.$$

Ex. 3. Clearing of fractions,

$$28-10-2x=14-9+x.$$

Uniting terms, $$3x=13;$$

$$x=4\tfrac{1}{3}.$$

Ex. 4. Clearing of fractions,

$$abm+bx=abn-abp-ax.$$

Transposing, $$ax+bx=ab(n-p-m).$$

Hence $$x=\frac{ab(n-p-m)}{a+b}.$$

Ex. 5. Multiply by 120,

$$16x-24-24x+54=32x-108-80x+405-27.$$

Uniting terms, $$40x=240;$$

$$x=6.$$

Ex. 6. Divide both sides by $a^3+a^2b+ab^2+b^3$, and we have

$$1=\frac{a-b}{x}.$$

Hence $$x=a-b.$$

Ex. 7. Suppress $a-b$ both in the numerator and denominator of the first fraction.

$$a^3+a^2b+ab^2+b^3-a^2x-b^3=2a^2b+ab^2.$$

Uniting terms, $a^2x=a^3-a^2b.$

Reducing, $x=a-b.$

Ex. 8. Multiply by 40,

$$24x-28x+30x-35x=-600.$$

Uniting terms, $9x=600;$

$$x=66\tfrac{2}{3}.$$

Ex. 9. Clearing of fractions,

$$92x=11x+535-40x-74.$$

Uniting terms, $121x=461;$

$$x=3\tfrac{98}{121}.$$

Ex. 10. $3a+x-5x=6.$

Transposing, $4x=3a-6.$

Ex. 11. Clearing of fractions,

$$3ac+cx=3a^2+ax+am-mx.$$

Transposing, $cx-ax+mx=a(m-3c+3a);$

$$x=\frac{a(m-3c+3a)}{c-a+m}.$$

Ex. 12. Clearing of fractions,

$$14-28x-28+35x=-13.$$

Uniting terms, $7x=1;$

$$x=\tfrac{1}{7}.$$

Ex. 13. $mn-mx-nx+x^2=px-pq+x^2-qx.$

Transposing, $mx+nx+px-qx=mn+pq;$

$$x=\frac{mn+pq}{m+n+p-q}.$$

Ex. 14. Clearing of fractions,

$$24x^2-84x+112x-392=24x^2-88x+126x-462.$$

Reducing, $10x=70;$

$$x=7.$$

Ex. 15. Clearing of fractions,

$$dex=ade+bce+cfx.$$

Transposing, $dex-cfx=e(ad+bc);$

$$x=\frac{e(ad+bc)}{de-cf}.$$

Ex. 16. Clearing of fractions,

$$cx-ac-adx+3a^2bc=0.$$

Transposing, $$cx-adx=ac-3a^2bc;$$

$$x=\frac{ac(1-3ab)}{c-ad}.$$

Ex. 17. Expanding,

$$64-48x+9x^2+16-32x+16x^2=81-90x+25x^2.$$

Reducing, $$10x=1;$$

$$x=\tfrac{1}{10}.$$

Ex. 18. Uniting terms, $$\frac{x-16}{177-9x}=\frac{1}{24}.$$

Clearing of fractions,

$$24x-384=177-9x.$$

Uniting terms, $$33x=561;$$

$$x=17.$$

Ex. 19. Uniting terms,

$$\frac{9x+10}{11x-12}=\frac{12}{13}+\frac{1}{5}=\frac{73}{65}.$$

Clearing of fractions,

$$585x+650=803x-876.$$

Uniting terms, $$218x=1526;$$

$$x=7.$$

Ex. 20. Since $(3-4x)(7-8x)=21-52x+32x^2$, we have

$$21-66x+48x^2-45+114x-72x^2=8-24x^2.$$

Reducing, $$48x=32;$$

$$x=\tfrac{2}{3}.$$

PROBLEMS INVOLVING EQUATIONS OF THE FIRST DEGREE WITH ONE UNKNOWN QUANTITY, PAGE 357.

Prob. 1. Let x denote the required period.

By the conditions,

$$50-x=27-x+24-x+19-x+16-x=86-4x.$$

Reducing, $$3x=36;$$

$$x=12.$$

Prob. 2. Let x denote the distance from D to E.

By the conditions,

$$27+x=8+x+5+x+x=13+3x.$$

Reducing, $2x=14;$

$$x=7.$$

Prob. 3. Let x denote the distance from B to C.

By the conditions,

$$8(37+x)+6x=11(34-x)+9(48-x).$$

Reducing, $296+8x+6x=374-11x+432-9x.$

Uniting terms, $34x=510;$

$$x=15.$$

Prob. 4. Let x denote the number of minutes supposed.

$\frac{20x}{3}$ will denote the number of qts. received in the first case;

$\frac{52x}{5}$ " " " " the second case.

By the conditions, $\frac{20x}{3}+40=\frac{52x}{5}-72.$

Clearing of fractions,

$$100x+600=156x-1080.$$

Reducing, $56x=1680;$

$$x=30 \text{ minutes};$$

$\frac{20x}{3}+40=240$ quarts, the capacity of the reservoir.

Prob. 5. Let x denote the excess above 888 cubic feet in the first case.

By the conditions, $\frac{888+x}{10}=\frac{888-x}{8\frac{1}{2}}.$

Clearing of fractions,

$$7548+8\tfrac{1}{2}x=8880-10x.$$

Uniting terms, $18\frac{1}{2}x=1332;$

$$x=72;$$

$$888-72=816 \text{ cubic feet}.$$

Prob. 6. The discount is $670+980-1594.41=\$55.59.$

$4\frac{4}{5}$ per cent. per year is $\frac{4}{10}$ of one per cent. per month.

Let x denote the number of months required.

By the conditions,

$$670\times\frac{4x}{1000}+980(x+4\tfrac{1}{2})\times\frac{4}{1000}=55.59.$$

Reducing, $2680x+3920x+17,640=55,590.$

Uniting terms, $6600x=37,950;$

$$x=5\tfrac{3}{4}.$$

Prob. 7. Let x denote the loss per cent.

The cost of a hogshead of oil is $\frac{3600}{108}$, or $\frac{3200}{100-x}$.

Hence $$\frac{36}{108}=\frac{32}{100-x},$$

or $$\frac{1}{3}=\frac{32}{100-x}.$$

Clearing of fractions, $100-x=96;$

$x=4$ per cent. *loss*.

Prob. 8. Let x denote the gain per cent.

The cost of a bag of coffee is $\frac{3900}{97\frac{1}{2}}$, or $\frac{4150}{100+x}$.

Hence $$40=\frac{4150}{100+x}.$$

Clearing of fractions,

$$400+4x=415;$$

$x=3\frac{3}{4}$ per cent. *gain*.

Prob. 9. Let x denote the number of months required.

By the conditions,

$$2007\times5+3395\times7+6740\times13=12,142x.$$

Uniting terms, $12,142x=121,420.$

Hence $x=10.$

Prob. 10. Let x denote the amount of the third sum.

By the conditions,

$$1013\times3\tfrac{1}{2}+431\times7\tfrac{1}{2}+11\tfrac{1}{2}x=(1444+x)6\tfrac{1}{4}.$$

Reducing, $3545\frac{1}{2}+3232\frac{1}{2}+11\frac{1}{2}x=9025+6\frac{1}{4}x.$

Uniting terms, $5\frac{1}{4}x=2247;$

$$x=428.$$

Prob. 11. Let x denote the quantity of the better sort.

Then will $64-x$ denote the quantity of the poorer sort.

By the conditions,

$$40x+24(64-x)=64\times 34.$$

Expanding, $40x+1536-24x=2176.$

Uniting terms, $16x=640;$

$$x=40.$$

Prob. 12. Let x denote the number of hogsheads of water.

He wishes to sell his vinegar at 4 cents per quart, or $4.80 per hogshead.

By the conditions,

$$(29\tfrac{1}{2}+x)\tfrac{480}{100}=29\tfrac{1}{2}\times 6=177.$$

Reducing, $29\tfrac{1}{2}+x=\tfrac{17700}{480}=\tfrac{295}{8}.$

Clearing of fractions,

$$236+8x=295;$$

$$x=7\tfrac{3}{8}.$$

Prob. 13. Let x denote the pounds of copper added.

The $94\tfrac{1}{2}$ pounds of the compound contain 54 pounds of copper, and $40\tfrac{1}{2}$ of silver.

By the conditions, $54+x:40\tfrac{1}{2}::7:2.$

Hence $108+2x=283\tfrac{1}{2};$

$$x=87\tfrac{3}{4}.$$

Prob. 14. The 255 pounds of spirit contain 102 pounds of water and 153 of alcohol.

Let x denote the pounds of water to be extracted.

By the conditions, $102-x:153::3:17,$

or $102-x:9::3:1.$

Hence $102-x=27;$

$$x=75.$$

Prob. 15. Let x denote the required number.

By the conditions,

$$(52-x)(45-x)=(66-x)(37-x).$$

Expanding, $2340-97x+x^2=2442-103x+x^2.$

Uniting terms, $6x=102;$

$$x=17.$$

Prob. 16. By the conditions,

$$x^2-1188=(x-6)^2=x^2-12x+36.$$

Uniting terms, $12x=1224;$

$$x=102.$$

Prob. 17. Let x denote the number of dollars on a side of the square in the first case.

By the conditions,

$$x^2-25=(x-2)^2+31.$$

Reducing, $4x=60;$

$$x=15.$$

$$15^2-25=200, \textit{ Ans.}$$

Prob. 18. Let $7x$ denote the number of plants on the longer side in the first case;

then will $5x$ denote the number of plants on the shorter side in the first case.

By the conditions,

$$35x^2+2832=(7x+14)(5x+10)+172$$
$$=35x^2+140x+140+172.$$

Reducing, $140x=2520;$

$$x=18;$$

$$7x=126, \text{ and } 5x=90.$$

$$126\times90+2832=14{,}172, \textit{ Ans.}$$

Prob. 19. Let x denote the number of pounds of powder.

$$\frac{2x}{3}+10=\text{the nitre};$$

$$\frac{x}{6}-4\tfrac{1}{2}=\text{the sulphur};$$

$$\frac{2x+30}{21}-2=\text{the charcoal.}$$

Hence $\frac{2x}{3}+\frac{x}{6}+\frac{2x+30}{21}+3\frac{1}{2}=x.$

Multiply by 42,

$$28x+7x+4x+60+147=42x.$$

Uniting terms, $3x=207;$

$$x=69.$$

Prob. 20. Let $3x$, $4x$, and $5x$ denote the three numbers.

By the conditions,

$$15x+16x+15x=690.$$

Uniting terms, $46x=690;$

$x=15.$

Prob. 21. Let $x-1$ denote the first part.

Then will $x-2$ " the second part,

$x+3$ " the third part,

$\frac{x}{4}$ " the fourth part,

$5x$ " the fifth part.

By the conditions, $8\frac{1}{4}x=165;$

$x=20.$

Prob. 22. Let x denote the rate at which the criminal traveled.

Then will $x+3$ denote the rate at which his pursuers traveled.

By the conditions,

$$(8+2\tfrac{2}{3})(x+3)=(10+8-2\tfrac{2}{3})x.$$

Reducing, $10\frac{2}{3}(x+3)=15\frac{1}{3}x.$

By division, $2(x+3)=3x;$

$x=6$ miles per hour.

The criminal has 60 miles' start, which would be gained in $\frac{60}{3}=20$ hours.

Prob. 23. Let x denote the required distance.

$\frac{x}{a}$ denotes the number of revolutions of the fore wheel;

$\frac{x}{b}$ " " " the hind wheel.

By the conditions, $\frac{x}{a}-\frac{x}{b}=n.$

Clearing of fractions,

$$bx-ax=abn;$$

$$x=\frac{abn}{b-a}.$$

Prob. 24. Let x denote the time required.

In one hour the first pipe will furnish $\frac{1}{a}$, the second $\frac{1}{b}$, etc.

By the conditions, $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=\frac{1}{x}.$

Clearing of fractions,

$$abcx+abdx+acdx+bcdx=abcd;$$

$$x=\frac{abcd}{abc+abd+acd+bcd}.$$

EQUATIONS OF THE FIRST DEGREE WITH SEVERAL UNKNOWN QUANTITIES, PAGE 360.

Ex. 1. Clearing of fractions, we have

$$6x+5y=540; \quad (1)$$

$$2x-y=84. \quad (2)$$

Multiply (2) by 3, $6x-3y=252.$ (3)

Subtract (3) from (1), $8y=288.$

Hence $y=36.$

From Eq. (2), $2x=84+36=120.$

Hence $x=60.$

Ex. 2. Clearing of fractions, and uniting terms,

$$x+5y=48; \quad (1)$$

$$7x+y=132. \quad (2)$$

Multiply (1) by 7, $7x+35y=336.$ (3)

Subtract (2) from (3), $34y=204.$

Hence $y=6.$

From Eq. (1), $x=48-30=18.$

Ex. 3. Clearing of fractions,

$$3x+2y=6; \quad (1)$$

$$4x+3y=12. \quad (2)$$

Multiply (1) by 3, $9x+6y=18.$ (3)

Multiply (2) by 2, $8x+6y=24.$ (4)

Subtract (4) from (3), $x=-6.$

From Eq. (1), $2y=6+18=24.$

Hence $y=12.$

Ex. 4. Clearing of fractions,

$$11x+y=781; \quad (1)$$

$$13y-x=793. \quad (2)$$

Multiply (2) by 11, $143y-11x=8723.$ (3)

Add (1) to (3), $144y=9504.$

Hence $y=66.$

From Eq. (2), $x=858-793=65.$

Ex. 5. Clearing of fractions, and uniting terms,

$$81y-14x=25; \quad (1)$$

$$x=4y. \quad (2)$$

Substitute (2) in (1), $81y-56y=25.$

Hence $y=1$ and $x=4.$

Ex. 6. Clearing of fractions, and reducing, we have

$$11x=7y; \quad (1)$$

$$8x=1+5y. \quad (2)$$

Multiply (1) by 5, $55x=35y.$ (3)

Multiply (2) by 7, $56x=7+35y.$ (4)

Subtract (3) from (4), $x=7.$

From Eq. (2), $5y=56-1=55.$

Hence $y=11.$

Ex. 7. $\frac{x}{a}+\frac{y}{b}=1;$ (1)

$$\frac{x}{a}+\frac{y}{2b}=2. \quad (2)$$

Subtract (2) from (1), $\frac{y}{2b}=-1.$

Hence $y=-2b.$

From Eq. (1), $\frac{x}{a}=1+2=3.$

Hence $x=3a.$

Ex. 8. Adding the two equations, we have

$$\frac{2x}{a+b}=\frac{1}{a+b}+\frac{1}{a-b}=\frac{2a}{a^2-b^2}.$$

Hence $x=\frac{a}{a-b}.$

Subtracting the first equation from the second, we have

$$\frac{2y}{a-b}=\frac{1}{a-b}-\frac{1}{a+b}=\frac{2b}{a^2-b^2}.$$

Hence $y=\frac{b}{a+b}.$

Ex. 9. Clearing of fractions, and uniting terms, we have

$$11x - 27y = -30; \qquad (1)$$
$$9x - 2y = 96. \qquad (2)$$

Multiply (1) by 9, $99x - 243y = -270.$ (3)

Multiply (2) by 11, $99x - 22y = 1056.$ (4)

Subtract (3) from (4), $221y = 1326.$

Hence $y = 6.$

From Eq. (2), $9x = 96 + 12 = 108.$

Hence $x = 12.$

Ex. 10. Uniting terms, we have from the first equation

$$\frac{3x-6y}{2x-8} = -\frac{3}{2}.$$

Clearing of fractions,

$$6x - 12y = -6x + 24.$$

Transposing, $12x = 12y + 24.$

Hence $x = y + 2.$ (1)

Also, from the second equation,

$$\frac{3y+5x}{4y-6} = 3.$$

Clearing of fractions,

$$3y + 5x = 12y - 18.$$

Hence $5x = 9y - 18.$ (2)

Substituting Eq. (1) in (2),

$$5y + 10 = 9y - 18.$$

Hence $4y = 28.$

Therefore $y = 7$ and $x = 9.$

Ex. 11. Add the three equations together, and divide by 2.

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 6. \qquad (4)$$

Subtract (1) from (4),

$$\frac{z}{c} = 2, \text{ whence } z = 2c.$$

Subtract (2) from (4),

$$\frac{y}{b} = 2, \text{ whence } y = 2b.$$

Subtract (3) from (4),

$$\frac{x}{a}=2, \text{ whence } x=2a.$$

Ex. 12. Multiply (3) by 6, $12x+6y+36z=276.$ (4)

Add (4) to (1), $17x+40z=291.$ (5)

Multiply (3) by 4, $8x+4y+24z=184.$ (6)

Subtract (2) from (6), $x+27z=165.$ (7)

Multiply (7) by 17, $17x+459z=2805.$ (8)

Subtract (3) from (8), $419z=2514.$

Hence $z=6.$

From Eq. (7), $x=165-162=3.$

From Eq. (3), $y=46-6-36=4.$

Ex. 13. Clearing of fractions, and transposing, we have

$$x+4y=21; \quad (1)$$

$$2x+3z=27; \quad (2)$$

$$2y+15z=128. \quad (3)$$

Multiply (1) by 2, $2x+8y=42.$ (4)

Subtract (2) from (4), $8y-3z=15.$ (5)

Multiply (5) by 5, $40y-15z=75.$ (6)

Add (3) to (6), $42y=203.$

Hence $y=\frac{29}{6}=4\frac{5}{6}.$

From Eq. (1), $x=21-\frac{58}{3}=\frac{5}{3}.$

From Eq. (2), $3z=27-\frac{10}{3}=\frac{71}{3}.$

Hence $z=\frac{71}{9}=7\frac{8}{9}.$

Ex. 14. Multiply (1) by 5, $5x+5y+5z=25.$ (4)

Add (2) to (4), $8x+12z=100.$ (5)

Multiply (5) by 9, $72x+108z=900.$ (6)

Multiply (3) by 8, $72x-88z=-80.$ (7)

Subtract (7) from (6), $196z=980.$

Hence $z=5.$

From Eq. (5), $8x=100-60=40.$

Hence $x=5.$

From Eq. (1), $y=5-5-5=-5.$

Ex. 15. Add together the three equations, and we have

$$14x=14 \text{ or } x=1.$$

Substituting in Eq. (2), $2y-3z=-5.$ (4)
Substituting in Eq. (3), $3y-z=3.$ (5)
Multiply (5) by 3, $9y-3z=9.$ (6)
Subtract (4) from (6), $7y=14.$
Hence $y=2.$
From Eq. 3, $z=4+6-7=3.$

Ex. 16. Multiply (1) by 2, $2x-4y+6z=12.$ (4)
Subtract (4) from (2), $7y-10z=8.$ (5)
Multiply (1) by 3, $3x-6y+9z=18.$ (6)
Subtract (6) from (3), $4y-4z=8.$ (7)
Hence $y-z=2.$ (8)
Multiply (8) by 7, $7y-7z=14.$ (9)
Subtract (5) from (9), $3z=6.$
Hence $z=2.$
From Eq. (8), $y=2+2=4.$
From Eq. (1), $x=6+8-6=8.$

Ex. 17. Multiply (1) by 7, $49x-21y=7.$ (5)
Multiply (2) by 3, $12z-21y=3.$ (6)
Subtract (6) from (5), $49x-12z=4.$ (7)
Multiply (4) by 7, $133x-21u=7.$ (8)
Multiply (3) by 3, $33z-21u=3.$ (9)
Subtract (9) from (8), $133x-33z=4.$ (10)
Multiply (10) by 12, $1596x-396z=48.$ (11)
Multiply (7) by 33, $1617x-396z=132.$ (12)
Subtract (11) from (12), $21x=84.$
Hence $x=4.$
From Eq. (1), $3y=28-1=27.$
Hence $y=9.$
From Eq. (2), $4z=63+1=64.$
Hence $z=16.$
From Eq. (4), $3u=76-1=75.$
Hence $u=25.$

Ex. 18. Multiply (2) by 5, $10x+15y=195.$ (5)
Multiply (3) by 2, $10x-14z=22.$ (6)
Subtract (6) from (5), $15y+14z=173.$ (7)
Multiply (7) by 3, $45y+42z=519.$ (8)
Multiply (4) by 14, $56y+42z=574.$ (9)

Subtract (8) from (9), $11y=55$.
Hence $y=5$.
From Eq. (1), $3u=2+10=12$.
Hence $u=4$.
From Eq. (2), $2x=39-15=24$.
Hence $x=12$.
From Eq. (4), $3z=41-20=21$.
Hence $z=7$.

Ex. 19. Add (1) to (3), $4u-3y+2z=43$. (5)
Subtract (2) from (5), $4u-7y=29$. (6)
Multiply (6) by 3, $12u-21y=87$. (7)
Multiply (4) by 4, $12u+20y=128$. (8)
Subtract (7) from (8), $41y=41$.
Hence $y=1$.
From Eq. (4), $3u=32-5=27$.
Hence $u=9$.
From Eq. (2) $2z=14-4=10$.
Hence $z=5$.
From Eq. (3), $2x=36-30=6$.
Hence $x=3$.

Ex. 20. Clearing of fractions,

$$20x+33y=1,260; \quad (1)$$
$$114x+145z=9,510; \quad (2)$$
$$328z+385u=41,496; \quad (3)$$
$$710u+801x=75,150. \quad (4)$$

Multiply (2) by 328, $37,392x+47,560z=3,119,280$. (5)
Multiply (3) by 145, $47,560z+55,825u=6,016,920$. (6)
Subtract (5) from (6), $55,825u-37,392x=2,897,640$. (7)
Multiply (7) by 142,
$$7,927,150u-5,309,664x=411,464,880. \quad (8)$$
Multiply (4) by 11,165,
$$7,927,150u+8,943,165x=839,049,750. \quad (9)$$
Subtract (8) from (9), $14,252,829x=427,584,870$.
Hence $x=30$.
From Eq. (1), $33y=1260-600=660$.
Hence $y=20$.

From Eq. (2), $145z = 9510 - 3420 = 6090.$

Hence $z = 42.$

From Eq. (3), $385u = 41{,}496 - 13{,}776 = 27{,}720.$

Hence $u = 72.$

Ex. 21. Multiply (4) by 3,

	$15z + 12u + 6v - 6x = 9.$	(6)
Multiply (5) by 2,	$-4y + 12u - 6v + 8x = 12.$	(7)
Add (6) to (7),	$15z + 24u - 4y + 2x = 21.$	(8)
Add (1) to (5),	$7x - 6y + 3z = 17.$	(9)
Add (3) to (4),	$-2x + 10y + 2z + 7u = 5.$	(10)
	$3x - 5y + 2z - 4u = 11.$	(2)
Multiply (9) by 2,	$14x - 12y + 6z = 34.$	(11)
Multiply (10) by 3,	$-6x + 30y + 6z + 21u = 15.$	(12)
Subtract (12) from (13),		
	$20x - 42y - 21u = 19.$	(13)
Multiply (9) by 5,	$35x - 30y + 15z = 85.$	(14)
Subtract (8) from (14),		
	$33x - 26y - 24u = 64.$	(15)
Subtract (10) from (2),	$5x - 15y - 11u = 6.$	(16)
Multiply (16) by 33,	$165x - 495y - 363u = 198.$	(17)
Multiply (15) by 5,	$165x - 130y - 120u = 320.$	(18)
Subtract (17) from (18),	$365y + 243u = 122.$	(19)
Multiply (16) by 4,	$20x - 60y - 44u = 24.$	(20)
Subtract (13) from (20),	$-18y - 23u = 5.$	(21)
Multiply (19) by 23,	$8395y + 5589u = 2806.$	(22)
Multiply (21) by 243,	$-4374y - 5589u = 1215.$	(23)
Add (22) to (23),	$4021y = 4021.$	

Hence $y = 1.$

From Eq. (21), $23u = -18 - 5 = -23.$

Hence $u = -1.$

From Eq. (16), $5x = 6 + 15 - 11 = 10.$

Hence $x = 2.$

From Eq. (2), $2z = 11 - 6 + 5 - 4 = 6.$

Hence $z = 3.$

From Eq. (4), $2v = 3 - 15 + 4 + 4 = -4.$

$v = -2.$

PROBLEMS INVOLVING EQUATIONS OF THE FIRST DEGREE WITH SEVERAL UNKNOWN QUANTITIES, PAGE 363.

Prob. 1. Let x and y denote the two sums of money.

By the conditions, $\frac{5x}{100}+\frac{4\frac{1}{2}y}{100}=284.40$; (1)

$\frac{4\frac{1}{2}x}{100}+\frac{5y}{100}=279.90$. (2)

Clearing of fractions, $10x+9y=56,880$; (3)

$9x+10y=55,980$. (4)

Add (3) to (4), and divide by 19, $x+y=5,940$. (5)

Multiply (5) by 9, $9x+9y=53,460$. (6)

Subtract (6) from (4), $y=2520$ and $x=3420$.

Prob. 2. Let x denote the left-hand digit, and y the right-hand digit.

By the conditions,

$10x+y=3(x+y)$; (1)

$30x+3y=(x+y)^2=9(x+y)$. (2)

From Eq. (2), $x+y=9$. (3)

Substitute (3) in (1), $9x+9=27$.

Hence $x=2$ and $y=7$.

Prob. 3. Let x and y denote the cost of the two bales.

By the conditions, $\frac{91\frac{1}{4}x}{100}+\frac{88\frac{3}{4}y}{100}=987.62$; (1)

$\frac{88\frac{3}{4}x}{100}+\frac{91\frac{1}{4}y}{100}=992.37$. (2)

Clearing of fractions, and reducing, we have

$73x+71y=79,010$; (3)

$71x+73y=79,390$. (4)

Add (3) to (4), and reduce, $x+y=1,100$. (5)

Multiply (5) by 71, $71x+71y=78,100$. (6)

Subtract (6) from (4), $2y=1,290$.

Hence $y=645$ and $x=455$.

Prob. 4. Let x denote the rate of A, and y that of B.

By the conditions, $\frac{57\frac{1}{2}-5\frac{3}{4}x}{x+y}=6\frac{1}{8};$ (1)

$\frac{57\frac{1}{2}-5\frac{3}{4}y}{x+y}=5\frac{5}{8}.$ (2)

Clearing of fractions, and transposing, we have

$95x+49y=460;$ (3)

$45x+91y=460.$ (4)

Subtract (4) from (3), and divide by 2,

$25x-21y=0.$ (5)

Multiply (4) by 5, $225x+455y=2300.$ (6)

Multiply (5) by 9, $225x-189y=0.$ (7)

Subtract (7) from (6), $644y=2300.$

Hence $7y=25;$

$y=3\frac{4}{7}.$

From Eq. (5), $25x=75.$

Hence $x=3.$

Prob. 5. Let x denote the carats of the first mass, and y of the second.

By the conditions, $10x+5y=11(10+5)=165;$ (1)

$7\frac{1}{2}x+1\frac{1}{2}y=10(7\frac{1}{2}+1\frac{1}{2})=90.$ (2)

Divide (2) by $1\frac{1}{2}$, $5x+y=60.$ (3)

Multiply (3) by 2, $10x+2y=120.$ (4)

Subtract (4) from (1), $3y=45.$

Hence $y=15.$

From Eq. (3), $5x=60-15=45.$

Hence $x=9.$

Prob. 6. Let x denote the number of oxen, and y the number of days the provender will last. Then xy will denote the days the provender would last one ox.

By the conditions, $(x-75)(y+20)=xy;$ (1)

$(x+100)(y-15)=xy.$ (2)

Reducing (1), $4x-15y=300.$ (3)

Reducing (2), $3x-20y=-300.$ (4)

Multiply (3) by 3, $12x-45y=900.$ (5)

Multiply (4) by 4, $12x-80y=-1200.$ (6)

Subtract (6) from (5), $35y=2100$.
Hence $y=60$.
From Eq. (3), $4x=300+900=1200$.
Hence $x=300$.

Prob. 7. Let x denote the number of laborers, and y the pounds carried by each at one time. Then xy will denote the pounds carried by all the laborers at one time; $6xy$ will denote the entire work done in 6 hours.

By the conditions, $5(x+2)(y+4)=6xy$; (1)
$8(x-3)(y-5)=6xy$. (2)
Reducing (1), $10y+20x+40=xy$. (3)
Reducing (2), $12y+20x-60=xy$. (4)
Subtract (3) from (4), $2y-100=0$.
Hence $y=50$.
Substituting in (3), $500+20x+40=50x$.
Reducing, $30x=540$.
Hence $x=18$.

Prob. 8. Let y denote the number of miles per hour the second wagon travels;
$y+1\frac{1}{4}$ will denote the miles per hour the first wagon travels;
$y+\frac{35}{12}$ " " " the third " "
Let x denote the distance from A to B.

By the conditions, $\frac{x}{y+1\frac{1}{4}}=\frac{x}{y}-4$; (1)

$\frac{x}{y}=\frac{x}{y+\frac{35}{12}}+7$. (2)

Reducing (1), $5x=16y^2+20y$. (3)
Reducing (2), $5x=12y^2+35y$. (4)
Subtract (4) from (3), $4y^2=15y$.
Hence $y=3\frac{3}{4}$.
From Eq. (4), $5x=300$.
Hence $x=60$.
The first wagon travels 5 miles per hour, the second $3\frac{3}{4}$, and the third $6\frac{2}{3}$.

Prob. 9. Let x denote the sum, and y the rate per cent.

By the conditions,

$$\frac{3}{4}\times\frac{xy}{100}=x-1{,}208; \quad (1)$$

$$\frac{5}{4}\times\frac{xy}{100}=x-1{,}160. \quad (2)$$

Reducing (1), $3xy=400x-483{,}200.$ (3)

Reducing (2), $xy=80x-92{,}800.$ (4)

Multiply (4) by 3, $3xy=240x-278{,}400.$ (5)

Subtract (5) from (3), $160x=204{,}800.$

Hence $x=1280.$

From Eq. (4), $1280y=9600.$

Hence $y=7\frac{1}{2}.$

Prob. 10. Let x denote a side of the smaller square; then will $x+118$ denote a side of the larger square.

By the conditions, $(x+118)^2-x^2=26{,}432.$

Reducing, $236x=12{,}508.$

Hence $x=53$ and $x+118=171.$

Prob. 11. Let x denote the difference of the two numbers; then $5x$ will denote their sum, and $18x$ their product.

Therefore $3x$ will be the greater number, and $2x$ the less.

Hence $6x^2=18x;$

$x=3.$

The required numbers are 9 and 6.

Prob. 12. Let $7x$ and $3x$ denote the two numbers.

By the conditions, $4x:21x^2::1:21.$

Hence $21x^2=84x;$

$x=4.$

The required numbers are 28 and 12.

Prob. 13. Let x denote the distance from A to B, y the distance from B to C, and z the distance from A to C.

By the conditions, $x+y=164;$ (1)

$y+z=194;$ (2)

$x+z=178.$ (3)

Take the half sum of the 3 equations,

$$x+y+z=268. \quad (4)$$

Subtract (1) from (4), $z=104$.
Subtract (2) from (4), $x=74$.
Subtract (3) from (4), $y=90$.

Prob. 14. Let y denote the original rate; then $\frac{3y}{5}$ will denote the rate after the accident, and $\frac{50}{y}$ denotes the time of running 50 miles at the former rate.

By the conditions, $\frac{50}{y}=\frac{50}{\frac{3}{5}y}-\frac{4}{3}$.

Clearing of fractions,

$$150=250-4y.$$

Hence $y=25$.

Let x denote the length of the railroad.

$\frac{x}{25}$ denotes the time at the former rate.

$$\frac{x}{25}+3=\text{the actual time}=1+1+\frac{x-25}{15}.$$

Hence $10x=1000$;

$$x=100.$$

Prob. 15. Let y denote the original rate; then will $\frac{y}{n}$ denote the rate after the accident.

$\frac{b}{y}$ denotes the time of running b miles at the original rate.

By the conditions, $\frac{b}{y}+a-c=\frac{b}{\frac{y}{n}}=\frac{nb}{y}$.

Clearing of fractions,

$$b+ay-cy=nb.$$

Hence $y=\frac{b(n-1)}{a-c}$.

Prob. 16. Let x denote the number of A's marbles, y the number of B's, and z the number of C's.

By the conditions, $x+5=2(y-5);$ (1)
$y+13=3(z-13);$ (2)
$z+3=6(x-3).$ (3)
Reduce Eq. (1), $2y-x=15.$ (4)
Reduce Eq. (2), $3z-y=52.$ (5)
Reduce Eq. (3), $6x-z=21.$ (6)
Multiply (5) by 2, $6z-2y=104.$ (7)
Add (4) to (7), $6z-x=119.$ (8)
Multiply (6) by 6, $36x-6z=126.$ (9)
Add (8) to (9), $35x=245.$
Hence $x=7.$
From Eq. (6), $z=42-21=21.$
From Eq. (5), $y=63-52=11.$

Prob. 17. Let x denote the first part, and y the second; then will $232-x-y$ denote the third part.

By the conditions,

$$x+\frac{232-x}{2}=y+\frac{232-y}{3}; \quad (1)$$

$$x+\frac{232-x}{2}=232-x-y+\frac{x+y}{4}. \quad (2)$$

Reducing Eq. (1), $3x+232=4y.$ (3)
Reducing Eq. (2), $5x-464=-3y.$ (4)
Multiply (3) by 3, $9x+696=12y.$ (5)
Multiply (4) by 4, $20x-1856=-12y.$ (6)
Add (5) to (6), $29x=1160.$
Hence $x=40.$
From Eq. (3), $4y=232+120=352.$
Hence $y=88.$

Prob. 18. Let x denote the distance from A to B, y the distance from B to C, z the distance from C to D, and v the distance from A to D.

By the conditions, $x+y+z=61;$ (1)
$y+z+v=55;$ (2)
$x+y-z-v=0;$ (3)
$x-y+v-z=-4.$ (4)
Add (2) to (4), $x+2v=51.$ (5)
Add (1) to (4), $2x+v=57.$ (6)
Add (5) to (6), and divide by 3, $x+v=36.$ (7)

Subtract (7) from (6), $x=21$.
From Eq. (7), $v=36-21=15$.
From Eq. (1), $z+y=40$.
From Eq. (3), $z-y=6$.
Hence $z=23$, and $y=17$.

Prob. 19. A, B, C, and D together have 256 dollars.

Let x denote the sum D had before commencing play.
$256-x$ denotes what the three others had.
After the first game D had $x-(256-x)$, or $2x-256$.
After the second game D had $2(2x-256)$.
After the third game D had $4(2x-256)$.
By the conditions, $8(2x-256)=64$.
Hence $2x-256=8$.
Therefore $x=132$.
Let y denote the sum C had before commencing play.
After the first game C had $2y$, and the three others had $256-2y$.
After the second game C had $2y-(256-2y)$, or $4y-256$.
After the third game C had $2(4y-256)$.
By the conditions, $4(4y-256)=64$.
Hence $4y-256=16$.
Therefore $y=68$.
Let z denote the sum B had before commencing play.
After the first game B had $2z$.
After the second game B had $4z$, and the three others had $256-4z$.
After the third game B had $4z-(256-4z)$, or $8z-256$.
By the conditions, $2(8z-256)=64$.
Hence $8z-256=32$.
Therefore $z=36$.

A had $256-132-68-36=20$ dollars.

Prob. 20. Let x denote the distance from the foot of the mountain to the summit.

Let y denote A's rate of walking, and z denote B's rate.

$\frac{x}{y}$ is the time in which A would reach the summit.

By the first condition, $\frac{x}{y}=\frac{x}{z}-\frac{1}{2}$. (1)

$\frac{2}{2y}$ is the time A spends over the needless mile and back.

By the second condition,

$$\frac{x}{y}+\frac{1}{y}=\frac{x}{z}-\frac{1}{10}. \quad (2)$$

By the third condition,

$$\frac{x}{2\frac{1}{7}}=\frac{x}{z}+\frac{1}{6}-\frac{1}{3}. \quad (3)$$

Subtract (1) from (2), $\frac{1}{y}=\frac{1}{2}-\frac{1}{10}=\frac{2}{5}$.

Hence $y=2\frac{1}{2}$.

Subtract (1) from (3),

$$\frac{x}{2\frac{1}{7}}-\frac{x}{2\frac{1}{2}}=\frac{1}{2}-\frac{1}{3}+\frac{1}{6}=\frac{1}{3}.$$

Reducing, $\frac{7x}{15}-\frac{2x}{5}=\frac{1}{3}$.

Hence $x=5$.

From Eq. (1), $2=\frac{5}{z}-\frac{1}{2}$.

Hence $\frac{5}{2}=\frac{5}{z}$.

Therefore $z=2$.

Prob. 21. Let x, y, and z denote the three numbers.

By the conditions, $x-6:y-6::2:3$; (1)

$x+30:z+30::3:4$; (2)

$y-10:z-10::4:5$. (3)

Reducing Eq. (1), $3x-2y=6$. (4)

Reducing Eq. (2), $4x-3z=-30$. (5)

Reducing Eq. (3), $5y-4z=10$. (6)

Multiply (4) by 4, $12x-8y=24$. (7)

Multiply (5) by 3, $12x-9z=-90$. (8)

Subtract (8) from (7), $9z-8y=114$. (9)

Multiply (9) by 5, $45z-40y=570$. (10)

Multiply (6) by 8, $40y-32z=80$. (11)

Add (10) to (11), $13z=650.$
Hence $z=50.$
From Eq. (6), $5y=10+200=210.$
Hence $y=42.$
From Eq. (4), $3x=6+84=90.$
Hence $x=30.$

Prob. 22. Let $3x$ denote A's daily work,
$2x$ " B's daily work,
y " C's daily work,
and z the number of days C worked.

The whole quantity of work done $=(3x+2x)12+yz$, which also equals $9(3x+2x+y)$; whence we obtain

$$15x+yz=9y. \quad (1)$$

Also, $3x+y:2x+y::8:7.$
Hence $x:2x+y::1:7.$
Therefore $y=5x. \quad (2)$
Substituting in (1), $3y+yz=9y.$
Hence $3+z=9.$
Therefore $z=6.$

EQUATIONS OF THE SECOND DEGREE WITH ONE UNKNOWN QUANTITY, PAGE 367.

Ex. 1. Clearing of fractions,

$$x^2+16x-36+x^2-16x-36=\tfrac{5}{3}(x^2-4).$$

Uniting terms, $2x^2-72=\tfrac{5}{3}(x^2-4).$
Reducing, $6x^2-216=5x^2-20.$
Transposing, $x^2=196.$
Extracting the square root,

$$x=\pm 14.$$

Ex. 2. By involution and transposition,

$$\frac{10}{x^2}-49=2\sqrt{\frac{25}{x^4}-49^2}.$$

By involution again,

$$\frac{100}{x^4}-\frac{980}{x^2}+49^2=\frac{100}{x^4}-4\times 49^2.$$

Reducing, $5\times 49^2=\frac{980}{x^2}$.

Divide by 5×49, $49=\frac{4}{x^2}$.

Extracting the square root,

$$7=\pm\frac{2}{x}.$$

Hence $x=\pm\frac{2}{7}$.

Ex. 3. Clearing of fractions,

$$2x^2+50=5x^2+20.$$

Uniting terms, $3x^2=30$.

Extracting the square root,

$$x=\pm\sqrt{10}.$$

Ex. 4. Clearing of fractions,

$$2x\sqrt{a+x^2}=a^2-a-2x^2.$$

By involution,

$$4x^2(a+x^2)=a^4-2a^3-4a^2x^2+a^2+4ax^2+4x^4.$$

Transposing, $4a^2x^2=a^4-2a^3+a^2$.

Reducing, $4x^2=a^2-2a+1$.

Extracting the square root,

$$\pm 2x=a-1.$$

Hence $x=\pm\frac{1}{2}(a-1)$.

Ex. 5. By involution,

$$\frac{3m^2}{x^2}+m^2-3=m^2+1+\frac{3m^2}{x^2}-2+2m-(2m+2)\sqrt{\frac{3m^2}{x^2}-2}.$$

Uniting terms, $2m+2=(2m+2)\sqrt{\frac{3m^2}{x^2}-2}$.

Reducing, $1=\sqrt{\frac{3m^2}{x^2}-2}$.

By involution, $\frac{3m^2}{x^2}-2=1$.

Clearing of fractions, $x^2=m^2$.

Extracting the square root,

$$x=\pm m.$$

Ex. 6. By involution,

$$\frac{560}{x^2}+29=\frac{560}{x^2}-34+49+14\sqrt{\frac{560}{x^2}-34}.$$

Reducing, $14=14\sqrt{\frac{560}{x^2}-34}.$

By involution, $\frac{560}{x^2}-34=1.$

Clearing of fractions, $35x^2=560.$

Reducing, $x^2=16.$

Extracting the square root,

$$x=\pm 4.$$

Ex. 7. Clearing of fractions,

$$1+\sqrt{1-x^2}-1+\sqrt{1-x^2}=\sqrt{3}.$$

Uniting terms, $2\sqrt{1-x^2}=\sqrt{3}.$

By involution, $4-4x^2=3.$

Hence $4x^2=1;$

$$x=\pm\tfrac{1}{2}.$$

Ex. 8. $27y^2-43=77-3y^2.$

Uniting terms, $30y^2=120.$

By evolution, $y=\pm 2=7-x.$

Hence $x=7\pm 2=9$ or $5.$

Ex. 9. Multiply both numerator and denominator of the first member by $a-\sqrt{a^2-x^2}$,

$$(a-\sqrt{a^2-x^2})^2=bx^2.$$

By evolution, $a-\sqrt{a^2-x^2}=\pm x\sqrt{b}.$

By transposition, $a\mp x\sqrt{b}=\sqrt{a^2-x^2}.$

Squaring, $a^2\mp 2ax\sqrt{b}+bx^2=a^2-x^2.$

Transposing, $(b+1)x^2=\pm 2ax\sqrt{b}.$

Hence $x=\pm\frac{2a\sqrt{b}}{b+1}.$

Ex. 10. Multiply both numerator and denominator of the first member by $\sqrt{x}+\sqrt{x-a}$,

$$(\sqrt{x}+\sqrt{x-a})^2=\frac{a^2b^2}{x-a}.$$

By evolution, $\sqrt{x}+\sqrt{x-a}=\pm\dfrac{ab}{\sqrt{x-a}}$.

Clearing of fractions,

$$\sqrt{x^2-ax}+x-a=\pm ab.$$

Transposing, $\sqrt{x^2-ax}=a(1\pm b)-x$.

Squaring, $x^2-ax=a^2(1\pm b)^2-2ax(1\pm b)+x^2$.

Reducing, $ax(1\pm 2b)=a^2(1\pm b)^2$.

Hence $x=\dfrac{a(1\pm b)^2}{1\pm 2b}$.

Ex. 11. $\sqrt{a+x}+\sqrt{a-x}=\dfrac{x}{\sqrt{b}}$.

Squaring, $a+x+a-x+2\sqrt{a^2-x^2}=\dfrac{x^2}{b}$.

Clearing of fractions,

$$2b\sqrt{a^2-x^2}=x^2-2ab.$$

Squaring, $4a^2b^2-4b^2x^2=x^4-4abx^2+4a^2b^2$.

Reducing, $x^2=4ab-4b^2$.

By evolution, $x=\pm 2\sqrt{ab-b^2}$.

Ex. 12. $\sqrt{1+x}-\sqrt{1-x^2}=\sqrt{1-x}+\sqrt{1-x^2}$.

By transposition,

$$\sqrt{1+x}-\sqrt{1-x}=2\sqrt{1-x^2}.$$

By involution,

$$1+x+1-x-2\sqrt{1-x^2}=4-4x^2.$$

By transposition, $4x^2-2=2\sqrt{1-x^2}$;

$$2x^2-1=\sqrt{1-x^2}.$$

By involution, $4x^4-4x^2+1=1-x^2$.

Uniting terms, $4x^4-3x^2=0$.

Dividing by x^2, $4x^2=3$.

Hence $x=\pm\frac{1}{2}\sqrt{3}$.

Ex. 13. $x^2-\dfrac{557x}{8}+\left(\dfrac{557}{16}\right)^2=\dfrac{310,249}{256}-\dfrac{185,640}{256}=\dfrac{124,609}{256}$.

By evolution,

$$x=\frac{557}{16}\pm\frac{353}{16}=\frac{910}{16}, \text{ or } \frac{204}{16}=56\tfrac{7}{8} \text{ or } 12\tfrac{3}{4}.$$

Ex. 14. $$x^2-\frac{x}{7}=\frac{1}{49}.$$

Completing the square,

$$x^2-\frac{x}{7}+\left(\frac{1}{14}\right)^2=\frac{5}{196}.$$

By evolution, $$x=\frac{1}{14}\pm\frac{1}{14}\sqrt{5}.$$

Hence $x=0.0714286\pm0.1597191.$

Ex. 15. Completing the square,

$$x^2-\frac{x}{48}+\left(\frac{1}{96}\right)^2=\frac{1}{9216}+\frac{16,128}{9216}.$$

By evolution,

$$x=\frac{1}{96}\pm\frac{127}{96}=\frac{128}{96}, \text{ or } -\frac{126}{96}=1\tfrac{1}{3} \text{ or } -1\tfrac{5}{16}.$$

Ex. 16. Completing the square,

$$x^2-\frac{19x}{6}+\left(\frac{19}{12}\right)^2=\frac{361}{144}+\frac{1160}{144}.$$

By evolution,

$$x=\frac{19}{12}\pm\frac{39}{12}=\frac{58}{12}, \text{ or } -\frac{20}{12}=4\tfrac{5}{6} \text{ or } -1\tfrac{2}{3}.$$

Ex. 17. Multiply by 105,

$$25x^2+25x-30x^2-15x+15=12x+12.$$

Reducing, $5x^2+2x=3.$

Completing the square,

$$x^2+\frac{2x}{5}+\frac{1}{25}=\frac{1}{25}+\frac{3}{5}.$$

By evolution, $$x=-\frac{1}{5}\pm\frac{4}{5}=-1 \text{ or } +\frac{3}{5}.$$

Ex. 18. Clearing of fractions,

$$6x^2-72x+216-6x^2+144x-864=5x^2-90x+360.$$

Reducing, $5x^2-162x=-1008.$

Completing the square,

$$x^2-\frac{162x}{5}+\left(\frac{81}{5}\right)^2=\frac{6561}{25}-\frac{5040}{25}.$$

By evolution, $$x=\frac{81}{5}\pm\frac{39}{5}=\frac{120}{5} \text{ or } \frac{42}{5}.$$

Ex. 19. Clearing of fractions,

$$3x^2+24x+48+3x^2-24x+48=10x^2-160.$$

Reducing, $4x^2=256.$

By evolution, $x=\pm 8.$

This example should have been included in Class A.

Ex. 20. Clearing of fractions,

$$4x^2+20x+24+5x^2+20x+15=12x^2+36x+24.$$

Reducing, $3x^2-4x=15.$

Completing the square,

$$x^2-\frac{4x}{3}+\left(\frac{2}{3}\right)^2=\frac{4}{9}+5.$$

By evolution, $x=\frac{2}{3}\pm\frac{7}{3}=3$ or $-\frac{5}{3}.$

Ex. 21. Clearing of fractions,

$$588x^3-5761x^2+10{,}373x+9650-560x^3+4672x^2-5765x+1158$$
$$=28x^3-555x^2+1737x-400.$$

Reducing, $534x^2-2871x=11{,}208.$

By division, $178x^2-957x=3736.$

Completing the square,

$$x^2-\frac{957x}{178}+\left(\frac{957}{356}\right)^2=\frac{3736}{178}+\frac{915{,}849}{126{,}736}.$$

By evolution,

$$x=\frac{957}{356}\pm\frac{1891}{356}=\frac{2848}{356}\text{ or }-\frac{934}{356};$$

$$x=8\text{ or }-2\tfrac{111}{178}.$$

Ex. 22. Clearing of fractions,

$$8x^2-24x+18+18x^2-60x+50=30x^2-95x+75.$$

Reducing, $4x^2-11x=-7.$

Completing the square,

$$x^2-\frac{11x}{4}+\left(\frac{11}{8}\right)^2=\frac{121}{64}-\frac{7}{4}.$$

By evolution, $x=\frac{11}{8}\pm\frac{3}{8}=\frac{7}{4}$ or 1.

Ex. 23. Clearing of fractions,

$$x^3-3x^2-x+3+x^3-2x^2-5x+6=2x^3+2x^2-10x+6.$$

Reducing, $7x^2-4x=3.$

Completing the square,

$$x^2-\frac{4x}{7}+\left(\frac{2}{7}\right)^2=\frac{4}{49}+\frac{21}{49}.$$

By evolution, $x=\frac{2}{7}\pm\frac{5}{7}=1$ or $-\frac{3}{7}$.

Ex. 24. Multiply by $7+4\sqrt{3}$,

$$x^2+(2+\sqrt{3})x=14+8\sqrt{3}.$$

Completing the square,

$$x^2+(2+\sqrt{3})x+\left(\frac{2+\sqrt{3}}{2}\right)^2=\frac{63+36\sqrt{3}}{4}.$$

By evolution, $x=-\frac{2+\sqrt{3}}{2}\pm\frac{6+3\sqrt{3}}{2}$.

Hence $x=\frac{4+2\sqrt{3}}{2}$ or $-\frac{8+4\sqrt{3}}{2}$.

Ex. 25. Clearing of fractions,

$$2x+882-84\sqrt{x}+2x=105\sqrt{x}-5x.$$

Uniting terms, $9x-189\sqrt{x}=-882$.

By division, $x-21\sqrt{x}=-98$.

Completing the square,

$$x-21\sqrt{x}+\left(\frac{21}{2}\right)^2=\frac{441}{4}-\frac{392}{4}.$$

By evolution, $\sqrt{x}=\frac{21}{2}\pm\frac{7}{2}=14$ or 7.

By involution, $x=196$ or 49.

Ex. 26. Completing the square,

$$\sqrt{x}+\sqrt[4]{x}+\tfrac{1}{4}=20\tfrac{1}{4}.$$

By evolution, $\sqrt[4]{x}=-\frac{1}{2}\pm\frac{9}{2}=4$ or -5.

By involution, $x=256$ or 625.

Ex. 27. Clearing of fractions,

$$abx=abx+b^2x+bx^2+a^2x+abx+ax^2+a^2b+ab^2+abx.$$

Reducing, $(a+b)x^2+(a+b)^2x=-(a+b)ab$.

By division, $x^2+(a+b)x=-ab$.

Completing the square,

$$x^2+(a+b)x+\left(\frac{a+b}{2}\right)^2=\frac{a^2+2ab+b^2-4ab}{4}.$$

By evolution, $x=-\frac{a+b}{2}\pm\frac{a-b}{2}=-a$ or $-b$.

Ex. 28. Multiplying by $(a+x)(b+x)$,

$$a^2-x^2-b^2+x^2=\frac{(a+x)(b+x)(a+b)}{a-b}.$$

By division, $$a-b=\frac{(a+x)(b+x)}{a-b}.$$

Clearing of fractions,

$$a^2-2ab+b^2=ab+ax+bx+x^2.$$

Transposing, $x^2+(a+b)x=a^2-3ab+b^2.$

Completing the square,

$$x^2+(a+b)x+\left(\frac{a+b}{2}\right)^2=\frac{5a^2-10ab+5b^2}{4}.$$

By evolution, $$x+\frac{a+b}{2}=\pm\frac{(a-b)\sqrt{5}}{2}.$$

Hence $$x=\pm\frac{(a-b)\sqrt{5}}{2}-\frac{a+b}{2}.$$

Ex. 29. $$a+x+\sqrt{2ax+x^2}=ab+bx.$$

By transposition,

$$\sqrt{2ax+x^2}=ab+bx-a-x.$$

Squaring,

$$2ax+x^2=a^2b^2+b^2x^2+a^2+x^2+2ab^2x-2a^2b-4abx-2bx^2+2ax.$$

Uniting terms,

$$(2b-b^2)x^2+(2b-b^2)2ax+(2b-b^2)a^2=a^2.$$

By division, $$x^2+2ax+a^2=\frac{a^2}{2b-b^2}.$$

By evolution, $$x+a=\pm\frac{a}{\sqrt{2b-b^2}}.$$

Hence $$x=\pm\frac{a}{\sqrt{2b-b^2}}-a.$$

Ex. 30. By Art. 264 we find

$$(x^2-2x)^2+3(x^2-2x)=18.$$

Put $$y=x^2-2x;$$

$$y^2+3y+\left(\frac{3}{2}\right)^2=\frac{81}{4};$$

$$y=-\frac{3}{2}\pm\frac{9}{2}=3 \text{ or } -6;$$

$$x^2-2x+1=3+1.$$

Hence $$x=1\pm2=3 \text{ or } -1.$$

Also, $$x^2-2x+1=1-6=-5.$$

Hence $$x=1\pm\sqrt{-5}.$$

PROBLEMS INVOLVING EQUATIONS OF THE SECOND DEGREE WITH ONE UNKNOWN QUANTITY, PAGE 370.

Prob. 1. Let $6x$, $4x$, and $3x$ denote the three numbers.

By the conditions,

$$36x^2+16x^2+9x^2=10{,}309.$$

Reducing, $$x^2=169.$$

Hence $$x=13.$$

Prob. 2. Let x denote the number of pounds of salt, $4x$ the pounds of sugar, and $8x$ the coffee.

Then $$64x^2+16x^2+x^2=324.$$

Hence $$9x=18;$$

$$x=2.$$

Prob. 3. Let x denote the number of feet by which the breadth was increased.

By the conditions,

$$(37+x)(259-7x)=9583-63.$$

Reducing, $$9583-7x^2=9520;$$

$$7x^2=63;$$

$$x^2=9.$$

Hence $$x=3.$$

Prob. 4. Let x denote the required number.

By the conditions, $x^3+x^2=9(x+1)$.

Dividing by $x+1$, $x^2=9$.

Hence $x=3$.

Prob. 5. Suppose A travels x miles per hour, B travels y miles per hour, and they meet z miles from New York. Let a denote the distance from New York to Chicago. The time they travel before meeting will be denoted by $\frac{z}{x}$ or $\frac{a-z}{y}$.

Hence $$zy=ax-zx;$$

$$z=\frac{ax}{x+y}.$$

When they meet, B has $\frac{ax}{x+y}$ miles to travel, and A has $a-\frac{ax}{x+y}$ miles to travel; that is, $\frac{ay}{x+y}$. Hence we have

$$\frac{ay}{x(x+y)}=16; \qquad (1)$$

$$\frac{ax}{y(x+y)}=36. \qquad (2)$$

Divide Eq. (2) by (1), $\frac{x^2}{y^2}=\frac{36}{16}$.

Hence $\frac{x}{y}=\frac{3}{2}$.

From Eq. (1), $\frac{a}{x}=16\left(\frac{x+y}{y}\right)=16\times\frac{5}{2}=40$ hours, which is the time in which A performs the journey.

From Eq. (2), $\frac{a}{y}=36\left(\frac{x+y}{x}\right)=36\times\frac{5}{3}=60$ hours, which is the time in which B performs the journey.

Prob. 6. Let x denote the length of one side of the vineyard.

By the conditions, $\left(\frac{x}{3\frac{1}{2}}\right)^2-\left(\frac{x}{4}\right)^2=8640.$

Clearing of fractions,

$$64x^2-49x^2=8640\times 49\times 16.$$

Reducing, $$15x^2=8640\times 49\times 16;$$

$$x^2=576\times 49\times 16.$$

By evolution, $$x=24\times 7\times 4=672.$$

Prob. 7. Let x denote the breadth of the frame.

By the conditions,

$$(33+2x)(22+2x)=33\times22\times2=1452.$$

Expanding, $726+110x+4x^2=1452.$

Transposing, $4x^2+110x=726.$

Completing the square,

$$x^2+\frac{55x}{2}+\left(\frac{55}{4}\right)^2=\frac{3025}{16}+\frac{2904}{16}.$$

By evolution, $x=-\frac{55}{4}\pm\frac{77}{4}=\frac{11}{2}.$

Prob. 8. By the conditions,

$$(a+2x)(b+2x)=ab(p+1).$$

Expanding, $ab+2ax+2bx+4x^2=abp+ab.$

Completing the square,

$$x^2+\frac{(a+b)}{2}x+\left(\frac{a+b}{4}\right)^2=\frac{a^2+2ab+b^2+4abp}{16}.$$

By evolution, $x+\frac{a+b}{4}=\frac{\sqrt{(a+b)^2+4abp}}{4}.$

Hence $x=\frac{\sqrt{(a+b)^2+4abp}-(a+b)}{4}.$

Prob. 9. Let x denote the price of a pound of the first kind, and y a pound of the second kind.

By the first condition,

$$60y-60x=240, \text{ whence } y=4+x.$$

By the second condition,

$$\frac{504}{x}-\frac{504}{y}=8,$$

whence $63y-63x=xy.$

By substitution, $252+63x-63x=4x+x^2.$

By transposition, $x^2+4x+4=256.$

By evolution, $x+2=16;$

$$x=14, \text{ and } y=18.$$

Prob. 10. Let x denote the price of the horse.

By the conditions, $144-x=\frac{x^2}{100}.$

Clearing of fractions, $14{,}400-100x=x^2$.

Completing the square,

$$x^2+100x+(50)^2=16{,}900.$$

By evolution, $x+50=130$.

Hence $x=80$.

Prob. 11. Let x denote the number of barrels.

By the conditions, $\frac{216}{x}+1=\frac{216}{x-3}$.

Clearing of fractions,

$$216x-648+x^2-3x=216x.$$

Completing the square,

$$x^2-3x+\left(\frac{3}{2}\right)^2=\frac{9}{4}+\frac{2592}{4}.$$

By evolution, $x=\frac{3}{2}\pm\frac{51}{2}=27$.

Prob. 12. Let x denote A's capital; then will $3400-x$ denote B's capital.

$2070-x$ denotes A's profits, and $1920-(3400-x)$ denotes B's profits.

By the conditions,

$$2070-x:x-1480::12x:16(3400-x).$$

Hence $3x^2-4440x=28{,}152{,}000-21{,}880x+4x^2$.

Reducing, $x^2-17{,}440x=-28{,}152{,}000$.

Completing the square,

$$x^2-17{,}440x+(8720)^2=47{,}886{,}400.$$

By evolution, $x=8720-6920=1800$.

Prob. 13. Let x denote the distance of the required point from the moon.

By the conditions, $(240{,}000-x)^2=80x^2$.

Reducing, $79x^2+480{,}000x=57{,}600{,}000{,}000$.

Completing the square,

$$x^2+\frac{480{,}000x}{79}+\left(\frac{240{,}000}{79}\right)^2=\frac{4{,}608{,}000{,}000{,}000}{79^2}.$$

By evolution,

$$x=\frac{2{,}146{,}625-240{,}000}{79}=24{,}134+.$$

Prob. 14. Let x denote the number of days required by the first mason, and $x+3$ by the second.

By the conditions, $\frac{5\frac{1}{2}}{x+3}+\frac{5\frac{1}{2}-1\frac{1}{2}}{x}=1.$

Clearing of fractions,

$$11x+8x+24=2x^2+6x.$$

Completing the square,

$$x^2-\frac{13x}{2}+\left(\frac{13}{4}\right)^2=\frac{169}{16}+\frac{192}{16}.$$

By evolution, $x=\frac{13}{4}\pm\frac{19}{4}=8.$

Prob. 15. Let x denote the distance from A to B.

The first courier goes $\frac{x}{14}$ miles an hour, and the second goes $\frac{x+10}{14}$ miles an hour.

By the conditions, $\frac{20\times 14}{x}-\frac{20\times 14}{x+10}=\frac{1}{2}.$

Clearing of fractions,

$$560x+5600-560x=x^2+10x.$$

Completing the square

$$x^2+10x+25=5625.$$

By evolution, $x=-5\pm 75=70.$

Prob. 16. Let x denote the time required by the fastest wagon to travel one mile, and $x+\frac{1}{32}$ by the slowest.

The distance traveled by the former is $\frac{10\frac{1}{2}}{x}$, and the distance traveled by the latter is $\frac{10\frac{1}{2}}{x+\frac{1}{32}}$.

Hence $\frac{10\frac{1}{2}}{x}+\frac{10\frac{1}{2}}{x+\frac{1}{32}}=104.$

Clearing of fractions,

$$672x+21+672x=6656x^2+208x.$$

Transposing, $6656x^2-1136x=21.$

Completing the square,

$$x^2-\frac{71x}{416}+\left(\frac{71}{832}\right)^2=\frac{5041}{692,224}+\frac{2184}{692,224}.$$

By evolution, $x=\frac{71}{832}\pm\frac{85}{832}=\frac{3}{16}$.

Prob. 17. Let $x+6$ denote the distance traveled by A before meeting, and $x-6$ the distance traveled by B. A's rate of travel is $\frac{x-6}{9}$, and B's rate of travel is $\frac{x+6}{16}$.

Hence $\frac{x-6}{9}:\frac{x+6}{16}::x+6:x-6$.

Reducing, $16x^2-192x+576=9x^2+108x+324$.

Transposing, $7x^2-300x=-252$.

Completing the square,

$$x^2-\frac{300x}{7}+\left(\frac{150}{7}\right)^2=\frac{22,500}{49}-\frac{1764}{49}.$$

Hence $x=\frac{150}{7}\pm\frac{144}{7}=42$;

$x+6=48$ miles; $x-6=36$ miles.

Prob. 18. Let C denote the point of meeting; let x denote the distance AC, and y the distance BC. Then A's rate of travel will be $\frac{x}{4\frac{1}{12}}$, and B's rate of travel will be $\frac{y}{2\frac{1}{12}}$. The time of A's traveling from C to B will be y divided by $\frac{x}{4\frac{1}{12}}$, that is, $\frac{4\frac{1}{12}y}{x}$; and the time of B's traveling from C to A will be x divided by $\frac{y}{2\frac{1}{12}}$, that is, $\frac{2\frac{1}{12}x}{y}$.

Hence $\frac{2\frac{1}{12}x}{y}=\frac{4\frac{1}{12}y}{x}$.

Clearing of fractions, $25x^2=49y^2$.

By evolution, $\frac{x}{y}=\frac{7}{5}$;

$\frac{2\frac{1}{12}x}{y}=2\frac{11}{12}$, A's time of traveling from C to B;

$2\frac{11}{12}+4\frac{1}{12}=7$, A's time of performing the whole journey;

$2\frac{11}{12}+2\frac{1}{12}=5$, B's " " " "

Prob. 19. Let x denote the rate of travel of the first, and y that of the second.

The first has a start of $56x$ miles.

$\frac{910-56x}{x+y}$ denotes the number of hours from the time the second starts to the time of meeting.

Hence $$\left(\frac{910-56x}{x+y}\right)x+56x=\frac{910}{2}. \quad (1)$$

Also, $$20x+20y=910-550=360.$$

Hence $$x+y=18. \quad (2)$$

Substitute (2) in (1),

$$\frac{(910-56x)}{18}x+56x=455.$$

Clearing of fractions, $56x^2-1918x=-8190.$

Completing the square,

$$x^2-\frac{137x}{4}+\left(\frac{137}{8}\right)^2=\frac{18,769}{64}-\frac{9360}{64}.$$

By evolution, $$x=\frac{137}{8}\pm\frac{97}{8}=5;$$

$$y=18-5=13.$$

The time required by the first is $\frac{910}{5}=182$ hours;

the time of the second is $\frac{910}{13}=70$ hours.

Prob. 20. Let x denote the quantity of brandy first drawn.

$20-x$ denotes the quantity of brandy remaining, or the quantity of water in the second cask.

$20:x::x:$ the quantity of brandy returned to the first cask $=\frac{x^2}{20}$.

The quantity of brandy in the second cask is $x-\frac{x^2}{20}$.

$20:20-x+\frac{x^2}{20}::\frac{20}{3}:$ the quantity of brandy in $6\frac{2}{3}$ gallons $=\frac{x^2-20x+400}{60}$.

Hence $$\frac{20x-x^2}{20}+\frac{x^2-20x+400}{60}=10.$$

Clearing of fractions,

$$60x-3x^2+x^2-20x+400=600.$$

Reducing, $x^2-20x+100=0.$

By evolution, $x-10=0.$

Prob. 21. Let x denote the number of yards sold by the first merchant, and $x+3$ the yards sold by the second.

The price per yard with the first merchant was $\frac{24}{x+3}$, and with the second $\frac{12\frac{1}{2}}{x}$.

Hence $$\frac{24x}{x+3}+\frac{12\frac{1}{2}(x+3)}{x}=35.$$

Clearing of fractions,

$$48x^2+25x^2+150x+225=70x^2+210x.$$

Reducing, $3x^2-60x=-225.$

Completing the square,

$$x^2-20x+100=25.$$

By evolution, $x=10\pm5=15$ or 5;

$$x+3=8 \text{ or } 18.$$

Prob. 22. Let x denote the number of miles A or B travels per hour.

The geese travel at the rate of $\frac{3}{2}$ miles per hour, and the wagon at the rate of $\frac{9}{4}$ miles.

B approaches the wagon at the rate of $x+\frac{9}{4}$, and he overtakes the geese $\frac{10}{3}$ hours after A.

B's distance from A is $\frac{10x}{3}-5$.

A meets the wagon $50-2x$ miles from Baltimore, and B meets it $31+\frac{2x}{3}$ miles from Baltimore. In the interval the wagon had traveled $\frac{8x}{3}-19$ miles, and therefore the interval $=\frac{4}{9}\left(\frac{8x}{3}-19\right)$.

Also, A's distance from $B=\frac{4}{9}\left(\frac{8x}{3}-19\right)\left(x+\frac{9}{4}\right)$.

Hence $$\frac{4}{9}\left(\frac{8x}{3}-19\right)\left(x+\frac{9}{4}\right)=\frac{10x}{3}-5.$$

Clearing of fractions,

$$32x^2-156x-513=90x-135.$$

Reducing, $x^2-\frac{123x}{16}=\frac{189}{16}.$

Completing the square,

$$x^2-\frac{123x}{16}+\left(\frac{123}{32}\right)^2=\frac{15,129}{1024}+\frac{12,096}{1024}.$$

By evolution, $x=\frac{123}{32}\pm\frac{165}{32}=9.$

A's distance from B is $\frac{10x}{3}-5=25$ miles.

EQUATIONS OF THE SECOND DEGREE WITH SEVERAL UNKNOWN QUANTITIES, PAGE 373.

Ex. 1. $169x^2+2y^2=177;$ (1)

$-13x^2+4y^2=3.$ (2)

Multiply (1) by 2, $338x^2+4y^2=354.$ (3)

Subtract (2) from (3), $351x^2=351.$

Hence $x=\pm1.$

From Eq. (2), $4y^2=13+3=16.$

Hence $y=\pm2.$

Ex. 2. From Eq. (1), $7x^2+7y^2=25x^2-25y^2.$

Transposing, $32y^2=18x^2.$

Hence $4y=\pm3x.$

From Eq. (2), $\frac{3x^2}{4}=48.$

Hence $x=\pm8;$

$y=\pm6.$

Ex. 3. $2z^2-5v^2=75;$ (1)

$7z^2+15v^2=1075.$ (2)

Multiply (1) by 3, $6z^2-15v^2=225.$ (3)

Add (2) to (3), $13z^2=1300.$

Hence $z=\pm10.$

From Eq. (1), $5v^2=200-75=125.$

Hence $v=\pm5;$

$x=z-4=6$ or $-14;$

$y=v+7=12$ or $2.$

Ex. 4. Multiply (1) by 4,

$$4(x+y)^2-8x^2=196. \qquad (3)$$

Subtract (3) from (1), $11x^2=176$.

Hence $x=\pm 4$.

From Eq. (1), $(x+y)^2=49+32=81$.

Hence $x+y=\pm 9$;

$$y=\pm 9\pm 4=\pm 5 \text{ or } \pm 13.$$

Ex. 5. From Eq. (1), $y=\dfrac{37-2x}{3}$. (3)

From Eq. (2), $45x+45y=14xy$. (4)

Substitute (3) in (4),

$$45x+15(37-2x)=\frac{14}{3}(37x-2x^2).$$

Reducing, $135x+1665-90x=518x-28x^2$.

Transposing, $28x^2-473x=-1665$.

Completing the square,

$$x^2-\frac{473x}{28}+\left(\frac{473}{56}\right)^2=\frac{223,729}{3136}-\frac{186,480}{3136}.$$

By evolution, $x=\dfrac{473}{56}\pm\dfrac{193}{56}=5 \text{ or } \dfrac{333}{28}$;

$$y=\frac{37-2x}{3}=9 \text{ or } \frac{185}{42}.$$

Ex. 6. $x^3+y^3=9xy$. (3)

Substitute (2) in (3),

$$y^3+216-108y+18y^2-y^3=54y-9y^2.$$

Reducing, $27y^2-162y=-216$.

Completing the square,

$$y^2-6y+9=1.$$

By evolution, $y=3\pm 1=2 \text{ or } 4$;

$$x=6-y=4 \text{ or } 2.$$

Ex. 7. From Eq. (2), $x^2+2xy+y^2=15,376$. (3)

Subtract (1) from (3), $2xy = 5,376$; (4)

$4xy = 10,752$. (5)

Subtract (5) from (3),

$$x^2-2xy+y^2= 4,624. \qquad (6)$$

By evolution, $x-y=\pm\ 68$. (7)

Add (7) to (2), $2x=192$ or 56;

$x=96$ or 28.

Subtract (7) from (2), $2y=56$ or 192;

$x=28$ or 96.

Ex. 8. From Eq. (1), $$x=\frac{36}{y}. \quad (3)$$

Substitute (3) in (2), $$\frac{6}{\sqrt{y}}+\sqrt{y}=5.$$

Clearing of fractions,

$$y-5\sqrt{y}=-6.$$

Completing the square,

$$y-5\sqrt{y}+\left(\frac{5}{2}\right)^2=\frac{25}{4}-\frac{24}{4}.$$

By evolution, $$\sqrt{y}=\frac{5}{2}\pm\frac{1}{2}=3 \text{ or } 2.$$

By involution, $y=9$ or 4;

$\sqrt{x}=5-3$ or $5-2$.

By involution, $x=4$ or 9.

Ex. 9. Put $v=3x+4y$, and $z=7x-2y$,

$$vz+v=44; \quad (1)$$

$$vz-z=30. \quad (2)$$

Subtract (2) from (1),

$v+z=14$; hence $v=14-z$. (3)

Substitute (3) in (2), $14z-z^2-z=30$. (4)

Reducing, $z^2-13z=-30$. (5)

Completing the square,

$$z^2-13z+\left(\frac{13}{2}\right)^2=\frac{169}{4}-\frac{120}{4}. \quad (6)$$

By evolution, $$z=\frac{13}{2}\pm\frac{7}{2}=10 \text{ or } 3; \quad (7)$$

$v=14-z=4$ or 11. (8)

Hence $3x+4y=4$ or 11; (9)

$7x-2y=10$ or 3. (10)

Multiply (10) by 2, $14x-4y=20$ or 6. (11)

Add (9) to (11), $17x \quad =24$ or 17.

Hence $x=\frac{24}{17}$ or 1.

From Eq. (10), $2y=7x-z=-\frac{2}{17}$ or 4;

$y=-\frac{1}{17}$ or 2.

Ex. 10. From Eq. (1),

$$(x-3y)^2-4(x-3y)+4=0. \quad (3)$$

By evolution, $(x-3y)-2=0. \quad (4)$

Hence $x=3y+2. \quad (5)$

Substitute (5) in (2),

$$9y^2+12y+4-6y^2-4y+3y^2-12y-8+5y=53.$$

Reducing, $6y^2+y=57.$

Completing the square,

$$y^2+\frac{y}{6}+\left(\frac{1}{12}\right)^2=\frac{1}{144}+\frac{57}{6}.$$

Hence $y=-\frac{1}{12}\pm\frac{37}{12}=3$ or $-\frac{19}{6}$.

From Eq. (5), $x=11$ or $-\frac{15}{2}$.

Ex. 11. From Eq. (1),

$$2x^3+2x^2y+2xy^2+2y^3=15xy. \quad (3)$$

Divide Eq. (2) by Eq. (1),

$$2(x^2-y^2)(x-y)=3xy. \quad (4)$$

Whence $2x^3-2x^2y-2xy^2+2y^3=3xy. \quad (5)$

Subtract (5) from (3), $4x^2y+4xy^2=12xy. \quad (6)$

Divide by $4xy$, $x+y=3$ or $x=3-y. \quad (7)$

Substitute (7) in (1), $2(x^2+y^2)=5xy. \quad (8)$

Substitute (7) in (8),

$$2(9-6y+y^2)+2y^2=15y-5y^2.$$

Transposing, $9y^2-27y=-18.$

Completing the square,

$$y^2-3y+\left(\frac{3}{2}\right)^2=\frac{9}{4}-\frac{8}{4}.$$

By evolution, $y=\frac{3}{2}\pm\frac{1}{2}=2$ or 1;

$x=3-y=1$ or 2.

Ex. 12. Divide Eq. (2) by Eq. (1),

$$(x^2+y^2)(x+y)=40xy. \quad (3)$$

Hence $x^3+x^2y+xy^2+y^3=40xy.$ (4)

From Eq. (1), $x^3-x^2y-xy^2+y^3=16xy.$ (5)

Subtract (5) from (4), $2x^2y+2xy^2=24xy.$ (6)

Hence $x+y=12$ or $x=12-y.$ (7)

Substitute (7) in (3),

$$12(144-24y+y^2+y^2)=40(12y-y^2).$$

Reducing, $216-36y+3y^2=60y-5y^2.$

Uniting terms, $8y^2-96y=-216.$

Completing the square,

$$y^2-12y+36=9.$$

By evolution, $y=6\pm3=3$ or $9;$

$$x=12-y=9 \text{ or } 3.$$

Ex. 13. Multiply Eq. (1) by y, and Eq. (2) by x,

$$27y=18x; \text{ hence } y=\frac{2x}{3}.$$

Multiply Eq. (2) by z, and Eq. (3) by y,

$$36y=18z; \text{ hence } z=2y=\frac{4x}{3}.$$

Substitute in Eq. (1),

$$x\left(x+\frac{2x}{3}+\frac{4x}{3}\right)=27.$$

Reducing, $3x^2=27.$

By evolution, $x=\pm3.$

Hence $y=\frac{2x}{3}=\pm2;$

$$z=2y=\pm4.$$

Ex. 14. Multiply together Equations (1), (2), and (4),

$$xy^3zv=bxzv,$$

or $y^3=b.$

By evolution, $y=b^{\frac{1}{3}}.$

Multiply together Equations (3) and (4),

$$xyv^2=abx.$$

Hence $yv^2=ab.$

By substitution, $v^2=ab^{\frac{2}{3}}$.

By evolution, $v=a^{\frac{1}{2}}b^{\frac{1}{3}}$.

From Eq. (2), $z=\frac{v}{y}=a^{\frac{1}{2}}$.

From Eq. (1), $x=\frac{z}{y}=\frac{a^{\frac{1}{2}}}{b^{\frac{1}{3}}}$.

Ex. 15. Multiply the four equations together,

$$x^3y^3z^3v^3=105\times135\times189\times315=27^3\times7^3\times5^3. \quad (5)$$

By evolution, $xyzv=27\times7\times5$. (6)

Divide (6) by (1), $v=9$.

Divide (6) by (2), $z=7$.

Divide (6) by (3), $y=5$.

Divide (6) by (4), $x=3$.

Ex. 16. Squaring Eq. (2),

$$x^2+\frac{x^4}{y^2}+y^2+\frac{2x^3}{y}+2x^2+2xy=196. \quad (3)$$

Subtract (1) from (3), $\frac{2x^3}{y}+2x^2+2xy=112$. (4)

Divide Eq. (4) by Eq. (2),

$$2x=8, \text{ or } x=4.$$

Substituting in Eq. (2), $4+\frac{16}{y}+y=14$.

Clearing of fractions, etc.,

$$y^2-10y+25=9.$$

By evolution, $y=5\pm3=8$ or 2.

Ex. 17. From Eq. (2), $2\sqrt{y-x}=3\sqrt{a-x}$.

By involution, $4y-4x=9a-9x$.

Transposing, $4y=9a-5x$. (3)

From Eq. (1), $2\sqrt{y}-2\sqrt{a-x}=3\sqrt{a-x}$.

Transposing, $2\sqrt{y}=5\sqrt{a-x}$.

By involution, $4y=25a-25x$. (4)

Comparing (3) with (4),

$$9a-5x=25a-25x.$$

Reducing, $20x=16a.$

Hence $x=\frac{4a}{5}.$

From Eq. (3), $4y=9a-4a=5a.$

Hence $y=\frac{5a}{4}.$

Ex. 18. Multiplying Eq. (1) by y, and Eq. (2) by x, we have

$$ay^2=bx^2, \text{ or } y^2=\frac{bx^2}{a}. \quad (3)$$

Substituting (3) in (1),

$$\left(x^2+\frac{bx^2}{a}\right)x=x\sqrt{ab}.$$

Clearing of fractions,

$$ax^2+bx^2=a\sqrt{ab}=\sqrt{a^3b}.$$

Hence $x^2=\frac{\sqrt{a^3b}}{a+b}.$

By evolution, $x=\pm\sqrt{\frac{\sqrt{a^3b}}{a+b}}.$

From Eq. (3), $y^2=\frac{bx^2}{a}=\frac{\sqrt{ab^3}}{a+b}.$

By evolution, $y=\pm\sqrt{\frac{\sqrt{ab^3}}{a+b}}.$

Ex. 19. From Eq. (1), $\sqrt{5z^2}+z^2=10;$

$z^2+z\sqrt{5}=10.$

Completing the square,

$$z^2+z\sqrt{5}+\left(\frac{\sqrt{5}}{2}\right)^2=\frac{5}{4}+\frac{40}{4}=\frac{45}{4}.$$

By evolution,

$$z=-\frac{\sqrt{5}}{2}\pm\frac{3\sqrt{5}}{2}=\sqrt{5} \text{ or } -2\sqrt{5}.$$

$$\left(\frac{5}{2}+v\right)^5=\frac{3125}{32}+\frac{3125v}{16}+\frac{1250v^2}{8}+\frac{250v^3}{4}+\frac{25v^4}{2}+v^5;$$

$$\left(\frac{5}{2}-v\right)^5=\frac{3125}{32}-\frac{3125v}{16}+\frac{1250v^2}{8}-\frac{250v^3}{4}+\frac{25v^4}{2}-v^5.$$

From Eq. (2), $\frac{3125}{16}+\frac{1250v^2}{4}+25v^4=275.$

Reducing, $v^4+\frac{50v^2}{4}=\frac{51}{16}.$

Completing the square,

$$v^4+\frac{50v^2}{4}+\frac{625}{16}=\frac{676}{16}.$$

By evolution, $v^2=-\frac{25}{4}\pm\frac{26}{4}=\frac{1}{4}$ or $-\frac{51}{4}.$

By evolution, $v=\pm\frac{1}{2}$ or $\pm\frac{1}{2}\sqrt{-51};$

$$\sqrt{x}=\frac{5}{2}+v=3 \text{ or } 2.$$

Hence $x=9$ or $4.$

Also $\sqrt{x}=\frac{5}{2}\pm\frac{1}{2}\sqrt{-51};$

$$x=\frac{-13\pm5\sqrt{-51}}{2};$$

$$\sqrt{y}=\frac{5}{2}-v=2 \text{ or } 3, \text{ or } \frac{5\mp\sqrt{-51}}{2};$$

$$y=4 \text{ or } 9, \text{ or } \frac{-13\mp5\sqrt{-51}}{2}.$$

The other case, where $z=-2\sqrt{5}$, may be solved in the same manner.

Ex. 20. Put $x=9+v$, and $y=9-v$.

By Eq. (1), $(81+v^2)(81-v^2)=6545.$

Reducing, $v^4=6561-6545.$

By evolution, $v=\pm2.$

Hence $x=9\pm2=11$ or $7;$

$y=9\mp2=7$ or $11.$

Ex. 21. Put $x=z+v$, and $y=z-v$.

From Eq. (1), $5(2z^2+2v^2)+4(z^2-v^2)=356.$ (3)

Hence $7z^2+3v^2=178.$ (4)

From Eq. (2), $2z^2+2v^2+2z=62.$ (5)

Multiply (5) by $1\frac{1}{2}$, $3z^2+3v^2+3z=93.$ (6)

Subtract (6) from (4), $4z^2-3z=85.$

Completing the square,

$$z^2-\frac{3z}{4}+\left(\frac{3}{8}\right)^2=\frac{9}{64}+\frac{1360}{64}.$$

By evolution, $z=\frac{3}{8}\pm\frac{37}{8}=5 \text{ or } -\frac{17}{4}.$

From Eq. (5), $v^2=31-25-5=1.$

By evolution, $v=\pm 1.$

Hence $x=5\pm 1=6 \text{ or } 4;$

$y=5\mp 1=4 \text{ or } 6.$

Ex. 22. From Eq. (1), $x^2+y^2=\frac{300}{xy}.$ (3)

By involution, $x^4+2x^2y^2+y^4=\frac{90,000}{x^2y^2};$ (4)

$x^4+y^4=337.$ (2)

Subtract (2) from (4), $2x^2y^2=\frac{90,000}{x^2y^2}-337.$ (5)

Clearing of fractions, $2x^4y^4=90,000-337x^2y^2.$ (6)

Completing the square,

$$x^4y^4+\frac{337x^2y^2}{2}+\left(\frac{337}{4}\right)^2=\frac{113,569}{16}+\frac{720,000}{16}. \quad (7)$$

By evolution, $x^2y^2=\frac{913}{4}-\frac{337}{4}=144.$ (8)

By evolution, $xy=\pm 12.$ (9)

From Eq. (1), $x^2+y^2=25.$ (10)

Add twice (9) to (10), $x^2+2xy+y^2=49.$

By evolution, $x+y=\pm 7.$

Subtract twice (9) from (10),

$x^2-2xy+y^2=1.$

By evolution, $x-y=\pm 1.$

Hence $x=\pm 4 \text{ or } \pm 3;$

$y=\pm 3 \text{ or } \pm 4.$

Ex. 23. From Eq. (2), $x^3+3x^2y+3xy^2+y^3=125.$

By transposition,

$x^3+y^3=125-3x^2y-3xy^2;$

$x^3+y^3=125-3xy(x+y).$

Substitute Eq. (2), $x^3+y^3=125-15xy$.
Also, from Eq. (2), $x^2+y^2=25-2xy$.
Hence, from Eq. (1), $(25-2xy)(125-15xy)=455$.
Reducing, $(25-2xy)(25-3xy)=91$.
Expanding, $625-125xy+6x^2y^2=91$.
Completing the square,

$$x^2y^2-\frac{125xy}{6}+\left(\frac{125}{12}\right)^2=\frac{15,625}{144}-\frac{534}{6}.$$

By evolution, $xy=\frac{125}{12}\pm\frac{53}{12}=6$ or $\frac{89}{6}$.

From Eq. (2), $x^2+2xy+y^2=25$;
$4xy=24$.
By subtraction, $x^2-2xy+y^2=1$.
By evolution, $x-y=\pm1$.
Hence $x=2$ or 3;
$y=3$ or 2.

Ex. 24. Put $x=vy$.
From Eq. (1), $v^2y^2+vy^2+y^2=14vy+14y$.
Reducing, $v^2y+vy+y=14v+14$. (3)
From Eq. (2), $v^2y^2-vy^2+y^2=18vy-18y$.
Reducing, $v^2y-vy+y=18v-18$. (4)
Subtract (4) from (3), $2vy=32-4v$.
Reducing, $y=\frac{16}{v}-2$. (5)
Substitute (5) in Eq. (3),

$$16v-2v^2+16-2v+\frac{16}{v}-2=14v+14.$$

Uniting terms, $-2v^2+\frac{16}{v}=0$.
Clearing of fractions, $v^3=8$.
By evolution, $v=2$.
Hence, from (5), $y=8-2=6$;
$x=6\times2=12$.

Ex. 25. Comparing (1) and (2),

$$\frac{91}{x^2+y^2}=\frac{133}{x^2+xy+y^2}$$

Clearing of fractions,

$$91x^2+91xy+91y^2=133x^2+133y^2.$$

Uniting terms, $42x^2-91xy=-42y^2.$

Completing the square,

$$x^2-\frac{91xy}{42}+\left(\frac{91y}{84}\right)^2=\frac{8281y^2}{7056}-y^2.$$

By evolution, $x=\frac{91y}{84}\pm\frac{35y}{84}=\frac{3y}{2}$ or $\frac{2y}{3}$.

Substitute $\frac{3y}{2}$ in Eq. (1),

$$\left(\frac{9y^2}{4}-\frac{3y^2}{2}+y^2\right)\left(\frac{9y^2}{4}+y^2\right)=91.$$

Reducing, $\frac{7y^2}{4}\times\frac{13y^2}{4}=91.$

Clearing of fractions, $y^4=16.$

By evolution, $y=\pm2$ and $x=\pm3.$

Substitute $\frac{2y}{3}$ in Eq. (1),

$$\left(\frac{4y^2}{9}-\frac{2y^2}{3}+y^2\right)\left(\frac{4y^2}{9}+y^2\right)=91.$$

Reducing, $\frac{7y^2}{9}\times\frac{13y^2}{9}=91.$

Clearing of fractions, $y^4=81.$

By evolution, $y=\pm3$, and $x=\pm2.$

Ex. 26. From Eq. (1), $(x^2+2xy+y^2)x^2y^2=900.$

From Eq. (2), $(x^2+y^2)x^2y^2=900-2x^3y^3=468.$

Hence $2x^3y^3=432.$

By evolution, $xy=6.$

From Eq. (1), $x+y=5.$

By involution, $x^2+2xy+y^2=25.$

By subtraction, $x^2-2xy+y^2=1.$

By evolution, $x-y=\pm1.$

Hence $x=2$ or $3.$

Ex. 27. From Eq. (1), $x^2-y^2+\sqrt{x^2-y^2}=12.$

Completing the square,

$$x^2-y^2+\sqrt{x^2-y^2}+\frac{1}{4}=\frac{49}{4}.$$

By evolution, $\sqrt{x^2-y^2}=\pm\frac{7}{2}-\frac{1}{2}=3$ or -4.

By involution, $x^2-y^2=9$ or 16.

From Eq. (2), $x^2+y^2=41$.

By addition, $2x^2=50$ or 57.

By evolution, $x=\pm 5$ or $\pm\frac{1}{2}\sqrt{114}$.

From Eq. (2), $y^2=41-x^2=16$ or $\frac{25}{2}$.

By evolution, $y=\pm 4$ or $\pm\frac{5}{2}\sqrt{2}$.

Ex. 28. $(x+y)^4+10(x+y)^2+25=9(x+y)^2+30(x+y)+25$.

By evolution, $(x+y)^2+5=3(x+y)+5$.

Reducing, $x+y=3$.

From Eq. (2), $x-y=1$.

Hence $x=2$ and $y=1$.

Ex. 29. By involution, $z^2+2zv+v^2=324$;

$z^2 \quad +v^2=212$.

Hence $2zv=112$.

By subtraction, $z^2-2zv+v^2=100$.

By evolution, $z-v=\pm 10$;

$z+v=18$.

Hence $z=14$ or 4, $v=4$ or 14;

$$x+\frac{1}{x}=4.$$

Clearing of fractions, $x^2-4x=-1$.

Completing the square,

$$x^2-4x+4=3.$$

By evolution, $x=2\pm\sqrt{3}$.

Also, $x+\frac{1}{x}=14$.

Clearing of fractions, $x^2-14x=-1$.

Completing the square,

$$x^2-14x+49=48.$$

By evolution, $x=7\pm 4\sqrt{3}$.

PROBLEMS INVOLVING EQUATIONS OF THE SECOND DEGREE WITH SEVERAL UNKNOWN QUANTITIES, PAGE 377.

Prob. 1. Let $\frac{x}{y}$ denote the fraction.

By the first condition, $\frac{x+2}{y-2}=\frac{y}{x}$.

Clearing of fractions,

$$x^2+2x=y^2-2y.$$

Completing the square,

$$x^2+2x+1=y^2-2y+1.$$

By evolution, $x+1=y-1$, or $x=y-2$. (1)

By the second condition,

$$\frac{x-2}{y+2}+\frac{16}{15}=\frac{y}{x}.$$

Clearing of fractions,

$$15x^2-30x+16xy+32x=15y^2+30y.$$

Reducing, $15x^2+16xy+2x=15y^2+30y.$ (2)

Substitute (1) in (2),

$$15y^2-60y+60+16y^2-32y+2y-4=15y^2+30y.$$

Reducing, $16y^2-120y=-56.$

Completing the square,

$$y^2-\frac{15y}{2}+\left(\frac{15}{4}\right)^2=\frac{225}{16}-\frac{56}{16}.$$

By evolution, $y=\frac{15}{4}\pm\frac{13}{4}=7$ or $\frac{1}{2}$.

Hence $x=5$ or $-\frac{3}{2}$.

Prob. 2. Let $2x$ denote the first part, $3x$ the third part, and $102-5x$ the second.

By the conditions,

$$6x^2=102(102-5x)=10,404-510x.$$

Completing the square,

$$x^2+85x+\left(\frac{85}{2}\right)^2=\frac{7225}{4}+\frac{6936}{4}.$$

By evolution, $x=\frac{119}{2}-\frac{85}{2}=17;$

$$2x=34;\ 3x=51;\ 102-5x=17.$$

Prob. 3. Let x and y denote the two digits.

By the first condition,

$$(10y+x)(10x+y)=5092. \qquad (1)$$

By the second condition, $\frac{x}{y}=1+\frac{1}{y}.$

Clearing of fractions, $x=y+1. \qquad (2)$

Substitute (2) in Eq. (1),

$$(11y+1)(11y+10)=5092.$$

Expanding, $121y^2+121y+10=5092.$

Reducing, $y^2+y=42.$

Completing the square,

$$y^2+y+\frac{1}{4}=\frac{169}{4}.$$

By evolution, $y=\frac{13}{2}-\frac{1}{2}=6.$

Hence $x=7.$

Prob. 4. Let x and y denote the two circumferences.

By the first condition,

$$\frac{5775}{x}-\frac{5775}{y}=165.$$

Reducing, $\frac{35}{x}-\frac{35}{y}=1.$

Clearing of fractions, $35y-35x=xy. \qquad (1)$

By the second condition,

$$\frac{5775}{x+2\frac{1}{2}}-\frac{5775}{y+2\frac{1}{2}}=112.$$

Reducing, $\frac{825}{x+2\frac{1}{2}}-\frac{825}{y+2\frac{1}{2}}=16.$

Clearing of fractions,

$$825y+2062\tfrac{1}{2}-825x-2062\tfrac{1}{2}=16xy+40x+40y+100.$$

Uniting terms, $785y-865x=16xy+100. \qquad (2)$

Substitute Eq. (1) in (2),

$$785y-865x=560y-560x+100.$$

Uniting terms, $225y-305x=100.$
Reducing, $45y-\ 61x=\ 20.$

Hence $x=\dfrac{45y-20}{61}.$ (3)

Substitute (3) in (1),

$$35y-35\left(\frac{45y-20}{61}\right)=\frac{45y^2-20y}{61}.$$

Clearing of fractions,

$$427y-315y+140=9y^2-4y.$$

Reducing, $9y^2-116y=140.$
Completing the square,

$$y^2-\frac{116y}{9}+\left(\frac{58}{9}\right)^2=\frac{3364}{81}+\frac{1260}{81}.$$

By evolution, $y=\dfrac{58}{9}\pm\dfrac{68}{9}=14.$

Hence $x=\dfrac{630-20}{61}=10.$

Prob. 5. Let x denote the length, and y the breadth.

By the first condition, $\dfrac{2x}{8}+\dfrac{2y}{16}=4\frac{1}{4}.$

Clearing of fractions, $2x+y=34.$ (1)
By the second condition,

$$xy-\left(\frac{7x}{8}\times\frac{15y}{16}\right)=5\tfrac{3}{4}.$$

Uniting terms, $\dfrac{23xy}{128}=\dfrac{23}{4}.$

Reducing, $xy=32.$ (2)
Substitute (1) in (2), $34x-2x^2=32.$
Completing the square,

$$x^2-17x+\left(\frac{17}{2}\right)^2=\frac{289}{4}-\frac{64}{4}.$$

By evolution, $x=\dfrac{17}{2}\pm\dfrac{15}{2}=16 \text{ or } 1.$

Hence $y=2 \text{ or } 32.$

Prob. 6. Let x denote the number of laborers, and y the number of pounds each carried at a time; and suppose that m journeys are made in an hour.

The total number of pounds removed is $8mxy$.

Hence $$8mxy=7m(x+8)(y-5); \quad (1)$$
$$8mxy=9m(x-8)(y+11). \quad (2)$$

From Eq. (1), $$8xy=7xy+56y-35x-280.$$

Reducing, $$xy=56y-35x-280. \quad (3)$$

From Eq. (2), $$8xy=9xy-72y+99x-792.$$

Reducing, $$xy=72y-99x+792. \quad (4)$$

Comparing (3) and (4),
$$56y-35x-280=72y-99x+792.$$

Uniting terms, $64x-16y=1072$.

Reducing, $$y=4x-67. \quad (5)$$

Substitute (5) in (3),
$$4x^2-67x=224x-3752-35x-280.$$

Reducing, $$x^2-64x=-1008.$$

Completing the square,
$$x^2-64x+(32)^2=1024-1008.$$

By evolution, $$x=32\pm4=28 \text{ or } 36.$$

Hence $$y=4x-67=45 \text{ or } 77.$$

Prob. 7. Let x denote the capital, and y the rate per cent.

By the first condition, $\dfrac{xy}{100}=123\frac{1}{2}$.

Hence $$x=\frac{12,350}{y}. \quad (1)$$

By the second condition,
$$(x+700)\frac{(y-\frac{1}{4})}{100}=135.$$

Reducing, $4xy+2800y-x-700=54,000. \quad (2)$

Substituting the values of x and xy,
$$49,400+2800y-\frac{12,350}{y}-700=54,000.$$

Uniting terms, $2800y-\dfrac{12,350}{y}=5300.$

Clearing of fractions, $56y^2-247=106y$.

Completing the square,

$$y^2-\frac{53y}{28}+\left(\frac{53}{56}\right)^2=\frac{2809}{3136}+\frac{13,832}{3136}.$$

By evolution, $y=\frac{53}{56}\pm\frac{129}{56}=\frac{13}{4}=3\frac{1}{4}.$

Hence $x=2800y-5300=3800.$

Prob. 8. Let x denote the number of shares, and y the rate per cent. discount.

He paid $\frac{20(100-y)}{100}$ dollars per share,

and received $\frac{20(100+y)}{100}$ dollars per share.

By the first condition,

$$\frac{20x(100-y)}{100}=1500.$$

Hence $100x-xy=7500.$ (1)

By the second condition,

$$\frac{20(x-60)(100+y)}{100}=1000.$$

Expanding, $xy-60y+100x-6000=5000.$

Reducing, $xy-60y+100x=11,000.$ (2)

Substitute (1) in (2),

$$100x-7500-60y+100x=11,000.$$

Reducing, $10x-3y=925.$

Hence $y=\frac{10x-925}{3}.$ (3)

Substitute (3) in (1),

$$100x-\frac{10x^2-925x}{3}=7500.$$

Clearing of fractions, $10x^2-1225x=-22,500.$

Completing the square,

$$x^2-\frac{245x}{2}+\left(\frac{245}{4}\right)^2=\frac{60,025}{16}-\frac{36,000}{16}.$$

By evolution, $x=\frac{245}{4}\pm\frac{155}{4}=100 \text{ or } 22\frac{1}{2}.$

Hence $y=\frac{1000-925}{3}=25$ per cent.

Prob. 9. Let x denote the diminution of length, and y the increase of breadth.

By the first condition,

$$y-x=12 \text{ or } y=x+12. \qquad (1)$$

By the second condition,

$$(119-x)(19+y)=119\times 19.$$

Expanding, $2261-19x+119y-xy=2261.$

Reducing, $119y-19x=xy. \qquad (2)$

Substitute (1) in (2),

$$119x+1428-19x=x^2+12x.$$

Completing the square,

$$x^2-88x+44^2=1936+1428.$$

By evolution, $x=44\pm 58=102$ or $-14.$

Hence $y=x+12=114$ or $-2.$

Prob. 10. Let x and y denote the two numbers.

By the first condition,

$$x+y+xy=47, \text{ or } x+y=47-xy. \qquad (1)$$

By the second condition,

$$x^2+y^2-x-y=62. \qquad (2)$$

From Eq. (1), $x^2+y^2=2209-96xy+x^2y^2.$

Substitute in (2),

$$2209-96xy+x^2y^2-47+xy=62.$$

Completing the square,

$$x^2y^2-95xy+\left(\frac{95}{2}\right)^2=\frac{9025}{4}-\frac{8400}{4}.$$

By evolution, $xy=\frac{95}{2}\pm\frac{25}{2}=60$ or $35.$

From Eq. (1), $x+y=47-35=12.$

By involution $x^2+2xy+y^2=144;$

$4xy \quad =140.$

By subtraction, $x^2-2xy+y^2= \quad 4.$

By evolution, $x-y=\pm 2.$

Hence $x=7$ or $5.$

Prob. 11. By the first condition,

$$x+y=a.$$

By the second condition, $\frac{1}{x}+\frac{1}{y}=b.$

Clearing of fractions, $x+y=bxy=a.$

Hence $$xy=\frac{a}{b}.$$

By involution, $$x^2+2xy+y^2=a^2.$$

By subtraction, $$x^2-2xy+y^2=a^2-\frac{4a}{b}.$$

By evolution, $$x-y=\pm\sqrt{a^2-\frac{4a}{b}}.$$

Hence $$x=\frac{a}{2}\pm\sqrt{\frac{a^2}{4}-\frac{a}{b}}.$$

Prob. 12. Let x denote the number of days in which B could reap the field, and y the days in which C could reap it.

Then $\frac{1}{9}+\frac{1}{x}:\frac{1}{x}::24:$ the number of dollars B would have received $=\frac{216}{9+x}$;

$\frac{5}{x}\times 24=\frac{120}{x}=$ the number of dollars he did receive.

By the conditions, $$\frac{216}{9+x}=\frac{120}{x}+1.$$

Clearing of fractions,
$$216x=1080+120x+9x+x^2.$$

Completing the square,
$$x^2-87x+\left(\frac{87}{2}\right)^2=\frac{7569}{4}-1080.$$

By evolution, $$x=\frac{87}{2}\pm\frac{57}{2}=15 \text{ or } 72.$$

Let $x=15$; then $$\frac{5}{9}+\frac{5}{15}+\frac{2}{y}=1.$$

Reducing, $$\frac{2}{y}=1-\frac{5}{9}-\frac{1}{3}=\frac{1}{9}.$$

Hence $$y=18.$$

The other value of x is excluded by the conditions of the problem.

Prob. 13. By the conditions, $x^3+y^3=35;$ (1)

$x^9+y^9=20{,}195.$ (2)

Involving (1), $x^9+3x^6y^3+3x^3y^6+y^9=42{,}875.$ (3)

Subtract (2) from (3), $3x^6y^3+3x^3y^6=22{,}680.$

By division, $x^3+y^3=\frac{7560}{x^3y^3}=35.$

Reducing, $x^3y^3=216.$

By evolution, $xy=6$, or $x=\frac{6}{y}.$ (4)

Substitute (4) in (1), $\frac{216}{y^3}+y^3=35.$

Completing the square,

$$y^6-35y^3+\left(\frac{35}{2}\right)^2=\frac{1225}{4}-\frac{864}{4}.$$

By evolution, $y^3=\frac{35}{2}\pm\frac{19}{2}=27$ or $8.$

By evolution, $y=3$ or $2.$

Prob. 14. $xy=300$, or $x=\frac{300}{y};$ (1)

$x^3-y^3=37(x-y)^3.$

Divide by $x-y$,

$$x^2+xy+y^2=37(x-y)^2=37x^2-74xy+37y^2.$$

Uniting terms, $36x^2+36y^2=75xy.$

Reducing, $12x^2+12y^2=25xy.$ (2)

Substitute (1) in (2),

$$\frac{1{,}080{,}000}{y^2}+12y^2=7500.$$

Clearing of fractions,

$$y^4-625y^2=-90{,}000.$$

Completing the square,

$$y^4-625y^2+\left(\frac{625}{2}\right)^2=\frac{390{,}625}{4}-\frac{360{,}000}{4}.$$

By evolution, $y^2=\frac{625}{2}\pm\frac{175}{2}=400$ or $225.$

By evolution, $y=20$ or $15.$

Prob. 15. Let x denote the first part, and $26,000-x$ the second part. Also, y the rate of interest of the first part, and z the rate of interest of the second part.

Then $$\frac{xy}{100}=\frac{(26,000-x)z}{100}, \text{ or } xy=26,000z-xz; \quad (1)$$

$$\frac{xz}{100}=720, \qquad xz=72,000; \quad (2)$$

$$\frac{(26,000-x)y}{100} = 980. \quad (3)$$

Hence $$y=\frac{98,000}{26,000-x}. \quad (4)$$

Substitute (2) and (4) in (1),

$$\frac{98,000x}{26,000-x}=26,000\times\frac{72,000}{x}-72,000.$$

Reducing, $$\frac{49x}{26,000-x}=26,000\times\frac{36}{x}-36.$$

Clearing of fractions,

$$49x^2=24,336,000,000-936,000x-936,000x+36x^2.$$

Reducing, $13x^2+1,872,000x=24,336,000,000.$

Completing the square,

$$x^2+144,000x+(72,000)^2=7,056,000,000.$$

By evolution, $x=84,000-72,000=12,000.$

Hence $$z=\frac{72,000}{12,000}=6 \text{ per cent.};$$

$$y=\frac{98,000}{14,000}=7 \text{ per cent.}$$

Prob. 16. Let x denote the number of feet in a side of the one, and y the feet in a side of the other.

$$x^3y+y^3x=820; \quad (1)$$

$$x^2-y^2=\ \ 9. \quad (2)$$

From Eq. (1), $$x^2+y^2=\frac{820}{xy}. \quad (3)$$

By involution, $$x^4+2x^2y^2+y^4=\frac{820^2}{x^2y^2}.$$

From Eq. (2), $$x^4-2x^2y^2+y^4=81.$$

By subtraction, $4x^2y^2=\frac{820^2}{x^2y^2}-81.$

Clearing of fractions,

$$x^4y^4+\frac{81x^2y^2}{4}=410^2=168{,}100.$$

Completing the square,

$$x^4y^4+\frac{81x^2y^2}{4}+\left(\frac{81}{8}\right)^2=168{,}100+\frac{6561}{64}.$$

By evolution, $x^2y^2=\frac{3281}{8}-\frac{81}{8}=400.$

By evolution, $xy=\pm 20.$

From Eq. (3), $x^2+y^2=\frac{820}{xy}=41.$

Hence $2xy=40.$

By addition, $x^2+2xy+y^2=81.$

By evolution, $x+y=\pm 9.$

By subtraction, $x^2-2xy+y^2=1.$

By evolution, $x-y=\pm 1.$

Hence $x=\pm 5$ or ± 4;

$y=\pm 4$ or $\pm 5.$

The larger mass cost $y^3x=5^3\times 4=500$ dollars; the smaller mass cost $x^3y=4^3\times 5=320$ dollars.

Prob. 17. Let x denote the length of the lot, and y the breadth, expressed in rods.

The perimeter, expressed in feet, is $33(x+y)$, and the cost of the lot was $330(x+y)$.

Hence $330(x+y)=660\sqrt{xy}+330.$

Reducing, $x+y=2\sqrt{xy}+1.$ (1)

Also, $\left(\frac{x+y}{2}\right)^2=xy+12\frac{1}{4}.$

Expanding, $x^2+2xy+y^2=4xy+49.$

Reducing, $x^2-2xy+y^2=49.$

By evolution, $x-y=\pm 7$, or $x=7+y.$ (2)

Substitute (2) in (1),

$$7+2y=2\sqrt{xy}+1.$$

Reducing, $3+y=\sqrt{xy}$.
By involution, $9+6y+y^2=xy=7y+y^2$.
Reducing, $y=9$.
Hence $x=16$.

Prob. 18. Let x denote the sum belonging to A; $2400-x$ the sum belonging to B.

Let y denote B's rate of interest, and $y+1$ denote A's rate of interest.

Then $$x(y+1)=\frac{5}{6}y(2400-x).$$

Expanding, $6xy+6x=12{,}000y-5xy$.

Uniting terms, $11xy+6x=12{,}000y$. (1)

Also, $$x+\frac{x(y+1)}{10}=\frac{5}{7}\left\{2400-x+\frac{y(2400-x)}{10}\right\}.$$

Expanding,
$$70x+7xy+7x=120{,}000-50x+12{,}000y-5xy.$$

Uniting terms,
$$12xy+127x-12{,}000y=120{,}000. \quad (2)$$

Subtract (1) from (2), $xy+121x=120{,}000$. (3)

Multiply (3) by 11, $11xy+1331x=1{,}320{,}000$. (4)

Subtract (1) from (4),
$$1325x=1{,}320{,}000-12{,}000y.$$

Hence $$x=\frac{52{,}800-480y}{53}. \quad (5)$$

Substitute (5) in (3),
$$\frac{52{,}800y-480y^2}{53}+\frac{121}{53}(52{,}800-480y)=120{,}000.$$

Reducing, $110y-y^2+13{,}310-121y=13{,}250$.

Uniting terms, $y^2+11y=60$.

Completing the square,
$$y^2+11y+\left(\frac{11}{2}\right)^2=\frac{121}{4}+\frac{240}{4}.$$

By evolution, $$y=\frac{19}{2}-\frac{11}{2}=4.$$

From Eq. (5), $$x=\frac{52{,}800-1920}{53}=960.$$

Prob. 19. Let x denote the number of yards of the better sort, and y of the poorer.

$x(x+y)$ denotes the cost of the best piece.

By the first condition,

$$x(x+y)+y(x-y)=63.$$

Expanding, $x^2+2xy-y^2=63.$ (1)

Also, $x(x+y)=6y(x-y).$

Expanding, $x^2-5xy \quad =-6y^2.$ (2)

Completing the square,

$$x^2-5xy+\left(\frac{5y}{2}\right)^2=\frac{25y^2}{4}-6y^2.$$

By evolution, $x=\frac{5y}{2}\pm\frac{y}{2}=3y$ or $2y$.

Substitute $2y$ in (1),

$$4y^2+4y^2-y^2=63.$$

Reducing, $y^2=9.$

By evolution, $y=3.$

Hence $x=6.$

Substitute $3y$ in (1),

$$9y^2+6y^2-y^2=63.$$

Reducing, $y^2=\frac{9}{2}.$

By evolution, $y=3\sqrt{\tfrac{1}{2}}.$

Hence $x=9\sqrt{\tfrac{1}{2}}.$

Prob. 20. Reducing Eq. (2), we have

$$3x^2-5xy=2y^2.$$

Completing the square,

$$x^2-\frac{5xy}{3}+\left(\frac{5y}{6}\right)^2=\frac{49y^2}{36}.$$

By evolution, $x=\frac{5y}{6}\pm\frac{7y}{6}=2y.$

Substitute this value in Eq. (1), and we have

$$\frac{2y^2}{3y}+2=\frac{6y^2-2y^2}{6y}+\frac{2y}{3}.$$

Hence $y=3$, and $x=6.$

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Ex. 1. $s=\frac{n}{2}(a+l)=500\times1001=500{,}500.$

Ex. 2. $s=\frac{n}{2}(a+l)=19\times2839=53{,}941.$

Ex. 3. $a=l-(n-1)d=24-21\times\frac{5}{7}=9;$

$s=\frac{n}{2}(a+l)=11\times33=363.$

Ex. 4. $n=\frac{l-a}{d}+1=21\times\frac{8}{7}+1=25;$

$s=\frac{n}{2}(a+l)=-22\frac{1}{2}\times\frac{25}{2}=-281\frac{1}{4}.$

Ex. 5. $n=\frac{2s}{a+l}=\frac{784}{28}=28;$

$d=\frac{l-a}{n-1}=\frac{18}{27}=\frac{2}{3};$

Ex. 6. $d=\frac{l-a}{m+1}=\frac{6}{9}=\frac{2}{3}.$

Ex. 7. Represent the series by a, $a+d$, $a+2d$, $a+3d$, etc.

$a+18d+a+42d+a+56d=827.$

Hence $3a+116d=827;$ (1)

$a+26d+a+57d+a+68d+a+72d=1581.$

Hence $4a+223d=1581.$ (2)

Multiply (2) by 3, $12a+669d=4743.$ (3)

Multiply (1) by 4, $12a+464d=3308.$ (4)

Subtract (4) from (3), $205d=1435.$

Hence $d=7.$

From Eq. (1), $3a+812=827.$

Hence $a=5.$

Ex. 8. $l=a+(n-1)d=3.24+499\times\frac{5}{100}=28.19;$

$s=\frac{n}{2}(a+l)=250\times31.43=7857.50.$

Ex. 9. $l=a+(n-1)d=16\frac{1}{12}+19\times32\frac{1}{6}=627\frac{1}{4};$

$s=\frac{n}{2}(a+l)=10\times643\frac{1}{3}=6433\frac{1}{3}.$

Ex. 10. Represent the four parts by

$$\frac{1}{4}-3d,\ \frac{1}{4}-d,\ \frac{1}{4}+d,\ \frac{1}{4}+3d.$$

By the conditions, $\frac{1}{16}+15d^2=\frac{1}{10}.$

Hence $d^2=\frac{1}{400}.$

By evolution, $d=\frac{1}{20};$

$$\frac{1}{4}-\frac{3}{20}=\frac{1}{10},\ \frac{1}{4}-\frac{1}{20}=\frac{2}{10},\ \frac{1}{4}+\frac{1}{20}=\frac{3}{10},\ \frac{1}{4}+\frac{3}{20}=\frac{4}{10}.$$

Ex. 11. After reduction, we have

$$x^2-17x=2378.$$

Completing the square,

$$x^2-17x+\left(\frac{17}{2}\right)^2=\frac{9801}{4}.$$

By evolution, $x=\frac{17}{2}\pm\frac{99}{2}=58.$

Last term $=172-144=28.$

Ex. 12. After reduction, we have

$$5y^2-91y=-366.$$

Completing the square,

$$y^2-\frac{91y}{5}+\left(\frac{91}{10}\right)^2=\frac{8281}{100}-\frac{7320}{100}.$$

By evolution, $y=\frac{91}{10}\pm\frac{31}{10}=6.$

$86-60=26=$ the number of men alive when the provisions were exhausted.

Ex. 13. $l=ar^{n-1}=2^{12}=4096.$

$$s=\frac{lr-a}{r-1}=8192-1.$$

Ex. 14. $l=ar^{n-1}=7\times 3^{10}=7\times 59{,}049=413{,}343;$

$$s=\frac{lr-a}{r-1}=\frac{1{,}240{,}029-7}{2}=620{,}011.$$

Ex. 15. $$a=\frac{(r-1)s}{r^n-1}=\frac{6\times 411,771}{823,542}=3;$$
$$l=ar^{n-1}=3\times 7^6=3\times 117,649.$$

Ex. 16. $$r=\left(\frac{l}{a}\right)^{\frac{1}{n+1}}=\sqrt[12]{\tfrac{1}{2}}=0.943874=\text{the first term};$$
$$r^2=0.89090=\text{the second term, etc.}$$

Ex. 17. $$A=P(1+r)^n=1200(1.04)^{36}=1200\times 4.10392=4924.70.$$

Ex. 18. $$r=\left(\frac{l}{a}\right)^{\frac{1}{n-1}}=(1,048,576)^{\frac{1}{10}}=4.$$

Ex. 19. $$x+xy+xy^2=42; \quad (1)$$
$$xy^2-2xy+x=6. \quad (2)$$

Subtract (2) from (1), $3xy=36$, or $xy=12$. (3)

Substitute (3) in (1), $x+12+12y=42$.

Hence $x=30-12y$. (4)

Substitute (4) in (3), $30y-12y^2=12$.

Completing the square,
$$y^2-\frac{5y}{2}+\left(\frac{5}{4}\right)^2=\frac{25}{16}-1.$$

By evolution, $y=\frac{5}{4}\pm\frac{3}{4}=2$ or $\frac{1}{2}$.

Hence $x=\frac{12}{2}=6$ or 24.

The numbers are 6, 12, and 24, or 24, 12, and 6.

Ex. 20. $$xy^2-x=24; \quad (1)$$
$$x^2y^4-x^2:x^2y^4+x^2y^2+x^2::5:7. \quad (2)$$

Reducing, $7y^4-7=5y^4+5y^2+5$.

Completing the square,
$$y^4-\frac{5y^2}{2}+\left(\frac{5}{4}\right)^2=\frac{25}{16}+\frac{96}{16}.$$

By evolution, $y^2=\frac{5}{4}\pm\frac{11}{4}=4$ or $-\frac{3}{2}$.

By evolution, $y=\pm 2$.

From Eq. (1), $x=\frac{24}{y^2-1}=\frac{24}{3}=8$.

Ex. 21. Denote the numbers by x, xy, xy^2; then

$$x^3y^3=216.$$

By evolution, $xy=6.$

We may then denote the series by

$$\frac{6}{y}, \ 6, \ 6y;$$

$$\frac{216}{y^3}+216+216y^3=1971.$$

Clearing of fractions,

$$8+8y^3+8y^6=73y^3.$$

Uniting terms, $8y^6-65y^3=-8.$

Completing the square,

$$y^6-\frac{65y^3}{8}+\left(\frac{65}{16}\right)^2=\frac{4225}{256}-1.$$

By evolution, $y^3=\frac{65}{16}\pm\frac{63}{16}=8$ or $\frac{1}{8}.$

By evolution, $y=2$ or $\frac{1}{2}.$

Hence $x=3$ or $12.$

The numbers are 3, 6, and 12, or 12, 6, and 3.

Ex. 22. $$x+xy+xy^2+xy^3=350; \qquad (1)$$

$$xy^3-x:xy^2-xy::37:12. \qquad (2)$$

Divide by $xy-x$, $y^2+y+1:y::37:12.$

Reducing, $12y^2+12y+12=37y.$

Completing the square,

$$y^2-\frac{25y}{12}+\left(\frac{25}{24}\right)^2=\frac{625}{576}-1.$$

By evolution, $y=\frac{25}{24}\pm\frac{7}{24}=\frac{4}{3}$ or $\frac{3}{4}.$

From Eq. (1), $x+\frac{4x}{3}+\frac{16x}{9}+\frac{64x}{27}=350.$

Clearing of fractions,

$$27x+36x+48x+64x=27\times350.$$

Uniting terms, $175x=27\times350.$

Hence $x=54.$

Ex. 23. Denote the numbers by

$$x,\ xy,\ xy^2,\ 2xy^2-xy.$$

Then $$2xy^2-xy+x=14; \quad (1)$$

$$xy^2+xy \quad =12. \quad (2)$$

Subtract (1) from twice (2),

$$3xy-x=10, \text{ or } y=\frac{10+x}{3x}. \quad (3)$$

Substitute (3) in (2),

$$\frac{(10+x)^2}{9x}+\frac{10+x}{3}=12.$$

Clearing of fractions,

$$100+20x+x^2+30x+3x^2=108x.$$

Completing the square,

$$x^2-\frac{29x}{2}+\left(\frac{29}{4}\right)^2=\frac{841}{16}-25.$$

By evolution, $$x=\frac{29}{4}\pm\frac{21}{4}=2 \text{ or } \frac{25}{2}.$$

Hence $$y=\frac{10+x}{3x}=2 \text{ or } \frac{3}{5}.$$

Ex. 24. Denote the numbers by $5-y,\ 5,\ 5+y$.

By the conditions, $(6-y)(24+y)=9^2$.

Expanding, $144-18y-y^2=81$.

Transposing, $y^2+18y+81=144$.

By evolution, $y=12-9=3$.

The numbers are $5-3$, 5, and $5+3$.

THE END.